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Daubechies Wavelet Charts to Control and Monitor Individual Observations

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Abstract

Traditional quality control charts (Shewhart charts), such as the individual observations chart using the moving average, did not provide charts specifically for controlling and monitoring the variation or differences in the produced material. Therefore, the researchers proposed creating new charts for the approximation (low-pass filter) and detail (high-pass filter) coefficients of the Daubechies wavelet for order (N = 2, 3, 4, 5, 6) corresponding to the Shewhart chart for individual observations. Also, the proposed charts are robust to noise data that affects the accuracy of the results of the traditional charts. The proposed charts use Daubechies wavelet analysis based on discrete wavelet transformation. One of the charts highlights approximation coefficients to track individual observations. At the same time, the other focuses on the detailed coefficients to identify the differences between the observations in the material being produced. To create the charts, simulated data was used as well as real data, specifically from the engineering tests that were done on the strength of the steel that Erbil Steel Factory uses. The results showed that the charts were highly effective not only in terms of accuracy but also in dealing with noisy data. Furthermore, it was proven that they were better than the traditional charts at spotting very small details in the production process. This can make a huge difference in maintaining higher quality, and the steel production process in the Erbil factory was under control, meaning that the steel strength met the required specifications.

Keywords: Quality Control Charts, Individual Observations Chart, Daubechies Wavelet, Discrete Wavelet Transformation, Approximate and Detail Coefficients.

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لوحات المويجات دوبيشيز للتحكم في المشاهدات المفردة ومراقبتها

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المستخلص

لم توفر لوحات الجودة التقليدية (لوحات شيوارت)، مثل لوحة المشاهدات المفردة باستخدام المتوسط المتحرك، لوحات خاصة للتحكم ومراقبة التباين أو الاختلافات في المادة المنتجة. لذلك، اقترح الباحثون إنشاء لوحات جديدة لمعاملات التقريب (مرشح الترددات المنخفضة) والتفصيل (مرشح الترددات العالية) لمويجات دوبيشيز حسب الرتبة (N = 2, 3, 4, 5, 6) المقابلة للوحة شيوهارت للمشاهدات المفردة. كما أن اللوحات المقترحة حصينة ضد ضوضائية البيانات التي تؤثر على دقة نتائج اللوحات التقليدية. تستخدم اللوحات المقترحة تحليل مويجات دوبيشيز بناءً على تحويل المويجات المتقطع. يسلط أحد اللوحات الضوء على معاملات التقريب حتى يتمكن من تتبع المشاهدات المفردة. في الوقت نفسه، تركز اللوحة الآخرئ على المعاملات التفصيلية لتحديد الاختلافات بين المشاهدات في المادة التي يتم إنتاجها. ولإنشاء هذه اللوحات، تم استخدام بيانات المحاكاة بالإضافة إلى بيانات حقيقية، وتحديدًا من الاختبارات الهندسية التي أجريت على قوة الفولاذ الذي ينتجه مصنع أربيل للصلب. وأظهرت النتائج أن اللوحات كانت فعالة للغاية ليس فقط من حيث الدقة ولكن أيضًا في التعامل مع البيانات المشوشة. وعلاوة على ذلك، ثبت أنها كانت أفضل من اللوحات التقليدية في اكتشاف التفاصيل الصغيرة جدًا في عملية الإنتاج. وهذا يمكن أن يحدث فرقًا كبيرًا في الحفاظ على الجودة العالية وأن عملية انتاج الفولاذ في مصنع أربيل كانت تحت السيطرة أي أن قوة الفولاذ مطابقة للمواصفات المطلوبة.

الكلمات المفتاحية: لوحات السيطرة النوعية، المشاهدات المفردة، المويجة دوبيشيز، التحويل المويجي المتقطع ومعاملات التقريب والتفصيل.

1. Introduction

There are several types of control charts, and the most widely used one is the Individual Observation Chart using Moving Averages. It analyzes individual data points by smoothing them over time, which helps in identifying trends and shifts in the process (Montgomery, 2011). Nonetheless, although these charts are widely used, they have significant limitations. One main limitation is that they do not effectively control variations that occur in the production processes. They are not sensitive enough to be able to detect small changes that have the potential to develop into bigger issues later. This could significantly affect the product quality. Their effectiveness is further reduced by the fact that they usually cannot distinguish between short-term noise and meaningful trends (Montgomery, 2020). These limitations paved the way for improvements in the field of monitoring and detection. To overcome the shortcomings of traditional methods, wavelet analysis emerged. These wavelets are mathematical functions that analyze signals at multiple scales, which makes them ideal in their ability to detect both universal trends and local changes. The use of wavelets was a revolution in our understanding and analyzing data, especially in identifying minor changes that would be missed otherwise. Among these wavelets, the Daubechies wavelets are highly regarded as efficient because of their local support, symmetry, and ability to analyze data with multiple resolutions (Daubechies, 1992; Ali & Esraa, 2016).

These mentioned properties of the Daubechies wavelets make them highly suitable for handling non-stationary data, which is very common in dynamic environments of production. Much research has been conducted to better understand and explain the application of wavelet analysis in control charts.

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Among these, many have shown the ability of wavelet-based techniques to provide higher sensitivity and accuracy in comparison to traditional methods (Sang et al., 2002; Kareem et al., 2020). Donoho (1995) made the most influential contribution to understanding the application of wavelets as he explained their potential in denoising signals and separating noise from meaningful data components. This contribution paved the way for the modern application of wavelets in quality control.

Daubechies (1992) made further advancements in the field as well by working on compact, symmetric wavelets with capabilities to do multi-resolution analysis. This also greatly enhanced the applicability of wavelet techniques in real-world industrial settings. Taking these advancements into consideration, this research paper focuses on the application of the Daubechies wavelet in improving control charts. The main objective of this research study is to explain how Daubechies wavelets can be employed to improve the sensitivity of control charts, which allows them to effectively detect minor and short-term changes in the production process. There is a gap between traditional quality control methods and modern techniques, and this research study aims to bridge this gap. So, the paper offers a comprehensive framework for improving industrial performance. Traditional charts have limitations, and wavelet analysis has an advantage over them (Donoho, 1995; Gao et al., 2018); thus, this research paper seeks to contribute to more efficient, accurate, and reliable production processes.

This article proposed creating new charts for the approximation (low-pass filter) and detail (high-pass filter) coefficients of the Daubechies wavelets corresponding to the Shewhart chart for individual observations. Also, the proposed charts are robust to noise data that affects the accuracy of the results of the traditional charts.

2. Quality Control Charts:

Quality control charts are tools to assess the condition of the production process: to determine if they are under control or over control. Thus, they assist in making well-informed decisions regarding the progression of the process in each stage of production. They are among the most widely used statistical techniques in production and service process quality control (Ali et al. 2018). Types of qualitative control charts of data handled by the production or service process determine the two main types of control charts. These are variable quality control charts. Features of a control chart control diagrams for characteristics basic parts of control charts Scientist Shewhart developed the basic design of control charts, which includes the control chart, which is the center border (Koetsier et al. 2012).

Limit of Central Control (CL) one option is the Central Control Limit:

$$\mathrm{CL} = Mean(Observations) \tag{1}$$

For example, it displays the average. It displays the highest limit of the control chart, which is symbolized by the following: (UCL, or Upper Control Limit) Maximum Control Point 2. By going over what is permitted and departing by three standard deviations from the central control limit, the process is out of control. It represents the lowest limit of the control chart, which is as follows: Lower Control Limit Lower Control Limit (LCL) 3:

$$\begin{matrix}\mathrm{UCL}\\ \mathrm{LCL}\end{matrix} = \mathrm{CL} \pm 3 \times \text{Standard Devation} \tag{2}$$

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The process readings cannot be less than the central control limit, and there are three standard deviations outside of it (Ali et al. 2017), as shown in Figure 1:

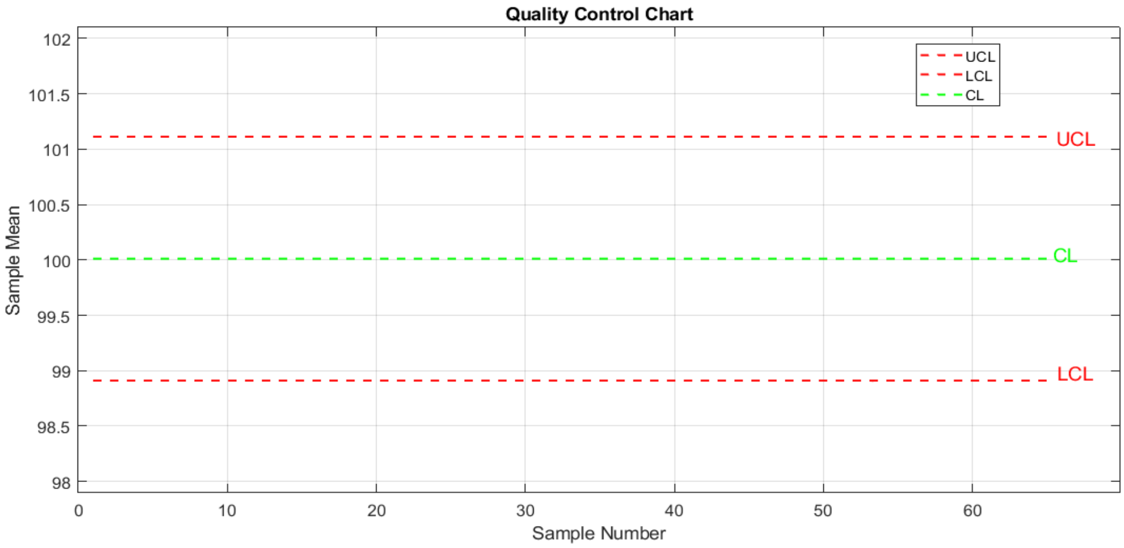

Figure 1. The hierarchical process for DWT coefficients

3. Individual Control Chart

One tool for tracking variable data is the individual control chart. It has two charts: one that shows the changing range (R) between the samples and the other that shows the results of individual samples (X). This chart is particularly useful for tracking procedures where data is not gathered often. It examines the evolution of individual sample results over time (Barlow & Irony, 1992). Considering when the results will be measured is crucial because rational subgrouping is not used here. The average on the "individuals" chart provides a reliable estimate of the overall average if the process is stable, and the average range aids in estimating the standard deviation (Samad et al. 2024). A quality control tool called an individual control chart plots individual measurement points to keep an eye on a particular process (Amin & Ethridge, 1998).

The horizontal axis (x-axis) plots the points against time or the sequential number of observations, while the vertical axis (y-axis) plots the points representing individual data points (e.g., time measurements, product weights, or dimensions). This chart makes it possible to track minor or major modifications to production processes or processes under observation, which aids in determining whether the process is under control (Ali, 2007). How to Find the Target Line and Target: The Target is the ideal or desired value for the parameter being measured. For instance, the goal can be 100 grams or a specific temperature if you are tracking a product's weight. The company's requirements or industry norms are used to determine this goal (Bakir, 2004). Target Line is a horizontal line at the target value is called the target line, and it is displayed on the chart. It stands for the ideal values that the procedure needs to strive for. The target line can alternatively be a predetermined goal based on specified requirements or, in some situations, the average value of the process (Roes et al. 1993). How to Calculate LCL and UCL: The upper limit that the plotted points shouldn't go above is known as the UCL (Upper Control Limit). Points above this threshold suggest that the procedure is unmanageable (Rigdon et al. 1994).

$$\mathrm{UCL} = \mu + 3 \times \sigma \tag{3}$$

Where:

μ represents the data's average (or desired value).

σ is the data's (or process's) standard deviation.

Lower Control Limit, or LCL: The lower number below which the plotted points should not fall is known as the LCL. If any point is under this threshold, the process is out of control. (Montgomery, 2020).

$$\mathrm{LCL} = \mu - 3 \times \sigma \tag{4}$$

4. Wavelets

An oscillation that begins at zero, peaks, and then falls back to zero is called a wavelet. It has a scale that explains its growth and contraction, a certain oscillation time, and a point at which it is strongest. By the 1990s, wavelet analysis had gained popularity in geophysics after making its debut in mathematics in the 1980s (Walker, 1999). Wavelets are useful for data compression, picture processing, and signal analysis. Wavelets aid in the separation of data at various sizes while preserving some sense of the location of events in space or time. For instance, wavelets are commonly used by the FBI to store and compress fingerprint data (Ricker, 1953).

Wavelets are excellent for researching fractal fields since they operate by altering one or two fundamental waveforms. They are particularly helpful when examining time series data that fluctuates over time, something that conventional Fourier analysis might not be able to accomplish. Wavelets are an efficient way to compress data from radar or satellite photos. A low-resolution rendition of the original image is produced by removing the highest frequencies while preserving crucial local information. However, because Fourier concentrates on global patterns rather than local ones, it is known to lose an image's distinguishable qualities if too many harmonics are eliminated (Mallat, 1998). In general, wavelets are viewed as a compromise between examining data in frequency space, which yields frequency insights but loses timing details and looking at data at precise times, which yields detailed timing information. Wavelet analysis is a helpful compromise since it lets us retain elements of both.

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5. Daubechies Wavelet

Daubechies wavelets are types of wavelets that were developed by mathematician Ingrid Daubechies in the late 1980s and are nowadays among the most famous and widely used wavelets in signal processing and data analysis. These wavelets are defined by a specific number of coefficients that is denoted by “N”. These coefficients control the shape of the wavelet. For example, Daubechies 2 (DB2) has two coefficients, Daubechies 4 (DB4) has four, and so on. The precision of the wavelet increases with the increase of coefficients. The higher the precision, the more accurate the representation of the signal, which allows for better extraction of details in the analysis process (Duaa et al. 2024).

Daubechies wavelets are mostly used to analyze non-stationary signals because they can effectively represent signals that have sharp transitions or clear edges. In this context, Daubechies wavelets are the best for multi-resolution analysis, also called MRA. MRA is a method that allows signals to be analyzed at different levels of detail. This efficiency makes it easier to focus on different and specific parts of the signal while the overall structure is maintained, which is very useful in handling signals whose frequencies are constantly changing (Ali and Awaz, 2017). One of the prominent distinguishing features of the Daubechies wavelets is that their length is finite, meaning that they are non-zero over some time and then become zero. This makes them very useful in analyzing time-domain signals that have distinct changes or sudden transitions. Unlike the Fourier Transform, which only relies on frequency components, the Daubechies wavelets can effectively represent signals through the domains of both time and

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frequency, making them perfect for applications like data compression and denoising (Antoniadis, 2007).

One of the widest uses of Daubechies wavelets is data compression, particularly in JPEG 2000 image compression format. The algorithms of JPEG 2000 rely on wavelets rather than Fourier Transforms in analyzing data, which results in better compression while the quality is maintained. Daubechies wavelets are highly effective in the compression of images that have sharp edges or fine details. In such images, capturing small details is highly crucial (Chui, 1997). In addition to their usage in compression, the Daubechies wavelets are also used to denoise signals. If employed, they can filter out noise from signals by decomposing the signal into different components and recovering merely the essential details. This makes the wavelets valuable tools in applications like audio and video processing and medical data analysis (Coifman and Wickerhauser, 1992).

Daubechies wavelets are also used in mathematical applications, such as solving partial differential equations and analyzing data that have sudden and irregular changes. One key feature of the Daubechies wavelets is that they can handle asymmetric signals. This characteristic is unique and distinguishes them from other wavelets that rely on symmetric representations of signals. These properties make the Daubechies wavelets a powerful tool for analyzing complex and non-stationary signals (Omer et al. 2024).

In summary, Daubechies wavelets are among the most important mathematical tools that are used in the processing of signals and data analysis (Sakar et al. 2024). Because of their ability to handle complex signals with sudden transitions and because of their effectiveness in data compression, denoising, and multiresolution analysis, Daubechies wavelets have remained the first choice in fields like audio and video processing, compression of images, and the analysis of medical data. The wavelets are named after the researcher Ingrid Daubechies, who is a pioneer researcher in the subject of wavelength and the normal orthogonal wavelets with an anchor in 1988. He was the one who made the discrete wavelet analysis applicable (Heyam and Saad, 2022). To further understand the abbreviations:

- D or db is the researcher's name, Daubechies
- N is the length of the candidate or rank
- L1 is the number of vanishing or ephemeral moments of the wavelet function

L1 indicates the same candidate who is the second-ranked candidate in this family and is associated with N in the following relations (Qais and Taha, 2020).

$$L1 = \frac{N}{2} \tag{5}$$

In general, (dbn) represents the family of small waves of order n (note that the small wave Haar is one of the members of this family and has the following characteristics:

- The anchor of the small wave (dbn) is on the period [0, 2n-1].
- A small wave (dbn) has n ephemeral moments, i.e.:

$$\left|\frac{d^k}{dX^k}\psi(X)\right| < \infty \quad \Rightarrow \quad \int X^k\psi(X)dX = 0\,,\ \ 1 \le \mathrm{k} \le \mathrm{n} \tag{6}$$

- Anomalies increase with rank, (dbn) have (rn) continuous derivatives (r is about 0.2)

6. Discrete Wavelet Transformation

Unlike Fourier theory, the continuous wavelet transformation is not as widely used as the discrete wavelet transformation. On the one hand, continuous wavelet transforms are mostly used in exploring theoretical properties (Abdel Qader and Heyam, 2021). On the other hand, discrete wavelet transformations are more practical for most computer-based applications. It is important to know that the "discrete" part of the discrete wavelet transform is not the same discrete that exists in the Fourier transform but is like the Fourier series. The term "discrete" only refers to the transform domain parameter (i.e. the frequency variable in the Fourier series or the scale and translation variables in the wavelet transform). So, it does not refer to the independent variable of the function that is being transformed, such as time or space (Heyam, 2012). In other terms, the scales and translations are discrete, but the independent variable (i.e. time or space) remains continuous. On a computer implementation, the independent variable is discretized, and the integration is approximated by summations. Due to this, the wavelet transform is described as the continuous-time wavelet series (CTWS), which is more accurate (Young, 1992). The discrete wavelet transform (DWT) is a highly applicable algorithm and is used in many different fields, such as science, engineering, mathematics, and computer science. It functions by breaking down an observation using scaled and shifted versions of a compactly supported basis function that is known as the mother wavelet. This results in a multiresolution representation of the observation (Cascio, 2007). For instance, if there is an observation vector y with n values and k is an integer, then the DWT of y can be expressed as formula (5):

$$W = wy \tag{7}$$

Here, w is a wavelet matrix with $n \times n$ dimensions W is a vector with $n \times 1$ dimensions where scaling and wavelet coefficients are both included. The vector of wavelet coefficients is organized into $k \times 1$ vectors.

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$W = [W_1, W_2, ..., W_k, V_{k0}]^T$. At each level of decomposition, the approximation coefficients are divided into bands using the same wavelet. Plus, the details from each decomposition level are added together, as shown in the following equation (Kadir et al. 2024):

$$y = W\, w^T = \sum_{k=1}^{k_0} W_k^T W_K + V_{k_0}^T V_{k_0} \quad (8)$$

At each level (k), we can reconstruct the observations from the denoise data by the inverse DWT (Gençay et al., 2001). The DWT coefficients for signal X can be described in a hierarchical process, as shown in Figure 2.

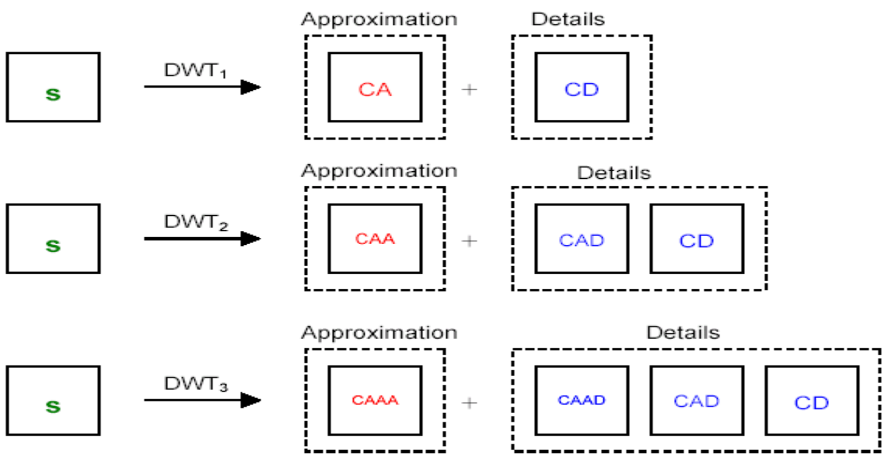

Figure 2. The hierarchical process for DWT coefficients

An alternative is to create a wavelet transform that is specifically designed to work on sample data. This leads to discrete wavelet transformation (DWT). For this, the data needs to be put in the form of $x_t = x(t\Delta)$, where, for convenience, we take the integer t to range from 0 up to $N - 1$. In other words, the data should be sampled at regular intervals. If the data is not already put in this form, one can use an interpolation technique to create a regularly spaced time series. This way, the DWT can be thought of as slicing through the continuous wavelet transform (CWT), although it is important to note that the DWT can function without needing a direct connection to the

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CWT. The slices are restricted to scales λ that assume the dyadic values $2^{j-1}\,\Delta$, where j = 1, 2, . . . ,J_0. Here J_0 is the total number of scales used and is usually dictated by the sample size N and/or by the application at hand. Typically, J_0 is set to a value no greater than $\log_2 N$, so the number of scales that we can meaningfully analyze is restricted by the amount of data available. It is convenient to define τ_j (Percival et al., 2004). The discrete wavelet transform (DWT) is defined by the following equation (Tzanetakis et al., 2001):

$$W(j,k) = \frac{1}{\sqrt{M}} \sum_{t=0}^{M-1} x(t)\, \psi_{j,k}(t) \tag{9}$$

$$t = 0,1, \ldots, M-1, \qquad j \geq j_0$$

7. Proposed Charts

To create new charts based on the discrete wavelet transformation coefficients of the Daubechies wavelet, specifically to create a chart of approximation (individual for low-pass filter) and detail (deference for high-pass filter) coefficients for the Daubechies wavelet corresponding to the Shewhart chart for individual observations. Suppose x is a data vector representing the observations (n). Perform a discrete wavelet transform for Daubechies wavelet of order (N = 2, 3, 4, 5, 6) for the variable of the data vector produce n/2+N-1 if n is even and (n+1)/2+N-1 of the approximate and detailed coefficients (two partitions) after symmetrically extending for the observations x ($x_{\{extended\}}$) as in the following formulas:

$$\mathrm{A[n]} = \sum_{k=0}^{L-1} \mathrm{x}_{\{\mathrm{extended}\}}[k] \times h[2n-k] \tag{10}$$

$$D[n] = \sum_{k=0}^{L-1} x_{\{extended\}}[k] \times g[2n - k] \quad (11)$$

Where n varies over the appropriate indices of the data. Filter length (L), for Daubechies wavelet with order N = 2, L = 4 thus h (low-pass filter) and g (high-pass filter) respectively are:

$$h[k] = \left[\frac{1+\sqrt{3}}{4\sqrt{2}}, \quad \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad \frac{1-\sqrt{3}}{4\sqrt{2}}\right] \quad (12)$$

$$g[k] = (-1)^k \times h[N-1-k] \quad (13)$$

The number of coefficients for Daubechies wavelet N = 2, 3, 4, 5, and 6 is L = 4, 6, 8, 10, and 12 (L=2N). Formula 8 represents the approximation coefficients (first part), scale or father function A[n] and with n/2+N-1 if n is even and (n+1)/2+N-1 coefficients, which are proportional to the qualitative characteristics of average observations. In contrast, formula 9 represents (second part) the detail coefficients D[n], the mother or wavelet function with the same number of approximate coefficients, which are proportional to the differences (variance) of the observations of the qualitative characteristic. The proposed charts are based on Daubechies wavelet analysis, which involves creating two charts, the first for controlling the Daubechies wavelet approximation coefficients for the individual observations (A), which represent the plotted points on approximate Daubechies quality chart (ADb chart), and the second for controlling the Daubechies wavelet detail coefficients for the individual observations (D), which represent the plotted points on detail Daubechies quality chart (DDb chart). Target Lines for ADb-Chart and DDb-Chart respectively are:

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Tartget Line (A)

$$= \sum_{i=1}^{n} A(i)/n \tag{14}$$

Target Line (D)

$$= \sum_{i=1}^{n} D(i)/n \tag{15}$$

The upper and lower control limits for the ADb-Chart and DDb-Chart, respectively, are:

$$\begin{matrix} \text{UCL (A)} \\ \text{LCL (A)} \end{matrix} = \text{Target Line (A)} \pm 3 \times \text{MMR(A)}/d2 \tag{16}$$

$$\begin{matrix} \text{UCL (D)} \\ \text{LCL (D)} \end{matrix} = \text{Target Line (D)} \pm 3 \times \text{MMR(D)}/d2 \tag{17}$$

Where d_2 = 1.128 from the statistical table (m = 2), MMR represents the mean moving range of coefficients (A and D):

$$\text{MMR(A)} = \sum_{i=1}^{n-1} \text{MRA}(i)/(n-1) \tag{18}$$

$$\text{MMR(D)} = \sum_{i=1}^{n-1} \text{MRD}(i)/(n-1) \tag{19}$$

And:

$$\text{MRA}(i) = |A(i+1) - A(i)| \tag{20}$$

$$\text{MRD}(i) = |D(i+1) - D(i)| \tag{21}$$

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For i = 1,2, …, n-1.

Note that when creating the proposed charts for the first time (Phase I), all the points drawn on them must be within the control limits. If several points fall outside the control limits, modified charts can be created for the ADb chart by replacing those values with the coefficients average and the modified DDb chart by zeroing out the values that fall outside the control limits, i.e. applying the kill or keep rule.

8. Simulation Study

To construct the proposed charts for the discrete wavelet transformation of the Daubechies wavelet at orders (2, 3, 4, 5, and 6) and compare them with the Shewhart chart for individual observations, 15 observations with normal distribution with the mean (5) and variance (1) were generated and wavelet analysis was performed to calculate the approximation and detail coefficients of the Daubechies wavelet at orders (2, 3, 4, 5, and 6) and the results in Figure 3:

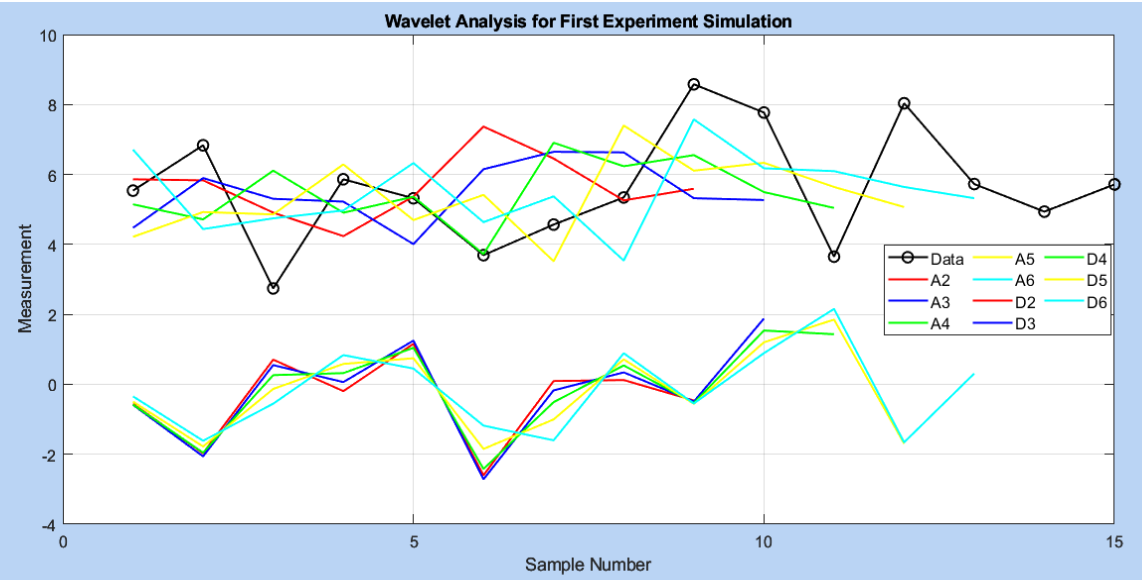

Figure 3. Wavelet Analysis for First Experiment Simulation

Figure 3 shows the wavelet analysis of the simulation data for the first experiment by performing the discrete wavelet transform of the Daubechies wavelet of orders (2, 3, 4, 5, and 6) after the convolutions to obtain the approximation coefficients with the low-pass filter that is proportional to the values of the individual observations and treatment data noise, where the number of coefficients (9, 10, 11, 12, and 13) is converging to the values of the original observations with the application of the data expansion method to obtain the centrality of the symmetric observations and result of down-sampled by depending on every second element of the filtered output (scale or father function), reducing the length of the coefficients by half plus N-1, this produces the approximation coefficients at a good resolution for vectors (A2, A3, A4, A5 and A6). The convolution process also creates detailed coefficients based on the mother or wavelet function or the high-pass filter coefficients. The detail coefficient vectors (D2, D3, D4, D5, and D6) are obtained using the same method as the previous one, which is proportional to the differences in the individual observations. They are centered around a zero mean, and the differences are measured above and below zero. The closer the coefficient is to zero, the better the individual observation is and under control. The simulation experiments (1000) times were repeated for several different sample sizes and variances, and the averages of the results of those charts were calculated as shown in Tables (1-6):

Table 1. Results Average for Simulation (Mean = 10, Variance = 1, and n = 20)

Chart	UCL	LCL	Target	Variance	D
ADb2	**11.6552**	**8.3658**	**10.0105**	**0.4650**	**3.2894**
ADb3	12.0029	8.0342	10.0185	0.4976	3.9687
ADb4	11.9842	8.0262	10.0052	0.4911	3.9580
ADb5	12.0213	7.9737	9.9975	0.4915	4.0476
ADb6	12.0630	7.9413	10.0022	0.5054	4.1217

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DDb2	**1.6556**	**-1.6543**	**0.0007**	**0.5096**	**3.3099**
DDb3	1.9851	-1.9860	-0.0005	0.4780	3.9710
DDb4	2.0152	-2.0151	0.0001	0.4814	4.0303
DDb5	2.0938	-2.0945	-0.0003	0.4861	4.1883
DDb6	2.0671	-2.0677	-0.0003	0.4688	4.1349
Individual	12.9410	7.0587	9.9998	0.9809	5.8823

Table 2. Results Average for Simulation (Mean = 10, Variance = 1, and n = 30)

Chart	UCL	LCL	Target	Variance	D
ADb2	**11.6582**	**8.3575**	**10.0079**	**0.4667**	**3.3006**
ADb3	12.0112	8.0263	10.0187	0.5084	3.9849
ADb4	12.0116	7.9985	10.0051	0.5043	4.0131
ADb5	12.0622	7.9318	9.9970	0.5100	4.1304
ADb6	12.0716	7.9319	10.0017	0.4982	4.1397
DDb2	**1.6622**	**-1.6622**	**0.0000**	**0.5144**	**3.3244**
DDb3	1.9872	-1.9877	-0.0002	0.4759	3.9750
DDb4	2.0414	-2.0417	-0.0001	0.4780	4.0831
DDb5	2.0766	-2.0767	-0.0000	0.4700	4.1533
DDb6	2.0952	-2.0955	-0.0001	0.4767	4.1907
Individual	12.9628	7.0345	9.9986	0.9890	5.9283

Table 3. Results Average for Simulation (Mean = 10, Variance = 1, and n = 50)

Chart	UCL	LCL	Target	Variance	D
ADb2	**11.6646**	**8.3489**	**10.0067**	**0.4731**	**3.3157**
ADb3	12.0057	8.0319	10.0188	0.4993	3.9739
ADb4	11.9983	8.0110	10.0046	0.5050	3.9873
ADb5	12.0402	7.9539	9.9971	0.5057	4.0863
ADb6	12.0473	7.9557	10.0015	0.4917	4.0916
DDb2	**1.6619**	**-1.6639**	**-0.0010**	**0.5132**	**3.3258**
DDb3	1.9824	-1.9833	-0.0004	0.4771	3.9657
DDb4	2.0385	-2.0397	-0.0006	0.4734	4.0782
DDb5	2.0801	-2.0802	-0.0000	0.4818	4.1603
DDb6	2.0780	-2.0785	-0.0002	0.4745	4.1565
Individual	12.9798	7.0143	9.9970	0.9928	5.9655

Table 4. Results Average for Simulation (Mean = 10, Variance = 2, and n = 20)

Chart	UCL	LCL	Target	Variance	D
ADb2	**12.3408**	**7.6889**	**10.0149**	**0.9300**	**4.6520**
ADb3	12.8325	7.2199	10.0262	0.9951	5.6126
ADb4	12.8060	7.2086	10.0073	0.9823	5.5974
ADb5	12.8586	7.1344	9.9965	0.9831	5.7242
ADb6	12.9175	7.0886	10.0031	1.0107	5.8289
DDb2	**2.3414**	**-2.3395**	**0.0009**	**1.0192**	**4.6809**

DDb3	2.8073	-2.8086	-0.0007	0.9559	5.6159
DDb4	2.8500	-2.8497	0.0001	0.9627	5.6997
DDb5	2.9611	-2.9621	-0.0005	0.9721	5.9231
DDb6	2.9234	-2.9242	-0.0004	0.9376	5.8476
Individual	14.1592	5.8403	9.9998	1.9618	8.3189

Table 5. Results Average for Simulation (Mean = 10, Variance = 2, and n = 30)

Chart	UCL	LCL	Target	Variance	D
ADb2	**12.3450**	**7.6772**	**10.0111**	**0.9334**	**4.6678**
ADb3	12.8442	7.2087	10.0265	1.0168	5.6355
ADb4	12.8449	7.1695	10.0072	1.0085	5.6754
ADb5	12.9164	7.0751	9.9958	1.0201	5.8412
ADb6	12.9297	7.0752	10.0025	0.9964	5.8545
DDb2	**2.3508**	**-2.3507**	**0.0000**	**1.0288**	**4.7015**
DDb3	2.8104	-2.8111	-0.0003	0.9517	5.6214
DDb4	2.8870	-2.8873	-0.0002	0.9560	5.7743
DDb5	2.9368	-2.9369	-0.0001	0.9400	5.8737
DDb6	2.9631	-2.9634	-0.0001	0.9535	5.9266
Individual	14.1900	5.8061	9.9981	1.9780	8.3839

Table 6. Results Average for Simulation (Mean = 10, Variance = 2, and n = 50)

Chart	UCL	LCL	Target	Variance	D
ADb2	**12.3541**	**7.6649**	**10.0095**	**0.9462**	**4.6892**
ADb3	12.8366	7.2167	10.0266	0.9987	5.6199
ADb4	12.8260	7.1871	10.0066	1.0100	5.6389
ADb5	12.8853	7.1064	9.9959	1.0114	5.7789
ADb6	12.8953	7.1090	10.0021	0.9834	5.7864
DDb2	**2.3503**	**-2.3531**	**-0.0014**	**1.0264**	**4.7034**
DDb3	2.8035	-2.8048	-0.0006	0.9543	5.6083
DDb4	2.8828	-2.8846	-0.0009	0.9469	5.7675
DDb5	2.9417	-2.9419	-0.0001	0.9636	5.8836
DDb6	2.9388	-2.9394	-0.0003	0.9490	5.8782
Individual	14.2141	5.7775	9.9958	1.9856	8.4366

9. Discussion Simulation Study

From Tables 1-6, All the proposed charts had lower D averages than the Shewhart chart, indicating the efficiency of the proposed charts compared to the Shewhart chart, the sensitivity of the proposed charts in detecting minor changes that may occur in the production process, and that they handled the data noise with variance average lower than the variance average of individual observations. The variance of the approximation and detail coefficients is always close to half the assumed variance for the simulation cases. The value of D increases as the assumed variance of the generated data increases. The target lines average of the ADb charts was close to the Shewhart chart, the DDb chart had target lines average close to zero, which is consistent with the discrete wavelet transform properties. To compare the ADb chart and DDb charts, it is noted that the ADb charts were more accurate than the DDb charts, that the Daubechies wavelet at order (2) for the ADb2 chart was better than the other ADb charts, and that the Daubechies wavelet at order (2) for the DDb2 chart was better than the other DDb charts for all simulation cases. Other different average simulations were tried, but the results did not add any new information other than average (10).

10. Real Data

The engineering units in the Erbil Steel Factory were reviewed to take different samples of the tensile strength of the steel to create the proposed and conventional charts for Phase I and then use them to control and monitor the production process in Phase II.

10.1. Phase I Operation and Interpretation of the Charts.

To construct the proposed charts for the ADb and DDb charts at orders (2, 3, 4, 5, and 6) and compare them with the Shewhart chart for individual observations, 20 observations represent the tensile strength of the steel. Wavelet analysis was performed to calculate the approximation and detail coefficients of the Daubechies wavelet at orders (2, 3, 4, 5, and 6), and the results are in Figure 4, which shows the wavelet analysis of the tensile strength of the steel by performing the discrete wavelet transform of the Daubechies wavelet of orders (2, 3, 4, 5, and 6) after the convolutions to obtain the approximation coefficients with $h(k)$ that is proportional to the values of the individual observations and treatment data noise, where the number of coefficients (4, 6, 8, 10, and 12) is converging to the values of the original observations with the application of the data expansion method to obtain the centrality of the symmetric observations and result of down-sampled by depending on every second element of the filtered output, reducing the length of the coefficients by half plus N-1, this produces the approximation coefficients at a good resolution for vectors (A2, A3, A4, A5, and A6). The convolution process also creates detail coefficients based on $g(k)$. The detail coefficient vectors (D2, D3, D4, D5, and D6) are obtained using the same method as the previous one, which is proportional to the differences in the individual observations. They are centered around a zero mean, and the differences are measured above and below zero. The closer the coefficient is to zero, the better the individual observation is and under control.

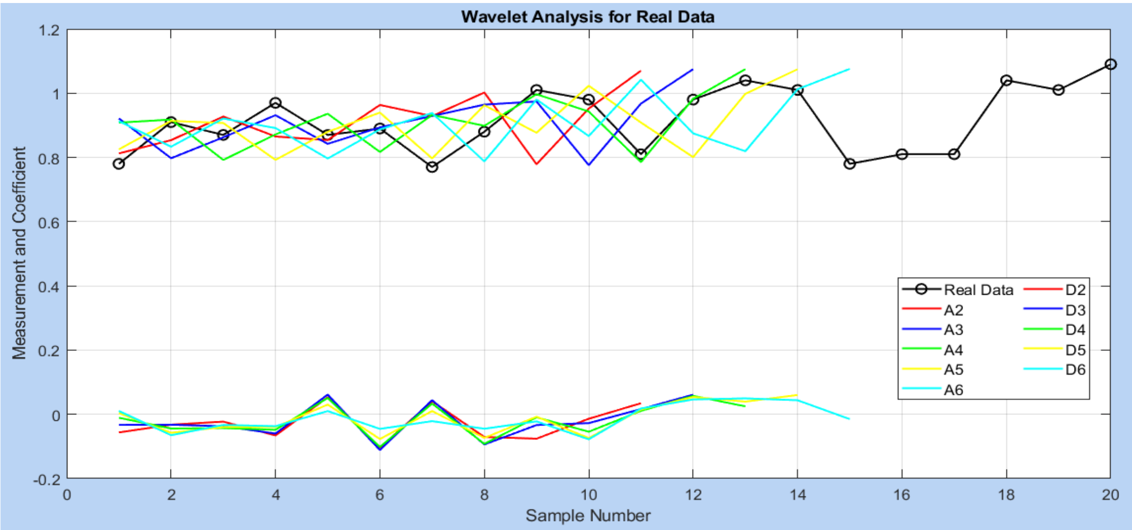

Figure 4. Wavelet Analysis for Tensile Strength of the Steel

Table 7 summarizes the results of the proposed charts, the Shewhart chart and the D distance values with variance for the tensile strength of the steel.

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All the proposed charts had lower D values than the Shewhart chart, indicating the efficiency of the proposed charts compared to the Shewhart chart, the sensitivity of the proposed charts in detecting minor changes that may occur in the production process, and that they handled the data noise with variance values lower than the variance of individual observations. The target lines of the ADb charts were close to the Shewhart chart, the DDb chart had target lines close to zero, which is consistent with the discrete wavelet transform properties. The lower approximation coefficient average, which was lower than those of individual observation charts, with the detail coefficient averages approaching zero, indicates a reduction in data noise and the achievement of highly efficient charts compared to the Shewhart chart. To compare the ADb chart and DDb charts, it is noted that the DDb charts were more accurate than the ADb charts (It depends on the noise of the data, whether in the values of the observations or the differences between the observations), that the Daubechies wavelet at order (3) for the ADb3 chart was better than the other ADb charts, and that the Daubechies wavelet at order (6) for the DDb6 chart was better than the other DDb charts, therefore these charts are drawn in Figures (5 -7):

Table 7. Charts Results for Tensile Strength of the Steel

Chart	UCL	LCL	Target	Variance	D
ADb2	1.1545	0.6650	0.9098	0.0074	0.4895
ADb3	**1.1473**	**0.6751**	**0.9112**	**0.0069**	**0.4722**
ADb4	1.1663	0.6577	0.9120	0.0068	0.5086
ADb5	1.1921	0.6220	0.9071	0.0076	0.5701
ADb6	1.2028	0.6158	0.9093	0.0077	0.5869
DDb2	0.1638	-0.2205	-0.0284	0.0027	0.3843
DDb3	0.1659	-0.2066	-0.0204	0.0032	0.3725
DDb4	0.1651	-0.1997	-0.0173	0.0026	0.3648
DDb5	0.1368	-0.1588	-0.0110	0.0024	0.2956
DDb6	**0.0890**	**-0.1133**	**-0.0122**	**0.0016**	**0.2023**
Individual	1.1689	0.6621	0.9155	0.0104	0.5067

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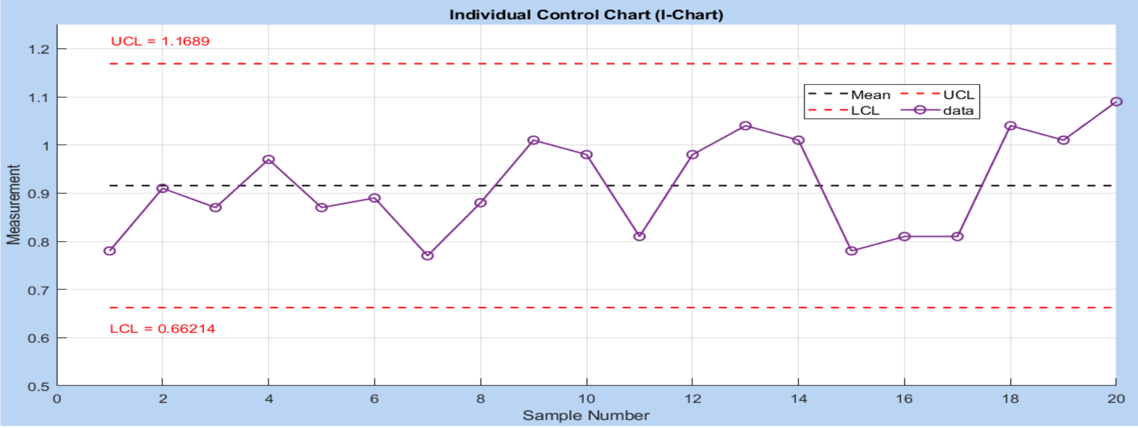

Figure 5. Individual Observations Chart for Tensile Strength of the Steel (Phase I)

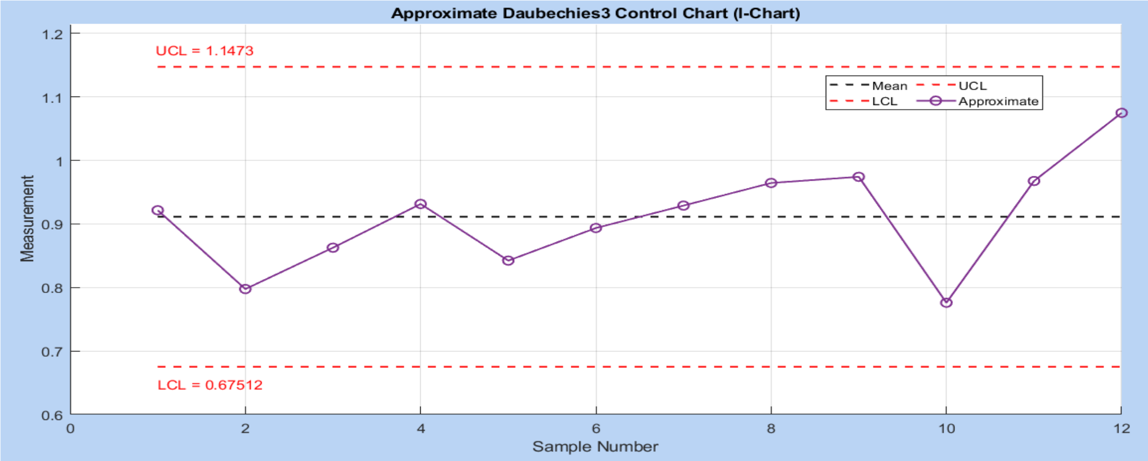

Figure 6. ADb3 Chart for Tensile Strength of the Steel (Phase I)

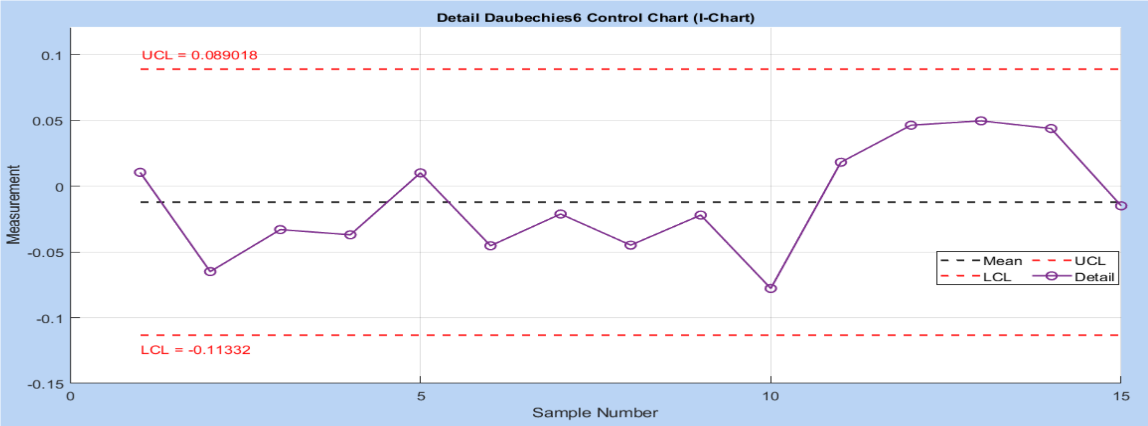

Figure 7. DDb6 Chart for Tensile Strength of the Steel (Phase I)

Figures 5 -7 of the proposed and classical charts for the first time (Phase I) show that all points are within the control limits. This means these charts can control qualitative characteristics (Tensile Strength of the Steel) in the future

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(Phase II). applying the kill or keep rule, if several points fall outside the control limits, modified charts can be created for the ADb chart by replacing those values with the coefficients average and the modified DDb chart by zeroing out the values that fall outside the control limits.

10.2. Phase II Operation and Interpretation of the Charts

The engineering units in the Erbil Steel Factory were reviewed to take a new sample of the steel's tensile strength (15 observations) to control and monitor the production process in Phase II. These new data are plotted in the individual observation, ADb4, and DDb2 charts developed in Phase I and explained in Figures 8-10.

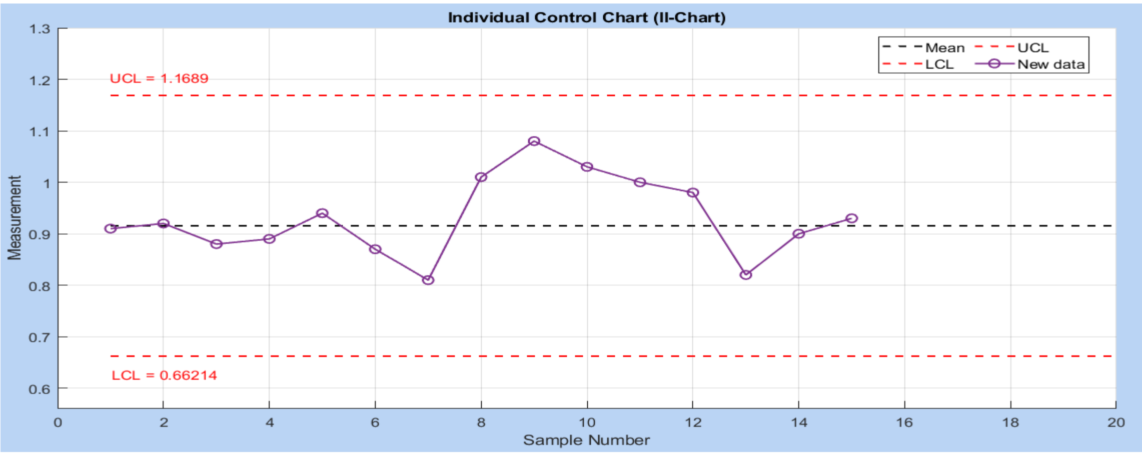

Figure 8. Individual Observations Chart for Tensile Strength of the Steel (Phase II)

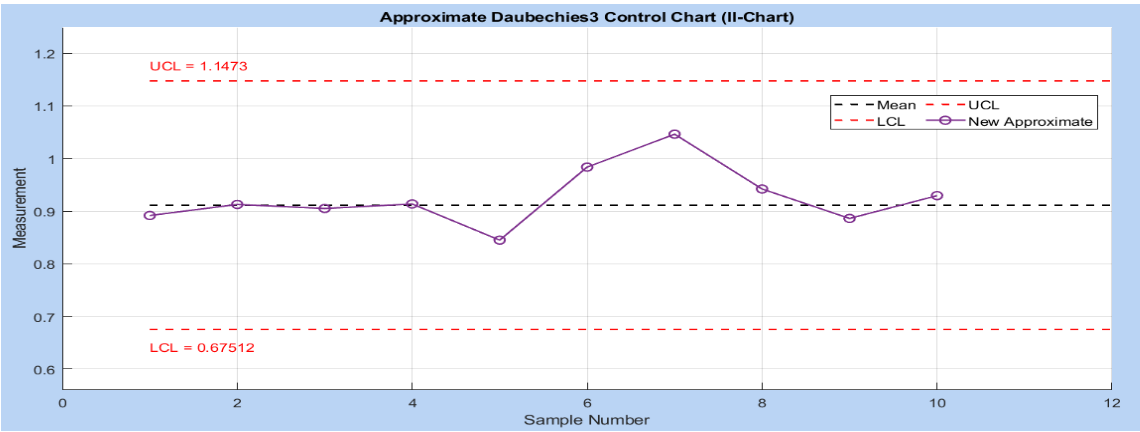

Figure 9. ADb3 Chart for Tensile Strength of the Steel (Phase II)

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Shewhart's individual observations chart (Figure 8) shows that the production process is controlled because all the points plotted on this chart are within the control limits. Therefore, the specific property of the tensile strength of steel in the Erbil factory is by the required specifications.

ADb3 chart (Figure 9) shows that the production process is under control because all the points plotted (number of approximate coefficients is 10, from (n+1)/2+N-1, where N = 3 and n = 15) on this chart are within the control limits and therefore the specific property of the tensile strength of steel (in terms of value) in the Erbil factory is by the required specifications.

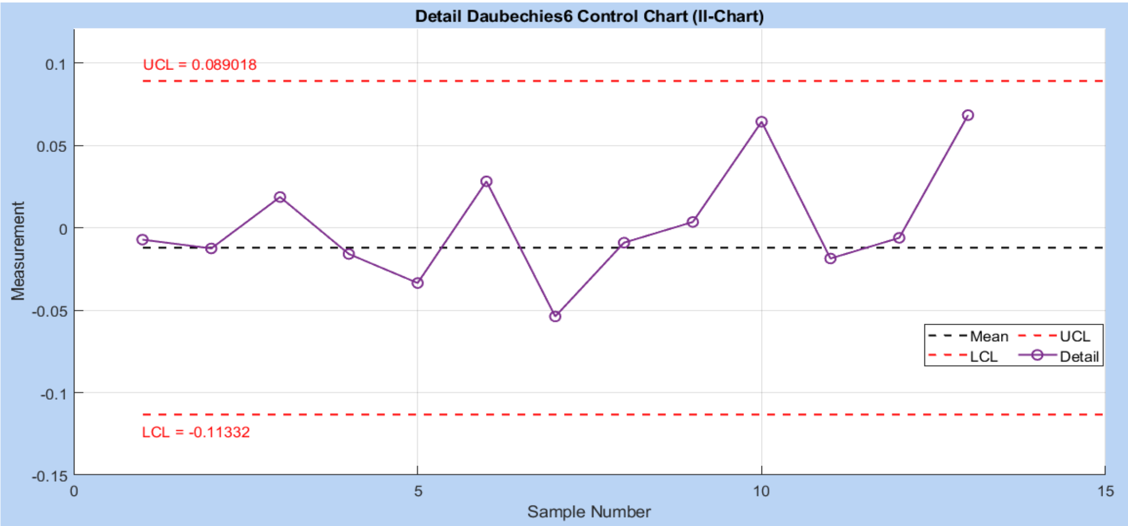

Figure 10. DDb6 Chart for Tensile Strength of the Steel (Phase II)

DDb6 chart (Figure 10) shows that the production process is under control because all the points plotted (number of detail coefficients is 13, from (n+1)/2+N-1, where N = 6 and n = 15) on this chart are within the control limits. Therefore, the specific property of the tensile strength of steel (in terms of difference or variance) in the Erbil Factory is by the required specifications.

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11. Conclusions

1. The proposed charts were more efficient than the Shewhart chart for individual observations depending on the difference between the upper and lower control limits and the minimum variance.
2. The proposed charts address the data noise problem using low and high pass filters.
3. Differences (or variance) in the quality of the produced material can be controlled and monitored through the proposed DDb chart, which is not available in the Shewhart chart for controlling individual observations.
4. The efficiency of the proposed charts varies for Daubechies wavelet at orders (2, 3, 4, 5, and 6) depending on the noise of the data used, so all orders must be applied and the best chart that gives the lowest D value must be chosen.
5. The proposed charts were more sensitive than classical charts in detecting subtle changes that may occur in the production process.
6. The ADb2 and DDb2 charts were the best and most sensitive to the slightest changes that could occur in the production process for all simulation cases.
7. The proposed and classical charts for the first time (Phase I) show that these charts can be used to control qualitative characteristics for the tensile strength of the Steel in the Erbil Steel Factory.
8. The specific property of the tensile strength of steel (in terms of value and difference or variance) in the Erbil Factory is by the required specifications.

12. Recommendations

1. Using the DWT coefficients for Daubechies wavelet in creating quality control charts to monitor the individual observations (corresponding to the Individual observations chart) and controlling and monitoring the differences (or variance) in the produced material, which is not available in traditional charts for Individual observations.
2. Conducting other studies to create charts of discrete wavelet transform coefficients for Dmey and Symlets wavelet.
3. Conducting other studies to create charts of maximal overlap discrete wavelet transform coefficients.

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