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On Multiplicative Regular Graphs of Commutative Rings

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ت حول الرسوم البيانية المنتظمة ضربيا للحلقات الابدالية

هفال امين محمد[1] و عادل قادر جبار[2]

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ABSTRACT

Background:

The zero-divisor graph $\Gamma_0(R)$ is central in algebraic graph theory. Let R be a commutative ring with identity $1 \neq 0$. In this work we define the multiplicative regular graph $\Gamma_{MR}(R)$ as a generalization that captures the behavior of regular elements those satisfying $a = aba$ or $b = bab$.

Materials and Methods:

The graph $\Gamma_{MR}(R)$ is defined using ring-theoretic conditions and graph-theoretic representations, Two distinct none-zero elements x and y are adjacent if and only if $xy \in Reg(R)$. A Python-based algorithm is used to construct and visualize $\Gamma_{MR}(R)$ for finite commutative rings.

Results:

We prove that $\Gamma_{MR}(R) \cong \Gamma_0(R)$ if and only if $R = \{0,1\}$, and $\Gamma_{MR}(R) \cong G_{Vnr^+}(R)$ when R is a regular ring. The study includes analysis of key graph invariants such as connectivity, diameter, girth, and regularity.

Conclusion:

The graph $\Gamma_{MR}(R)$ offers a generalization of existing graphs of commutative rings. By emphasizing regular elements, this model uncovers new structural relationships within the ring and explores ring-theoretic properties through graph-theoretic methods.

Keywords:

Multiplicative regular graph, connectivity of a graph, regularity of a graph, diameter of a graph, girth of a graph.

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1. INTRODUCTION

Obtaining graphs from algebraic structures is one of the extended branches of mathematics recently; Beck took the first steps in this area in 1988. He introduced the zero-divisor graph $\Gamma_0(R)$ of a commutative ring $V(\Gamma_0(R)) = R$, his work became the bridge between commutative algebra and graph theory and was mainly interested in coloring of graphs. In 1999 Anderson and Livingston associated the graph $\Gamma(R)$ to R with elements of R excluding zero. The zero divisor graphs are also studied and generalized in **[3-7]**.

Throughout this paper, R is a commutative ring with identity $1 \neq 0$, unless otherwise stated. The set of zero divisors, units, idempotent elements, nilpotent elements, regular elements of R denoted by $\mathrm{Z(R)}, \mathrm{U(R)}, \mathrm{Idem(R)}, \mathrm{Nil(R)}, \mathrm{Reg(R)}$ respectively.

In the present work, we introduce a multiplicative regular graph of R, denoted by $\Gamma_{MR}(R)$, which is undirected with $V(\Gamma_{MR}(R)) = R$. We focus on various properties of $\Gamma_{MR}(R)$, including diameter, cycles, girth, chromatic number, connectivity, regularity, and when the graph does become Eulerian. As an introduction to the graph theory, we start with the concept of an undirected simple graph, which is a fundamental object in graph theory. An undirected graph G consists of a non-empty finite set of vertices $V(G)$, and a finite set of edges $E(G)$, where each edge is an unordered pair of distinct vertices. These edges indicate a mutual relationship with no direction between the connected vertices. Two vertices v and w are said to be adjacent if they are joined by an edge, typically written as vw, and both are said to be incident to that edge. The degree of a vertex v, denoted by $\deg(v)$, is the number of edges connected to it. A vertex with degree 0 is called an isolated vertex, while one with degree 1 is known as an end-vertex.

A sub graph is any graph formed by selecting a subset of vertices and edges from a larger graph G. If a graph has no edges at all, it is called a null graph. A graph where every pair of distinct vertices is adjacent is known as a complete graph, denoted by K_n for n vertices. A cycle graph C_n is a connected graph where each vertex has degree 2, forming a closed loop. If an edge is removed from C_n , the resulting graph is a path graph, denoted P_n . A wheel graph W_n is obtained by connecting a new central vertex to all vertices of a cycle C_{n-1} . A graph is called regular if all vertices have the same degree; specifically, if each vertex has degree r, it is called an r-regular graph.

If $V(G)$ could be partitioned into two disjoint subsets A and B such that an edge in G joins a vertex in A to a vertex in B, then G is called a bipartite graph **[16]**. The distance between two distinct vertices x and y, denoted by $d(x,y)$, is the length of the shortest path between two x and y, and if there is no such path, then $d(x,y) = \infty$, and $d(x,x) = 0$. The graph G is connected if any two distinct vertices are joined by a path **[17]**, the diameter of G, is defined as $diam(\mathrm{G}) = \sup\{\mathrm{d(x,y)}: x,y \in V(G)\}$. Two graphs G and H are isomorphic if there is a bijective function (one – to – one and onto) $f : (G) \rightarrow (H)$ with $(G) = (H)$, $(G) = (H)$ and for $u,v \in V(G)$, $(u,v) \in E(G)$, if and only if $(f(u), f(v)) \in E(H)$.

The length of the shortest cycle in G is called the girth of G and denoted by $\mathrm{gr(G)}$, if G has no cycles, then $\mathrm{gr(G)} = \infty$ **[9,10]**.

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2. MULTIPLICATIVE REGULAR GRAPH OF A COMMUTATIVE RING *R*

An element a in R is said to be regular, (in the sense of von Neumann), if $a = aba$ for some $b \in R$ and R is called regular if each element of R is regular. It is clear that every unit of R is regular.

Examples 2.1: We provide some examples to understand our definition of the multiplicative regular graph $\Gamma_{MR}(R)$ and it consolidates the definition we present in this research. In **[14]**, P. A. Rashid and H. S. Rashid defined an undirected graph G for R, denoted by $\mathfrak{R}_\partial(R)$, with $V(\mathfrak{R}_\partial(R)) = R\backslash\{0\}$ and $a \neq b \in R$ are adjacent if and only if $a = aba$ or $b = bab$ and in **[15]**, Ali Jafari Taloukolaei and S. Sahebi defined an undirected graph, $G_{Vnr^+}(R)$, with $V(G_{Vnr^+}(R)) = R$, and $a \neq b \in R$ are adjacent if and only if $a + b$ is regular. In this work we introduce a new version of undirected graphs for R, called a multiplicative regular graph, denoted by $\Gamma_{MR}(R)$ with $V(\Gamma_{MR}(R)) = R$ and $a \neq b \in R$ are adjacent if and only if ab is regular. We give some examples to show that our definition is different from the both above two mentioned definitions. Now clearly, $\mathfrak{R}_\partial(R)$ is a subgraph of $\Gamma_{MR}(R)$ since, $V(\mathfrak{R}_\partial(R)) = R\backslash\{0\} \subset R = V(\Gamma_{MR}(R))$ and if $(a, b) \in E(\mathfrak{R}_\partial(R))$, then $a = aba$ or $b = bab$. In both cases we get ab is regular, so that $(a, b) \in E(\Gamma_{MR}(R))$, and thus $E(\mathfrak{R}_\partial(R)) \subseteq E(\Gamma_{MR}(R))$. Hence, $\mathfrak{R}_\partial(R)$ is a sub graph of $\Gamma_{MR}(R)$. Now, in $\mathbb{Z}_{12}$, we have $2.10 = 8 \in Reg(\mathbb{Z}_{12})$, so that $(2, 10) \in E(\Gamma_{MR}(\mathbb{Z}_{12}))$, but we have $2 \neq 2.10.2$ and $10 \neq 10.2.10$, so that $(2, 10) \notin E(\mathfrak{R}_\partial(R))$ and this proves that our definition is different from that P. A. Rashid and H. S. Rashid defined in **[14]**. In fact, $\Gamma_{MR}(R)$ is a generalization of $\mathfrak{R}_\partial(R)$ in the sense that, $V(\mathfrak{R}_\partial(R)) \subset V(\Gamma_{MR}(R))$ and $E(\mathfrak{R}_\partial(R)) \subseteq E(\Gamma_{MR}(R))$. Next, we consider the ring $\mathbb{Z}_8$. We have $2.4 = 0$ which is a regular element in $\mathbb{Z}_8$, but $2 + 4 = 6$, which is not regular in $\mathbb{Z}_8$. On the other hand, we have $2 + 6 = 0$ which is regular in $\mathbb{Z}_8$, but $2.6 = 4$, which is not regular in $\mathbb{Z}_8$ and this proves that our definition is independent with that Ali Jafari Taloukolaei and S. Sahebi defined in **[15]**. Finally, if x and y are regular, then one can easily show that xy is also regular. In $\mathbb{Z}_{12}$, we have 2 and 10 are not regular, but $2.10 = 8$, which is regular in $\mathbb{Z}_{12}$, that means it is possible that a given ring contains non regular elements but their multiplication is regular. All these examples give the validity of our basic definition in this research and now we are able to introduce our main definition.

Definition 2.2: The multiplicative regular graph of R denoted by $\Gamma_{MR}(R)$, it is undirected graph with $V(\Gamma_{MR}(R)) = R$ and $x \neq y \in R$ are adjacent if and only if $xy \in Reg(R)$.

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Examples 2.3: (1) clearly 0 is a regular element. For all $0 \neq x \in R$, $0x = 0$ is regular, that means 0 is adjacent to all nonzero elements of R in $\Gamma_{MR}(R)$.

(2) The multiplicative regular graphs of $\mathbb{Z}_n$, for some positive integers n.

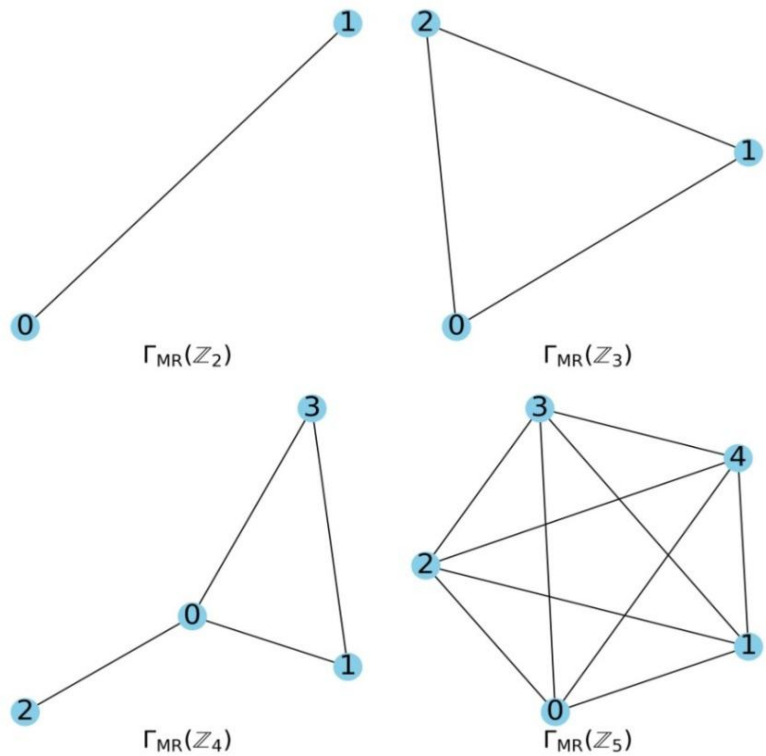

Figure 1

Remark 2.4: $\Gamma_0(R)$ is always a subgraph of $\Gamma_{MR}(R)$. In fact, $V(\Gamma_0(R)) = R = V(\Gamma_{MR}(R))$ and if $(a,b) \in E(\Gamma_0(R))$, then $ab = 0$ and as 0 is regular, we have ab is regular, so that $(a,b) \in E(\Gamma_{MR}(R))$. Hence, $\Gamma_0(R)$ is a sub graph of $\Gamma_{MR}(R)$. In **Example 2.1**, we see that $(2,10) \in E(\Gamma_{MR}(\mathbb{Z}_{12}))$ but $(2,10) \notin \Gamma_0(\mathbb{Z}_{12})$ and this makes us able to consider the multiplicative regular graph, $\Gamma_{MR}(R)$ as a generalization of $\Gamma_0(R)$ in some sense.

Now, we prove that the multiplicative regular graph of a ring with only additive and multiplicative identities is nothing just $\Gamma_0(\mathbb{Z}_2)$.

Proposition 2.5: Let $R = \{0,1\}$, then $\Gamma_{MR}(R) \cong \Gamma_0(\mathbb{Z}_2)$.

Proof: If we define $f: R \to \mathbb{Z}_2$ as $f(0) = 0$ and $f(1) = 1$. In R, since we have $0.1 = 0$ which is regular, so that 0 is adjacent to 1, so $(0,1) \in E(\Gamma_{MR}(R))$ which is the only edge in $E(\Gamma_{MR}(R))$, so that $E(\Gamma_{MR}(R)) = \{(0,1)\}$. On the other hand, in $\mathbb{Z}_2$, $0.1 = 0$, so that $(0 = f(0), 1 = f(1)) \in E(\Gamma_0(\mathbb{Z}_2))$ which is the only edge in $E(\Gamma_0(\mathbb{Z}_2))$, so that $E(\Gamma_0(\mathbb{Z}_2)) = \{(0,1)\}$. Now, we have

(i) $|V(\Gamma_{MR}(R))| = 2 = |V(\Gamma_0(\mathbb{Z}_2))|$.

(ii) $|E(\Gamma_{MR}(R))| = 1 = |E(\Gamma_0(\mathbb{Z}_2))|$.

(iii) $(0,1) \in E(\Gamma_{MR}(R))$, then $\big(f(0), f(1)\big) = (0,1) \in E(\Gamma_0(\mathbb{Z}_2))$. Also, $(0,1) \in E(\Gamma_0(\mathbb{Z}_2))$, then $(0,1) = \big(f^{-1}(0), f^{-1}(1)\big) = (0,1) \in E(\Gamma_{MR}(R))$. Hence, we get $\Gamma_{MR}(R) \cong \Gamma_0(\mathbb{Z}_2)$.

Corollary 2.6: Let $R = \{0,1\}$, then $\Gamma_{MR}(R)$ is regular graph with degree one (1-regular) and complete graph K_2.

Proof: By **Proposition 2.5**, we have $\Gamma_{MR}(R) \cong \Gamma_0(\mathbb{Z}_2)$, as, $\Gamma_0(\mathbb{Z}_2)$ is complete graph K_2 and regular graph with degree one (1-regular), so that $\Gamma_{MR}(R)$ is a complete graph K_2 and regular graph with degree one.

Theorem 2.7: Let R be regular. $\Gamma_{MR}(R) \cong \Gamma_0(R)$ if and only if $R = \{0,1\}$.

Proof: Let $\Gamma_{MR}(R) \cong \Gamma_0(R)$. If $R \neq \{0,1\}$, then there exists $x \in R$ such that $x \neq 0$ and $x \neq 1$. As R is regular, we have x is regular, then $x.1 = x \in Reg(R)$. Hence $(x,1) \in E(\Gamma_{MR}(R))$, as $\Gamma_{MR}(R) \cong \Gamma_0(R)$, we get $(x,1) \in E(\Gamma_0(R))$, so that $x.1 = 0$, that is $x = 0$, which is a contradiction. Hence, $R = \{0,1\}$. next let $R = \{0,1\}$. Clearly, $V(\Gamma_{MR}(R)) = R = V(\Gamma_0(R))$, Then, $0.1 = 0 \in Reg(R)$, so that $(0,1) \in E(\Gamma_{MR}(R))$ and $(0,1) \in E(\Gamma_0(R))$. Clearly, $E\big(\Gamma_{MR}(R)\big) = \{(0,1)\} = E(\Gamma_0(R))$, Hence $\Gamma_{MR}(R) \cong \Gamma_0(R)$.

Corollary 2.8: If R is a field, then $\Gamma_{MR}(R) \cong \Gamma_0(R)$ if and only if $R = \{0,1\}$.

Proof: As a field is regular, the proof is obvious.

Example 2.9: Consider the ring $\mathbb{Z}_6$, which is regular and since $\mathbb{Z}_6 \neq \{0,1\}$, so that by (**Theorem 2.7**) $\Gamma_{MR}(\mathbb{Z}_6) \ncong \Gamma_0(\mathbb{Z}_6)$. It is clear from their graphs as given in below, that $\Gamma_0(\mathbb{Z}_6)$ is a sub graph of $\Gamma_{MR}(\mathbb{Z}_6)$.

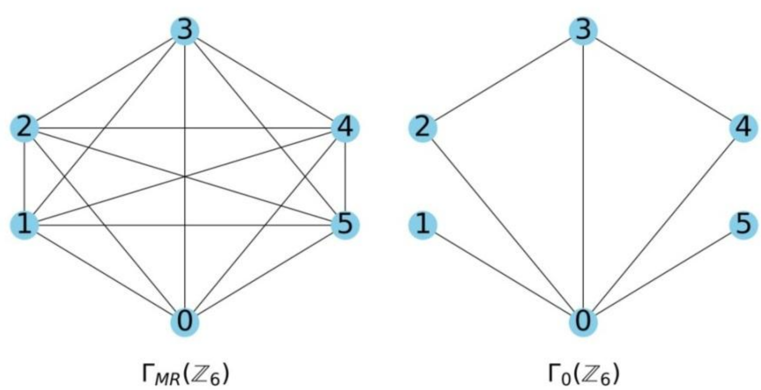

Figure 2

Proposition 2.10: If R is regular, then $\Gamma_{MR}(R)$ and $G_{Vnr^+}(R)$ are complete and $\Gamma_{MR}(R) \cong G_{Vnr^+}(R)$.

Proof: Let $x, y \in R$, then xy and $x + y$ are regular. Hence, each pair of vertices in both graphs $\Gamma_{MR}(R)$ and $G_{Vnr^+}(R)$ are adjacent, thus $\Gamma_{MR}(R)$ and $G_{Vnr^+}(R)$ are complete graphs. Now,

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$V(\Gamma_{MR}(R)) = R = V(G_{Vnr^+}(R))$. Then, the identity map $I: R \to R$ is the required isomorphism between $\Gamma_{MR}(R)$ and $G_{Vnr^+}(R)$. Hence, $\Gamma_{MR}(R) \cong G_{Vnr^+}(R)$.

Corollary 2.11: For a field R, $\Gamma_{MR}(R)$, $G_{Vnr^+}(R)$ are complete graphs and $\Gamma_{MR}(R) \cong G_{Vnr^+}(R)$.

Proof: Since a field is regular, the proof is obvious.

If R is not regular, $\Gamma_{MR}(R)$ and $G_{Vnr^+}(R)$ may not be complete and may not be isomorphic. Now, we give some examples to determine the differences between the graphs $\Gamma_{MR}(R)$, $G_{Vnr^+}(R)$ and $\Re_\partial(R)$.

Example 2.12: Consider the ring $\mathbb{Z}_8$.

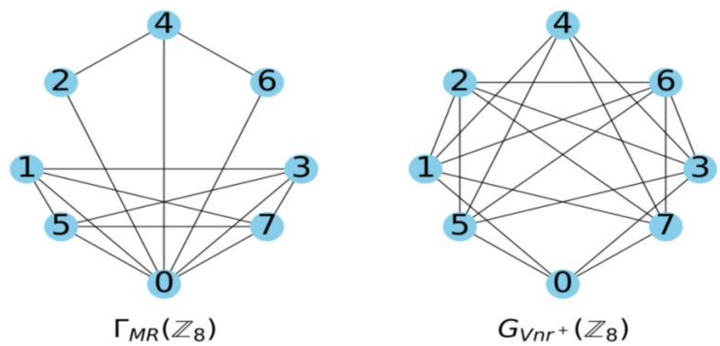

Figure 3

In the above figure we see that, the ring $\mathbb{Z}_8$ is not regular (2 is not a regular element). Now, $(1,5) \in E(\Gamma_{MR}(\mathbb{Z}_8))$ but $(1,5) \notin G_{Vnr^+}(\mathbb{Z}_8)$ and $(3,6) \in G_{Vnr^+}(\mathbb{Z}_8)$ but $(3,6) \notin E(\Gamma_{MR}(\mathbb{Z}_8))$. Hence, the two graphs are independent.

For the graph $\Re_\partial(R)$, even R is regular or not, then the graph $\Re_\partial(R)$ neither be complete nor isomorphic to $\Gamma_{MR}(R)$.

Example 2.13: We consider the regular ring $\mathbb{Z}_6$ and the irregular ring $\mathbb{Z}_8$.

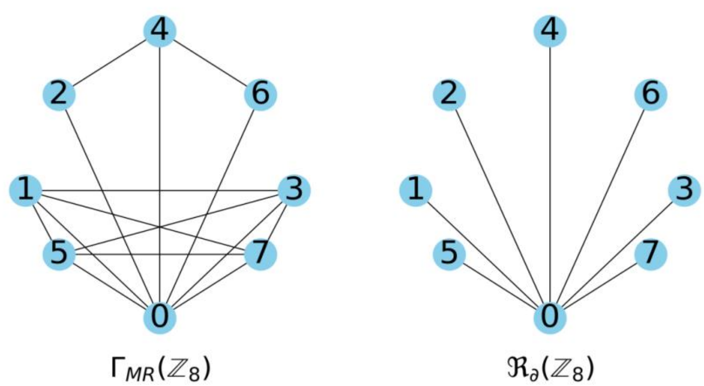

Figure 4

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We see that both graphs for the irregular ring $\mathbb{Z}_8$ are not complete and clearly, they are also not isomorphic, but $\Re_\partial(\mathbb{Z}_8)$ is a subgraph of $\Gamma_{MR}(\mathbb{Z}_8)$.

For the regular ring $\mathbb{Z}_6$ as given in below, we see in **Figure 5** that the graph $\Gamma_{MR}(\mathbb{Z}_6)$ is complete (since $\mathbb{Z}_6$ is regular), while the graph $\Re_\partial(\mathbb{Z}_6)$ is neither complete nor isomorphic to $\Gamma_{MR}(\mathbb{Z}_6)$ but $\Re_\partial(\mathbb{Z}_6)$ is a sub graph of $\Gamma_{MR}(\mathbb{Z}_6)$.

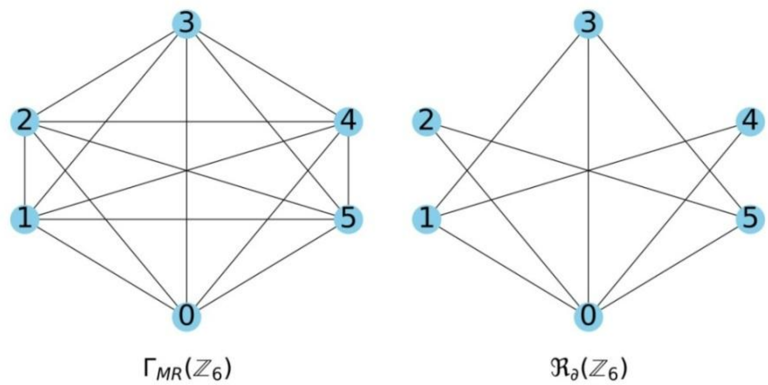

Figure 5

Remark 2.14: If $\Gamma_1(R)$ and $\Gamma_2(R)$ are two graphs with $V(\Gamma_1(R)) = V(\Gamma_2(R))$ and $E(\Gamma_1(R)) = E(\Gamma_2(R))$, then $\Gamma_1(R) \cong \Gamma_2(R)$. In fact, the identity mapping between $V(\Gamma_1(R)), V(\Gamma_2(R))$ is the required isomorphism.

Theorem 2.15: $\Gamma_{MR}(R) \cong \Gamma_0(R)$ if and only if 0 and 1 are the only regular elements in R.

Proof: Let $\Gamma_{MR}(R) \cong \Gamma_0(R)$. If R contains a non-zero non unit regular element $x \in R$. then $x.1 = x \in Reg(R)$. Hence, $(x,1) \in E(\Gamma_{MR}(R))$ and since $\Gamma_{MR}(R) \cong \Gamma_0(R)$, we get $(x,1) \in E(\Gamma_0(R))$, so that $x.1 = 0$, that is $x = 0$, which is a contradiction. Hence, 0 and 1 are the only regular elements in R. Next, assume that 0 and 1 are the only regular elements in R. Let $\Gamma_{MR}(R) \ncong \Gamma_0(R)$. Since $\Gamma_0(R)$ is a subgraph of $\Gamma_{MR}(R)$, we get $E(\Gamma_0(R)) \subseteq E(\Gamma_{MR}(R))$. If $E(\Gamma_{MR}(R)) \subseteq E(\Gamma_0(R))$, then $E(\Gamma_{MR}(R)) = E(\Gamma_0(R))$ and as $V(\Gamma_{MR}(R)) = R = V(\Gamma_0(R))$, by **Remark 2.14**, $\Gamma_{MR}(R) \cong \Gamma_0(R)$, which is a contradiction. Hence, we get $E(\Gamma_{MR}(R)) \nsubseteq E(\Gamma_0(R))$, so that there exist distinct elements $a, b \in R$ with $(a,b) \in E(\Gamma_{MR}(R))$ but $(a,b) \notin E(\Gamma_0(R))$, this gives that ab is regular in R and $ab \neq 0$, so that $a \neq 0$ and $b \neq 0$. As R has no regular elements other than 0 and 1, we get $ab = 1$, so that a and b are units and hence they are regular elements in R. We have:

(i) ($a = 0$ and $b = 0$), which is a contradiction.

(ii) ($a = 0$ and $b = 1$), which is a contradiction.

(iii) ($a = 1$ and $b = 0$), which is a contradiction.

(iv) ($a = 1$ and $b = 1$), this gives that $a = 1 = b$, which is a contradiction. Hence, we must have that $\Gamma_{MR}(R) \cong \Gamma_0(R)$.

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Proposition 2.16: $\Gamma_{MR}(R)$ is connected.

Proof: Let $x, y \in V(\Gamma_{MR}(R))$ with $x \neq y$. If $x = 0$, then $y \neq 0$, and clearly $xy = 0y = 0$ which is regular, so $(x, y) = (0, y) \in V(\Gamma_{MR}(R))$ and if $y = 0$, similarly we get $(x, y) = (x, 0) \in V(\Gamma_{MR}(R))$. Next, let $x \neq 0$ and $y \neq 0$. As $x0 = 0$ and $0y = 0$ are regular, so $(x, 0), (0, y) \in V(\Gamma_{MR}(R))$. Hence, $x - 0 - y$ is a path in $\Gamma_{MR}(R)$, that is $\Gamma_{MR}(R)$ is connected.

Theorem 2.17: $Diam(\Gamma_{MR}(R)) \leq 2$.

Proof: Let $Diam(\Gamma_{MR}(R)) > 2$, then there exist $x, y \in R$ with $d(x, y) > 2$. If $x = y$, then $d(x, y) = d(x, x) = 0$ which is a contradiction, hence $x \neq y$ and if $x = 0$ or $y = 0$, then clearly $xy = 0$ which is regular, so that $(x, y) \in E(\Gamma_{MR}(R))$ and as $\Gamma_{MR}(R)$ is connected by **Proposition 2.16**, we get $d(x, y) = 1$ which is again a contradiction, so that x and y are distinct non zero vertices. Now, we get $x - 0 - y$ is a path in $\Gamma_{MR}(R)$, that means $d(x, y) = 2$, which is a contradiction. Thus, we get $Diam(\Gamma_{MR}(R)) \leq 2$.

Theorem 2.18: If R contains at least three elements, then $gr(\Gamma_{MR}(R)) = \infty$ or $gr(\Gamma_{MR}(R)) = 3$.

Proof: If $\Gamma_{MR}(R)$ contains no cycle, then $gr(\Gamma_{MR}(R)) = \infty$. Now, suppose that $\Gamma_{MR}(R)$ contains a cycle $C: v_1 - v_2 - v_3 - \cdots - v_n - v_1$ with length $n \geq 3$. If $v_1 = 0$, then, we have $0 - v_2 - v_3 - 0$ is a cycle with length 3 in $\Gamma_{MR}(R)$ and if $v_n = 0$, we have $v_{n-1} - 0 - v_1 - v_2$ is a cycle with length 3 in $\Gamma_{MR}(R)$ and if $v_i = 0$ for some $2 \leq i \leq n - 1$, then we get that $v_1 - 0 - v_n - v_1$ is a cycle of length 3 in $\Gamma_{MR}(R)$ and the last case is that when $v_i \neq 0$ for all $1 \leq i \leq n$ and for this case we have $v_1 - v_2 - 0 - v_1$ is a cycle of length 3 in $\Gamma_{MR}(R)$, that means $\Gamma_{MR}(R)$ contains a cycle with length 3 and since the girth of any graph never be less that 3, hence we have $gr\left(\Gamma_{MR}(R)\right) = 3$.

Remark 2.19: If R is regular, $\Gamma_{MR}(R)$ is complete (by **Proposition 2.10**), that means the degree of all vertices of R are equal. Hence, $\Gamma_{MR}(R)$ is regular.

Corollary 2.20: If R is a field, $\Gamma_{MR}(R)$ is complete graph as well as a regular graph.

Proof: Is obvious from **Remark 2.19**.

Corollary 2.21: If p is prime, then $\Gamma_{MR}(\mathbb{Z}_p)$ is complete and $(p - 1)$ −regular.

Proof: As $\mathbb{Z}_p$ is a field, by **Remark 2.19**, $\Gamma_{MR}\left(\mathbb{Z}_p\right)$ is a regular graph and hence each vertex in $\Gamma_{MR}\left(\mathbb{Z}_p\right)$ is adjacent to all the remaining $p - 1$ vertices and hence the degree of each vertex in $\Gamma_{MR}\left(\mathbb{Z}_p\right)$ is $p - 1$, so that $\Gamma_{MR}\left(\mathbb{Z}_p\right)$ is a $(p - 1)$ −regular graph.

Theorem 2.22: The multiplicative regular graph, $\Gamma_{MR}(R)$ of every regular ring R with at least three distinct vertices is a Hamiltonian graph.

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Proof: By **Remark 2.19**, $\Gamma_{MR}(R)$ is complete and as R contains at least three distinct vertices, so that $\Gamma_{MR}(R)$ contains a Hamiltonian cycle, then $\Gamma_{MR}(R)$ is Hamiltonian.

Recall that a connected graph G is Eulerian if and only if all its vertices have even degree **[9, Th. 6.2]**.

Proposition 2.23: If R is a field with an odd number of elements, then $\Gamma_{MR}(R)$ is Eulerian.

Proof: Let $R = \{a_1, a_2, \dots, a_n\}$, where n is an odd number greater than 1, then by **Corollary 2.20**, we get $\Gamma_{MR}(R)$ is a regular graph, so that each vertex in $\Gamma_{MR}(R)$ is adjacent to the all remaining $n-1$ verticies and as n is odd, we get $n-1$ is even, so that the degree of each vertex in $\Gamma_{MR}(R)$ is even and hence $\Gamma_{MR}(R)$ is Eulerian.

Corollary 2.24: If $p > 2$ is prime, then $\Gamma_{MR}(\mathbb{Z}_p)$ is Eulerian.

Proof: As $\mathbb{Z}_p$ is a field and $p > 2$ is prime, so that p is an odd number, so that by **Proposition 2.23**, we get $\Gamma_{MR}(\mathbb{Z}_p)$ is Eulerian.

The chromatic polynomial is the number of possible ways a graph can be colored using no more than a given number of colors, for chromatic polynomial of a graph see **[3]**.

Proposition 2.25: Let R be a field with n elements. The chromatic polynomial of the graph $\Gamma_{MR}(R)$ is the polynomial $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda - n + 1)$.

Proof: By **Corollary 2.20**, we have $\Gamma_{MR}(R)$ is complete. By [**3, Th. 1**] the proof will be done.

Corollary 2.26: If p is prime, the chromatic polynomial of $\Gamma_{MR}(\mathbb{Z}_p)$ is $P_p(\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda - p + 1)$.

Proof. As $\mathbb{Z}_p$ is a field, the proof is obvious from **Proposition 2.25**.

Example 2.27: Consider the ring $\mathbb{Z}_3$. as $\mathbb{Z}_3$ is a field, by **Corollary 2.20**, we get that the multiplicative regular graph $\Gamma_{MR}(\mathbb{Z}_3)$ is a complete graph.

Remark 2.28: If $n > 1$, then the totient function (or φ −function) is the number of positive integers less than n and relatively prime with n **[11]**. If $n = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$, is the prime factorization of n, by [**13, Th. 7.3**], we have $\varphi(n) = (p_1^{s_1} - p_1^{s_1-1})(p_2^{s_2} - p_2^{s_2-1}) \dots (p_k^{s_k} - p_k^{s_k-1})$. If $a < n$ is any relatively prime number to n, then $ab = 1(mod n)$ for some $b \in \mathbb{Z}$, this means that a is a unit in $\mathbb{Z}_n$ and since every unit in a ring is non-zero and regular, this means that any relatively prime integer to n which is less than n is a non-zero regular element in $\mathbb{Z}_n$.

Theorem 2.29: For $n \geq 5$, $\Gamma_{MR}(\mathbb{Z}_n)$ contains at least one K_5 with 0 as one of its vertices.

Proof: Let $n = 5$. As 5 is prime, by **Corollary 2.21**, $\Gamma_{MR}(\mathbb{Z}_5)$ is a complete graph and as $\mathbb{Z}_5$ contains 5 vertices which are $0, 1, 2, 3, 4$, we get $\mathbb{Z}_5$ itself is a K_5. Next, let $n = 6$. Since $\mathbb{Z}_6$ is a regular ring, so by **Remark 2.19**, $\Gamma_{MR}(\mathbb{Z}_6)$ is a complete graph. Hence, any five vertices of $\mathbb{Z}_6$ will form a complete subgraph of $\Gamma_{MR}(\mathbb{Z}_6)$, so that $\{0, 1, 2, 3, 4\}$ is one of these K_5´s in $\Gamma_{MR}(\mathbb{Z}_6)$.

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Now, let $n > 6$, then n can be factorized into $n = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$. We have two cases, either the prime factorization of n contains at least a prime number $p_i \geq 5$ or all the primes in the prime factorization of n are less than 5. For the first case, suppose that for some $1 \leq i \leq k$, p_i is a prime number with $p_i \geq 5$, then $p_i^{s_i} - p_i^{s_i - 1}$ is a factor of $\varphi(n)$. If $p_i = 5$, then $p_i^{s_i} - p_i^{s_i-1} = p_i^{s_i-1}(p_i - 1) = 4p_i^{s_i-1} \geq 4$, and if $p_i > 5$, then we get $p_i^{s_i} - p_i^{s_i-1} = (p_i - 1)p_i^{s_i-1} > (5-1)p_i^{s_i-1} = 4p_i^{s_i-1} \geq 4$, so that in this case there is at least 4 relatively prime numbers to n. For the second case, suppose that all the prime numbers in the factorization of n are less than 5, this means that $p_i < 5$, for every $1 \leq i \leq k$, and thus for each $1 \leq i \leq k$, we have $p_i = 2$ or $p_i = 3$. Hence, we get $n = 2^{m_1} 3^{m_2}$, where $m_1 \geq 1$ and $m_2 \geq 1$, then $\varphi(n) = (2^{m_1} - 2^{m_1-1})(3^{m_2} - 3^{m_2-1})2^{m_1-1}3^{m_2-1}(2-1)(3-1) = 2.2^{m_1-1}3^{m_2-1}$. If $m_1 = 1 = m_2$, then $n = 2.3 = 6$, which is a contradiction, since $n > 6$. Hence, we have $m_1 \geq 2$ or $m_2 \geq 2$. If $m_1 \geq 2$, then $\varphi(n) = 2.2^{m_1-1}3^{m_2-1} \geq 2.2.3^{m_2-1} = 4.3^{m_2-1} \geq 4$, and if $m_2 \geq 2$, then we get $\varphi(n) = 2.2^{m_1-1}3^{m_2-1} \geq 2.2^{m_1-1}.3 = 6.2^{m_1-1} \geq 6 > 4$. Hence, all cases implies that there is at least 4 relatively prime numbers to n. Then, by **Remark 2.28**, every relatively prime number to n is a unit in $\mathbb{Z}_n$, and as every unit is nonzero and regular, so that $\mathbb{Z}_n$ contains at least four distinct nonzero regular elements, say v_1, v_2, v_3, v_4 and as 0 is always regular, so that $0, v_1, v_2, v_3, v_4$ are five distinct regular elements in $\mathbb{Z}_n$ and since the product of any two regular element is also regular, so that the set $\{0, v_1, v_2, v_3, v_4\}$ will form a complete subgraph of $\Gamma_{MR}(\mathbb{Z}_n)$ which is, in fact a K_5.

It is clear that $\Gamma_{MR}(\mathbb{Z}_3)$ and $\Gamma_{MR}(\mathbb{Z}_4)$ are planer graphs. In the following corollary, we prove that $\Gamma_{MR}(\mathbb{Z}_n)$ is not planer for all $n \geq 5$.

Corollary 2.30: For $n \geq 5$, the multiplicative regular graph, $\Gamma_{MR}(R)$ is not planer.

Proof: By **Theorem 2.29**, $\Gamma_{MR}(R))$ contains at least one K_5. Hence, $\Gamma_{MR}(R)$ cannot be planer graph. [**9**, **Th. 2.21**] □.

In view of **Theorem 2.29**, we can discuss end-regular sub-graphs of multiplicative regular graphs. Let G and H be graphs. A mapping $F: V(G) \to V(H)$ is called a homomorphism from G to H if for any $a, b \in V(G)$, and then a adjacent to b imply that $F(a)$ is adjacent to $F(b)$. A homomorphism from G to itself is called an endomorphism of G. $End(G)$ is the set of all endomorphisms of G. A graph G is called end-regular if $End(G)$ is regular and it is called independent if no two vertices in G are adjacent and it is called split if there is a partition $V(G) = K \cup U$ of its vertex set into a complete set K (every two elements are adjacent) and an independent set U. In the next theorem we prove that for all $n \geq 5$, the multiplicative regular graph $\Gamma_{MR}(R)$ contains at least one end-regular subgraph.

Theorem 2.31: For $n > 6$, $\Gamma_{MR}(\mathbb{Z}_n)$ contains at least one end-regular subgraph.

Proof: By **Theorem 2.29**, we get $\Gamma_{MR}(\mathbb{Z}_n)$ contains at least one K_5. Now, let G be a subgraph of $\Gamma_{MR}(\mathbb{Z}_n)$ such that $G = K_5 \cup S$, where $V(K_5) = \{0, v_1, v_2, v_3, v_4\}$ and $S = \{v_i, v_j\}$ for $v_i \neq v_j$ and $v_i, v_j \notin V(K_5)$. Now K_5 is a complete set, assume that v_i is not adjacent with v_j, this means S is independent set, now $|K_5| = 5$, v_i adjacent with 0, means at least the degree of v_i is one i.e. $d(v_i) = 1$, now $1 \in \{1,2,3,4,5\}$ and $v_i \in S$, Then by **[12, Lemma 1.2]** we get end-regular sub-graph of $\Gamma_{MR}(R)$. □

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3. An algorithm for constructing $\Gamma_{MR}\,(\mathbb{Z}_n)$

Now, we propose an algorithm to visualize the multiplicative regular graph of $\mathbb{Z}_n$. The construction process is explained in a detailed, step-by-step manner, where each step is carefully elaborated to demonstrate how the graph is drawn according to the specific conditions defined in our graph's definition. We construct an algorithm to construct a multiplicative regular graph of $\mathbb{Z}_n$ in which the graph depends on regular elements. We call the algorithm a multiplicatively regular graph. We must import networkx:

GraphRegular(var$\Gamma_{MR}\,(\mathbb{Z}_n)$:graph)

Import library networkx, matplotlib. pyplot and math

Step one:

For $a, b \in \mathbb{Z}_n$

If $a^2 * b = a$

Add a to $Reg(\mathbb{Z}_n)$

Else

Go to step one;

End for;

Step two:

For $i \in \mathbb{Z}_n$

Add i to V ($\Gamma_{MR}\,(\mathbb{Z}_n)$

For $j \in \mathbb{Z}_n$ do

If $i \neq j$

If $i * j \in Reg(\mathbb{Z}_n)$

Add $i \sim j$ to E ($\Gamma_{MR}\,(\mathbb{Z}_n)$

Else

Go to step two:

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Else

Go to step two:

End for:

Step three:

Draw $\Gamma_{MR}(\mathbb{Z}_n)$

End;

First we start the algorithm by importing library networkx, matplotlib.pyplot and math. Begin with step one, as the algorithm checks for the regular elements of $\mathbb{Z}_n$. In $\mathbb{Z}_n$ if there is a positive integer b such that $a^2 * b = a$ for any a in $\mathbb{Z}_n$ then a will be a regular element of $\mathbb{Z}_n$. Add a to $Reg(\mathbb{Z}_n)$. Otherwise if such b does not exist, the process returns to the privies loop in step one. For the step two, it is the step of determining the vertex set of the graph i.e. $V(\Gamma_{MR}(\mathbb{Z}_n))$. Each i in $\mathbb{Z}_n$ becomes the vertex of the graph, add i to $V(\Gamma_{MR}(\mathbb{Z}_n))$. Then the algorithm checks the vertex adjacency, that is, if for $i \in V(\Gamma_{MR}(\mathbb{Z}_n))$, there exists $j \in V(\Gamma_{MR}(\mathbb{Z}_n))$, where $i \neq j$, and $i * j \in Reg(\mathbb{Z}_n)$, then i and j are adjacent. Adds i to edge set of the graph i.e. $E(\Gamma_{MR}(\mathbb{Z}_n))$, Else it returns to the loop in step two. Here we have the prepared set of vertices and edges of the graph; at the end of the algorithm we have the last step which is drawing the multiplicative regular graph of $\mathbb{Z}_n$.

Example 3.1: Constructing multiplicative regular graph of $\mathbb{Z}_n$, for $n = 3, 4$ and 5. If $n = 3$, then $\mathbb{Z}_3 = \{0,1,2\}$. In the first step of algorithm, the process checks for regular elements of the ring. That is for an element a in $\mathbb{Z}_3$ if there is another element a in $\mathbb{Z}_3$ such that $a^2 * b = a$ then a is regular element of the ring $\mathbb{Z}_3$. Here in $\mathbb{Z}_3$ we have $0, 1$ and 2 are regular elements of $\mathbb{Z}_3$ that is $Reg(\mathbb{Z}_3) = \{0,1,2\}$. Step two is the step of determining vertices $V(\Gamma_{MR}(\mathbb{Z}_3))$. Means all elements in $\mathbb{Z}_3$ will be added to vertices of the graph. That is $V(\Gamma_{MR}(\mathbb{Z}_3)) = \{0,1,2\}$ and in this step the algorithm determines what vertices are adjacent? That is, for i in $V(\Gamma_{MR}(\mathbb{Z}_3))$ if there exists j when $i \neq j$, and $i * j \in Reg(\mathbb{Z}_3)$, then i is adjacent to j. Here in $\mathbb{Z}_3$, we have $0{\sim}1$, $0{\sim}2$ and $1{\sim}2$ and the edges will be added to $E(\Gamma_{MR}(\mathbb{Z}_3))$. The last step draws the multiplicative regular graph of $\mathbb{Z}_3$. As shown in **Figure 1**.

For $n = 4$ and 5 algorithm works in similar procedure and determines the sets of regular elements of the rings $\mathbb{Z}_4$ and $\mathbb{Z}_5$. As $Reg(\mathbb{Z}_4) = \{0,1,3\}$ and $Reg(\mathbb{Z}_5) = \{0,1,2,3,4\}$. In step two the algorithm determines set of vertices of both multiplicative regular graphs that we want to draw, i.e. $V(\Gamma_{MR}(\mathbb{Z}_4))$ and $V(\Gamma_{MR}(\mathbb{Z}_5))$. Also in this step the algorithm checks for the adjacency of any two distinct vertices in the two vertex sets. That is for any i in $V(\Gamma_{MR}(\mathbb{Z}_4))$ if there is j in $V(\Gamma_{MR}(\mathbb{Z}_4))$ whenever $i \neq j$ and $i * j \in Reg(\mathbb{Z}_4)$ for $\Gamma_{MR}(\mathbb{Z}_4)$ and $i * j \in Reg(\mathbb{Z}_5)$ for $\Gamma_{MR}(\mathbb{Z}_5)$ then i is adjacent to j. And $i{\sim}j$ will be added to $E(\Gamma_{MR}(\mathbb{Z}_4))$ for $\Gamma_{MR}(\mathbb{Z}_4)$ and $E(\Gamma_{MR}(\mathbb{Z}_5))$ for $\Gamma_{MR}(\mathbb{Z}_5)$. For $\mathbb{Z}_4$ we have $0{\sim}1$, $0{\sim}2$, $0{\sim}3$ and $1{\sim}3$, and for $\mathbb{Z}_5$ we have $0{\sim}1$, $0{\sim}2$, $0{\sim}3$, $0{\sim}4$, $1{\sim}2$, $1{\sim}3$, $1{\sim}4$, $2{\sim}3$, $2{\sim}4$ and $3{\sim}4$ and the third step of the algorithm is the step of drawing both graphs $\Gamma_{MR}(\mathbb{Z}_4)$ and $\Gamma_{MR}(\mathbb{Z}_5)$, as shown in **Figure 1**.

Example 3.2: if $n = 8$, then $\mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\}$. In the first step of algorithm, the process checks for regular elements of the ring. That is for an element a in $\mathbb{Z}_8$ if there is another element b in $\mathbb{Z}_8$ such that $a^2 * b = a$ then a is regular element of the ring $\mathbb{Z}_8$. We have $0, 1, 3, 5,7$ are regular elements of $\mathbb{Z}_8$ that is $Reg(\mathbb{Z}_8) = \{0,1,3,5,7\}$. Step two is the step of determining vertices $V\big(\Gamma_{MR}\,(\mathbb{Z}_8)\big)$, this means that all elements in $\mathbb{Z}_8$ will be added to vertices of the graph, so that $V\big(\Gamma_{MR}\,(\mathbb{Z}_8)\big) = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and in this step the algorithm checks for vertices that are adjacent, That is for i in $V\big(\Gamma_{MR}\,(\mathbb{Z}_8)\big)$ if there exists j when $i \neq j$, such that $i * j \in Reg(\mathbb{Z}_8)$, then i is adjacent to j, for example for 0 there is 1 such that $0.1 = 0 \in Reg(\mathbb{Z}_8)$, so that $0{\sim}1$, similarly $0{\sim}2$, $0{\sim}3$, $0{\sim}4$, $0{\sim}5$, $0{\sim}6$, $0{\sim}7$, $1{\sim}3$, $1{\sim}5$, $1{\sim}7$, $2{\sim}4$, $3{\sim}5$, $3{\sim}7$, $4{\sim}6$, $5{\sim}7$ and the edges will be added to $E\big(\Gamma_{MR}\,(\mathbb{Z}_8)\big)$. The last step of algorithm draws the multiplicative regular graph of the ring $\mathbb{Z}_8$, as shown in **Figure 6**.

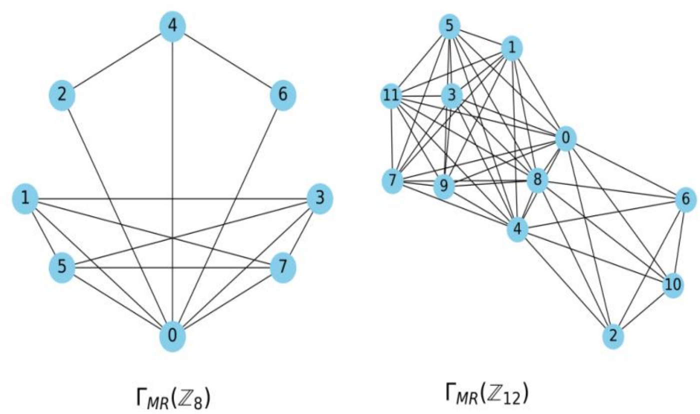

Figure 6

Example 3.3: if $n = 12$, then $\mathbb{Z}_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$. In the first step of algorithm, the process checks for regular elements of the ring. That is for an element a in $\mathbb{Z}_{12}$ if there is another element b in $\mathbb{Z}_{12}$ such that $a^2 * b = a$ then a is regular element of the ring $\mathbb{Z}_{12}$. We have $0, 1, 3, 4,5,7,8,9,11$ are regular elements of $\mathbb{Z}_{12}$ that is $Reg(\mathbb{Z}_{12}) = \{0,1,3,4,5,7,8,9,11\}$. Step two determines vertices $V\big(\Gamma_{MR}\,(\mathbb{Z}_{12})\big)$, this means that all elements in $\mathbb{Z}_{12}$ will be added to vertices of the graph, so that $V\big(\Gamma_{MR}\,(\mathbb{Z}_{12})\big) = \{0, 1, 2, 3, 4, 5, 6, 7,8,9,10,11\}$ and in this step the algorithm checks for vertices that are adjacent, That is for i in $V\big(\Gamma_{MR}\,(\mathbb{Z}_{12})\big)$ if there exists j when $i \neq j$, and $i * j \in Reg(\mathbb{Z}_{12})$, then i is adjacent to j, for the vertex 0 there is the vertex 1 such that $0.1 = 0 \in Reg(\mathbb{Z}_{12})$, so that $0{\sim}1$, similarly

$0{\sim}2, 0{\sim}3, 0{\sim}4, 0{\sim}5, 0{\sim}6, 0{\sim}7, 0{\sim}8, 0{\sim}9, 0{\sim}10, 0{\sim}11, 1{\sim}3, 1{\sim}4, 1{\sim}5, 1{\sim}7, 1{\sim}8, 1{\sim}9, 1{\sim}11, 2{\sim}4,$ $2{\sim}6\ 2{\sim}8, 2{\sim}10, 3{\sim}4, 3{\sim}5, 3{\sim}7, 3{\sim}8, 3{\sim}9, 3{\sim}11, 4{\sim}5, 4{\sim}6, 4{\sim}7, 4{\sim}8, 4{\sim}9, 4{\sim}10, 4{\sim}11, 5{\sim}7, 5{\sim}8,$ $5{\sim}9, 5{\sim}11, 6{\sim}10, 7{\sim}8, 7{\sim}9, 7{\sim}11, 8{\sim}6, 8{\sim}9, 8{\sim}10, 8{\sim}11, 9{\sim}11$ And the edges will be added to $E\big(\Gamma_{MR}\,(\mathbb{Z}_{12})\big)$. The last step of algorithm draws the multiplicative regular graph of the ring $\mathbb{Z}_{12}$, as shown in **Figure 6**.

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4. CONCLUSIONS

The graph $\Gamma_{MR}(R)$ is a generalization of both $\Gamma_0(R)$ and $\mathfrak{R}_\partial(R)$. For commutative ring R, the equivalence $\Gamma_{MR}(R) \cong \Gamma_0(R)$ holds if and only if the set of regular elements of R is $\{0, 1\}$. If R is a commutative regular ring, $\Gamma_{MR}(R)$ is isomorphic to $G_{Vnr^+}(R)$. Additionally, for $n \geq 5$, the graph $\Gamma_{MR}(\mathbb{Z}_n)$ contains a complete subgraph K_5 with 0 as one of its vertices.

Conflict of interests.

There are non-conflicts of interest.

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الخلاصة

المقدمة:

يعد الرسم البياني ذو القاسم الصفري $\Gamma_0(R)$ محوريا في نظرية الرسوم البيانية الجبرية. لتكن (R) حلقة ابدالية ذات العنصر المحايد $(1 \neq 0)$. في هذه العمل نعرف الرسوم البيانية المنتظمة ضربيا $(\Gamma_{MR}(R))$ بأنه تعميم للرسم البياني ذو القاسم الصفري $\Gamma_0(R)$ اللذي يظهر سلوك العناصر المنتظمة التي تحقق $a = aba$ او $b = bab$.

طرق العمل:

يعرف الرسم البياني $(\Gamma_{MR}(R))$ بإستخدام شروط نظرية الحلقات وتمثيلات نظرية الرسوم البيانية. عنصران مختلفان غير صفريين x و y متجاوران اذا و فقط اذا $xy \in Reg(R)$. استخدمت خوارزمية قائمة على بايثون لبناء و تصور الرسوم البيانية $(\Gamma_{MR}(R))$ للحلقات محدودة الابدالية.

النتائج:

أثبتنا أن $\Gamma_{MR}(R) \cong \Gamma_0(R)$ اذا و فقط اذا كانت $R = \{0,1\}$ و $\Gamma_{MR}(R) \cong G_{Vnr^+}(R))$ اذا كانت (R) منتظمة. تتظمن الدراسة تحليلا لمتغيرات الرسم البياني الرئيسية مثل الأرتباط , القطر, المحيط و الانتظام.

الاستنتاجات:

يقدم الرسم البياني $(\Gamma_{MR}(R))$ تعميما للرسوم بيانية الحالية للحلقات ابدالية. من خلال التركيز على العناصر المنتظمة. يكشف هذا النموذج عن علاقات هيكلية جديدة داخل الحلقة. و يستكشف خصائص نظرية الحلقات من خلال أساليب نظرية رسوم البيانية.

الكلمات المفتاحية:

رسم بياني منتظم ضربيا، ترابط رسم بياني، انتظام رسم بياني، قطر رسم بياني، محيط رسم بياني.