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Comparison of Some Methods To Estimate Reliability Function of Transmuted Inverted Topp-Leone distribution

ASS.Prof. Dr. Raissan A. Zalan

Basrah University- Basrah, Iraq, College of Administration and Economics
ressan.zalan@uobasrah.edu.iq

Abstract :
It is clear that studying lifetime distributions which are related to the survival function or reliability data, is an important matter in our lives, because these distributions are important in many areas in real life, for example, survival data analysis. Since this data is often complex, therefore, representing it through single distributions is less efficient than compound, mixed, or transformed distributions, which are more suitable for such data. Therefore, researchers are trying to find new distributions through transformation or composition, so that they contain more parameters so that they absorb this complexity in the data. Therefore, the researcher found a new distribution, which is the transformed inverse Top Lyon distribution. The properties of this distribution were found, as well as its parameters were estimated, and the reliability function was found using the least squares, maximum likelihood, and weighted least squares methods, and comparing them to find the best method for estimation.
Keywords The lifetime distribution, the transmuted Inverted Topp-Leone distribution, Reliability Function, Method of maximum likelihood, Method of Least-Squares, Method of Weighted Least-Squares

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مقارنة بعض الطرائق لتقدير دالة المعولية لتوزيع توب ليون المعكوس المحول

ريسان عبدالامام زعلان
جامعة البصرة، كلية الادارة والاقتصاد
ressan.zalan@uobasrah.edu.iq

المستخلص:
من الواضح ان التطرق الى توزيعات الحياة والتي ترتبط ببيانات دالة البقاء او المعولية من الأمور المهمة في حياتنا وذلك لان هذه التوزيعات تكون مهمة في العديد من المجالات في الحياة الحقيقية ومنها مثلا تحليلات بيانات البقاء وبما ان هذه البيانات تكون في الغالب معقدة وبالتالي يكون تمثيل التوزيعات المفردة لها اقل كفاءة من التوزيعات المركبة او المختلطة او المحولة والتي تكون اكثر ملائمة لمثل هذه البيانات لذا يحاول الباحثون في إيجاد توزيعات جديدة عن طريق التحويل او التركيب بحيث تحتوي على معلمات اكثر بحيث انها تمتص هذا التعقيد في البيانات , لذا قام الباحث بإيجاد توزيع جديد وهو توزيع توب ليون المعكوس المحول كما تم إيجاد خصائص هذا التوزيع وكذلك تم تقدير معلماته وإيجاد دالة المعولية باستعمال طريق المربعات الصغرى والامكان الأعظم والمربعات الصغرى الموزونة والمقارنة بينها لإيجاد افضل طريقة في التقدير.
الكلمات المفتاحية: توزيع العمر الافتراضي، توزيع توب ليون المقلوب المتحول، دالة الموثوقية، طريقة الاحتمالية القصوى، طريقة المربعات الصغرى، طريقة المربعات الصغرى المرجحة

1-INTRODUCTION

In order to obtain a transformer distribution, a new parameter is added λ to the cumulative distribution function (CDF) where ($0 \le \lambda \le 1$), Let it be CDF G(x) of any random variable X, then the cumulative distribution function F(x) of transmuted distribution is given by Shaw and Buckley[1] as:

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, \qquad 0 \le \lambda \le 1 \quad (1)$$

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And the probability function (PDF) is given as:

$$f(x) = [\,(1+\lambda) - 2\lambda G(x)]g(x) \quad (2)$$

Where $g(x)$ is the probability function (PDF) of base distribution.

Several different distributions have been created. [2] He developed a new generalized distribution for the Weibull distribution, called the transformed Weibull distribution. [3] He studied various structural properties of the transformed Rayleigh distribution. [4] He introduced the transformed modified Weibull distribution. A transformation to the Lomax distribution was performed by [5]. [6] He proposed the transformed Pareto distribution. A new distribution, the transformed generalized linear exponential, was proposed by [7], among others.
In this paper, the researcher introduced a new inverted Top Lyon distribution, called the transformed inverted Top Lyon distribution.
The paper is organized as follows: Part 2 discusses the transformed inverted Top Lyon distribution (TITL). Part 3 discusses some statistical properties of the TITL distribution, such as moment rth, quintile function, mode, and the distribution of ordered statistics for the TITL distribution. Part 4 proposes estimating the parameters of the TITL distribution using the maximum likelihood estimation method and the weighted least squares method. Part 5 presents a simulation of the new TITL distribution to compare estimation methods. Part 6 discusses the results and finally. Part 7 presents the conclusion of the study.

2-Transmuted Inverted Topp-Leone distribution

The inverted Top-Lyon (ITL) distribution, which has only one shape parameter (θ), introduced by Hassan et al. [10], is an important and modern model among the known inverted distributions. The density functions and hazard functions take several forms depending on the value of d, such as unimodal , right-skewed, increasing, decreasing, and inverted. The probability density function (PDF) and cumulative distribution function (CDF) of the ITL distribution are determined

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according to the following formulas:

$$G(x) = 1 - \frac{(1+2\,x)^{\theta}}{(1+\,x)^{2\theta}} \,, \quad x \ge 0\,, \theta > 0 \qquad (3)$$

$$g(x) = 2\,\theta\,x \frac{(1+2\,x)^{\theta-1}}{(1+\,x)^{2\theta+1}} \,, \quad x \ge 0\,, \theta > 0 \qquad (4)$$

Some authors have studied and developed new extensions and generalizations to the ITL distribution, such as the power ITL distribution by Abushal et al. [11], the Kumaraswamy ITL distribution by Hassan et al. [12], the alpha-power ITL distribution by Ibrahim et al. [13], the modified Kies ITL distribution by El-Mutawali et al. [14], the individual Weibull ITL distribution proposed by El-Mutawali [15], and the semi-logistic ITL distribution by Pantan et al. [16]. Here we introduced a new ITL distribution by replacing (3) in (1) to get the CDF of (TITL) as follow:

$$F(x) = (1+\lambda)\left(1 - \frac{(1+2\,x)^{\theta}}{(1+\,x)^{2\theta}}\right) - \lambda\left(1 - \frac{(1+2\,x)^{\theta}}{(1+\,x)^{2\theta}}\right)^{2} \,, -1 \le \lambda \le 1\,, \theta > 0 \qquad (5)$$

And PDF of the (TITL) is:

$$f(x) = \left(2x(1+x)^{2\theta}\theta(\lambda-1) + 4x(1+2x)^{\theta}\theta\lambda\right)\frac{(1+2\,x)^{\theta-1}}{(1+\,x)^{4\theta+1}} \qquad (6)$$

where θ and λ are the shape parameters.

The function of reliability and hazard are given respectively by:

$$R(x) = \frac{(1+2x)^{\theta}}{(1+x)^{4\theta}}\left(-(1+x)^{2\theta}(-1+\lambda) + (1+2x)^{\theta}\lambda\right) \qquad (7)$$

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$$h(x) = \frac{2x\theta((1+x)^{2\theta}(-1+\lambda) - 2(1+2x)^{\theta}\lambda)}{(1+x)(1+2x)((1+x)^{2\theta}(-1+\lambda) - (1+2x)^{\theta}\lambda)} \tag{8}$$

3-Graphs of the density function and hazard function for the distribution of TITL

From the following graphs, we notice that the shape of the density function for the title distribution is skewed to the right (Figure 1). We also notice that the hazard function is shown in Figure (2).

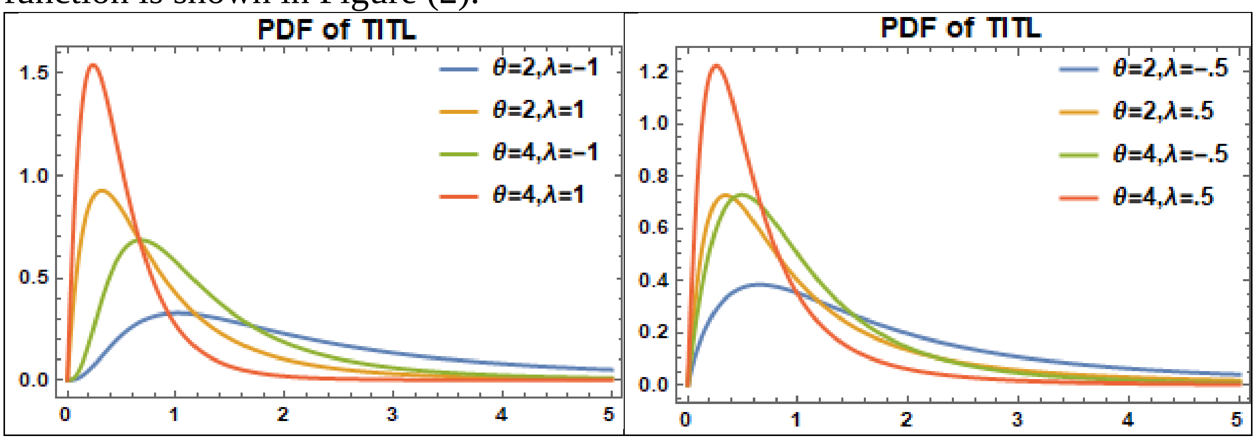

Figure1. The density function of TITL.

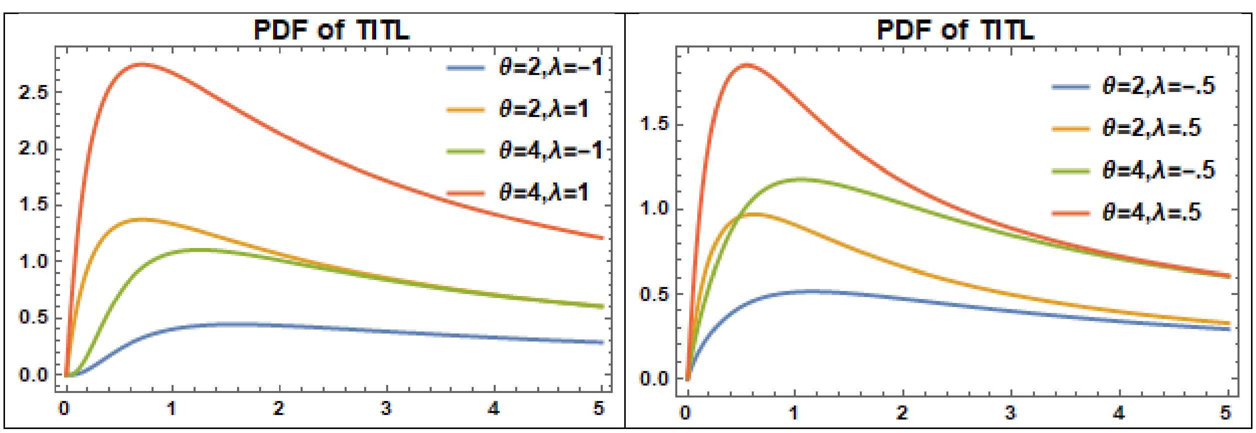

Figure2. The hazard function of TITL.

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4- Statistical Properties of TITL

This section is dedicated to highlighting and finding some statistical properties of the TITL distribution, such as the quintile moment rth, the median, and the mode.

4.1- The rth raw moment of a continuous random variable x denoted by μ'_r is defined as

The generalized binomial expansion, for > 0 is real non integer and $|z| < 1$ is

$$\mu'_r = E(x^r) = \int_0^\infty x^r f(x)dx \quad (9)$$

$$E(x^r) = \int_0^\infty x^r \left[\left(2x(1+x)^{2\theta}\theta(\lambda-1) + 4x(1+2x)^{\theta}\theta\lambda\right)\frac{(1+2\,x)^{\theta-1}}{(1+\ x)^{4\theta+1}}\right]dx$$

$$E(x^r) = 2\theta(\lambda-1)\int_0^\infty x^{r+1}((1+x)^{-2\theta-1}(1+2x)^{\theta-1}dx$$
$$+4\theta\lambda\int_0^\infty x^{r+1}((1+x)^{-4\theta-1}\,(1+2x)^{2\theta-1}dx$$

$$= 2\theta(\lambda-1)\int_0^\infty x^{r+1}((1+x)^{-2\theta-1}(1+x+x)^{\theta-1}dx$$
$$+4\theta\lambda\int_0^\infty x^{r+1}((1+x)^{-4\theta-1}(1+x+x)^{2\theta-1}dx$$

$$= 2\theta(\lambda-1)\int_0^\infty x^{r+1}((1+x)^{-\theta-2}(1+\frac{x}{1+x})^{\theta-1}dx$$
$$+4\theta\lambda\int_0^\infty x^{r+1}((1+x)^{-2\theta-2}(1+\frac{x}{1+x})^{2\theta-1}dx \quad (10)$$

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$$(1+z)^{\theta-1} = \sum_{j=0}^{\infty} \binom{\theta-1}{j} z^j \quad (11)$$

By using the binomial theorem (11) in (10), since it is considered a real number, we have

$$\mu'_r = 2\theta(\lambda-1)\sum_{j=0}^{\infty}\binom{\theta-1}{j}B(r+j+2,\theta-r) + 4\,\theta\,\lambda\sum_{j=0}^{\infty}\binom{2\theta-1}{j}B(r+j+2,2\theta-r-1) \quad (12)$$

where $B(.,.)$ is the beta function.

Furthermore, the r^{th} central moment of X is given by

$$\mu_m = E(X-\mu'_1)^r = \sum_{i=0}^{\infty}\binom{r}{j}(-\mu'_r)^j(\mu'_{r-j}) \quad (13)$$

The measure of symmetry is the skewness which describes the symmetry of the distribution and defined as

$$SK = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2{\mu'_1}^3}{(\mu'_2 - {\mu'_1}^2)^{\frac{3}{2}}} \quad (14)$$

The kurtosis which describes of the distribution and defined as follows:

$$KU = \frac{\mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2{\mu'_1}^2 - 3{\mu'_1}^4}{\left(\mu'_2 - {\mu'_1}^2\right)^2} \quad (15)$$

4.2- The quintile function by inverting the CDF equation in (5), we have the quintile function of TITL distribution as follows:

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$$x_u = \frac{1}{2}(-1 + 2^{-1/\theta}\left(\frac{-1+\lambda-\sqrt{1+2\lambda-4u\lambda+\lambda^2}}{\lambda}\right)^{\frac{1}{\theta}} - \sqrt{-3 + 2^{1-\frac{1}{\theta}}\left(\frac{-1+\lambda-\sqrt{1+2\lambda-4u\lambda+\lambda^2}}{\lambda}\right)^{\frac{1}{\theta}} + 2^{-2/\theta}\left(\frac{-1+\lambda-\sqrt{1+2\lambda-4u\lambda+\lambda^2}}{\lambda}\right)^{2/\theta}}) \quad (16)$$

5-Estimation Methods

This section uses three different estimation methods is: maximum likelihood, least-square, and weighted least-square to analyze the estimation problem of the TITL distribution parameters and reliability function.

5.1 Maximum Likelihood Estimators

The maximum likelihood estimates (MLEs) enjoy desirable properties that can be used when constructing confidence intervals and deliver simple approximations that work well in finite samples.

Let $x_1, x_2, \dots, x_n$ be independent and identically distributed observed random sample of size n from the TITL distribution. Then, the log-likelihood function based on observed given by

$$L = \prod_{i=1}^{n}(1+x_i)^{-1-4\theta}(1+2x_i)^{-1+\theta}(-2\theta(-1+\lambda)x_i(1+x_i)^{2\theta} + 4\theta\lambda x_i(1+2x_i)^{\theta}) \quad (17)$$

The equivalent log-likelihood function l = log(L) is

$$Log(L) = -\left((1+4\theta)\sum_{i=1}^{n}\text{Log}[1+x_i]\right) + (-1+\theta)\sum_{i=1}^{n}\text{Log}[1+2x_i]$$

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$$+\sum_{i=1}^{n}\text{Log}\left[-2\theta(-1+\lambda)x_i(1+x_i)^{2\theta} + 4\theta\lambda x_i(1+2x_i)^{\theta}\right] \qquad (18)$$

The maximum likelihood estimates of θ and λ are obtained by maximizing the log-likelihood function given in (18). For doing so, taking the derivatives with respect to unknown parameters and further proceed as follow:

$$\frac{\partial \log \text{L}}{\partial \theta} = -4\sum_{i=1}^{n}\text{Log}[1+x_i] + \sum_{i=1}^{n}\text{Log}[1+2x_i] + \sum_{i=1}^{n}\frac{-2(-1+\lambda)x_i(1+x_i)^{2\theta} - 4\theta(-1+\lambda)\text{Log}[1+x_i]x_i(1+x_i)^{2\theta} + 4\lambda x_i(1+2x_i)^{\theta} + 4\theta\lambda\text{Log}[1+2x_i]x_i(1+2x_i)}{-2\theta(-1+\lambda)x_i(1+x_i)^{2\theta} + 4\theta\lambda x_i(1+2x_i)^{\theta}}$$

$$\frac{\partial \log \text{L}}{\partial \theta} = \sum_{i=1}^{n}\frac{-2\theta x_i(1+x_i)^{2\theta} + 4\theta x_i(1+2x_i)^{\theta}}{-2\theta(-1+\lambda)x_i(1+x_i)^{2\theta} + 4\theta\lambda x_i(1+2x_i)^{\theta}} \qquad (20)$$

The MLE of the distribution parameters θ and λ can be calculated by finding the maximum value for Equation (18) by finding the first derivative for it with respect to θ and λ. The Mathematica software can be used to optimize the log-likelihood and obtain the MLE by using the Newton-Rapshon method.

5.2- Least-Squares and Weighted Least-Squares Methods

The least squares (LS) and weighted least squares (WLS) methods are used to estimate the parameters of different distributions. Thus, let us assume that $x_1 < x_2 < \ldots < x_n$ be a random sample with θ and λ parameters from the TITL distribution. LS estimators (LSE) and WLS estimators (WLSE) of the can be obtained by minimizing the following equation:

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$$w(\Theta)=\sum_{i=1}^{n} w_i\left(+(1+\lambda)\left(1-\left(x_i+\frac{1}{1+x_i}\right)^{\theta}\right)-\lambda\left(1-\left(x_i+\frac{1}{1+x_i}\right)^{\theta}\right)^2-\frac{i}{1+n}\right)^2 \qquad (21)$$

Where $w_i = 1$ for LSE and $w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ for WLSE with respect to θ and λ Furthermore,
by resolving the nonlinear equations, the LSE and WLSE follow:

$$\frac{\partial w(\Theta)}{\partial\theta}=\sum_{i=1}^{n}\Bigg(-2\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{\theta}+\frac{2i\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{\theta}}{1+n}+2\lambda\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{\theta}-\frac{2i\lambda\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{\theta}}{1+n}+2\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}-8\lambda\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}+\frac{4i\lambda\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}}{1+n}+2\lambda^2\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}+6\lambda\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{3\theta}-6\lambda^2\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{3\theta}+4\lambda^2\mathrm{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{4\theta}\Bigg) \qquad (22)$$

$$\frac{\partial w(\Theta)}{\partial\lambda}=\sum_{i=1}^{n}\Bigg(2\left(x_i+\frac{1}{1+x_i}\right)^{\theta}-\frac{2i\left(x_i+\frac{1}{1+x_i}\right)^{\theta}}{1+n}-4\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}+\frac{2i\left(x_i+\frac{1}{1+x_i}\right.}{1+n} +2\lambda\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}+2\left(x_i+\frac{1}{1+x_i}\right)^{3\theta}-4\lambda\left(x_i+\frac{1}{1+x_i}\right)^{3\theta}+2\lambda\left(x_i+\frac{1}{1+x_i}\right)^{4\theta}\Bigg) \qquad (23)$$

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And $$\frac{\partial w(\Theta)}{\partial\theta}=\sum_{i=1}^{n}\frac{1}{i(1-i+n)}(1+n)^2(2+n)\Bigg(-2\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{\theta}+\frac{2i\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{\theta}}{1+n}+2\lambda\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{\theta}-\frac{2i\lambda\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{\theta}}{1+n}+2\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}-8\lambda\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}+\frac{4i\lambda\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}}{1+n}+2\lambda^2\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}+6\lambda\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{3\theta}-6\lambda^2\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{3\theta}+4\lambda^2\text{Log}\left[x_i+\frac{1}{1+x_i}\right]\left(x_i+\frac{1}{1+x_i}\right)^{4\theta}\Bigg) \quad (24)$$

$$\frac{\partial w(\Theta)}{\partial\lambda}=\sum_{i=1}^{n}\frac{(1+n)^2(2+n)\left(2\left(x_i+\frac{1}{1+x_i}\right)^{\theta}-\frac{2i\left(x_i+\frac{1}{1+x_i}\right)^{\theta}}{1+n}-4\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}+\frac{2i\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}}{1+n}+2\lambda\left(x_i+\frac{1}{1+x_i}\right)^{2\theta}+2\left(x_i+\frac{1}{1+x_i}\right)^{3\theta}-4\lambda\left(x_i+\frac{1}{1+x_i}\right)^{3\theta}+2\lambda\left(x_i+\frac{1}{1+x_i}\right)^{4\theta}\right)}{i(1-i+n)} \quad (25)$$

6-Simulation Experiment

In this section, the details of simulation experiments are presented in terms of taking the real values of the parameters θ and λ the generated sample sizes, and replicating them through the following stages:
Samples of size (n = 25, 50, 75, and 100) were taken, and the sample was repeated (n = 1000).

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2- More than one value was taken for the parameters (θ and λ), as shown in Table 1.
3. Choose life time for estimating reliability, $0 < t_i \leq 1$, such that t_1 = 0.1, and t_{i+1}= t_1+ 0.1,
i = 1, 2, . . .,
4. At this stage, random data was generated using (16) and using Mathematical language version 12.2. After stage 4, find the estimators and reliability function, and finding $MSE(\hat{\theta}) = \frac{\sum_{i=1}^{n}(\hat{\theta}_i - \theta)^2}{N}$ and $MSE(\hat{\lambda}) = \frac{\sum_{i=1}^{n}(\hat{\lambda}_i - \lambda)^2}{N}$ where ($\hat{\theta}$) and ($\hat{\lambda}$) are an estimators of parameters θ and λ. Tables 2 to 5 show the values of the mean and MSE for the parameter estimators and the reliability function, as follows:

Table1: Default values of parameters

Experiment	θ	λ
E1	0.5	-0.5
E2	1.5	0.1
E3	1	0.2
E4	0.1	-1

Table2: Estimating values for parameters (θ and λ) and reliability function (R) using (E1, where R=0.9637332)

n	Estimators	Methods			Best
		MLE	LSE	WLSE	
25	$\hat{\theta}$ $MSE(\hat{\theta})$	0.515800 0.016962	0.466972 0.019151	0.476065 0.018435	MLE
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	-0.502894 0.209966	-0.350129 0.275616	-0.378192 0.263343	MLE
	$\hat{R}$ $MSE(\hat{R})$	0.966849 0.000530	0.962883 0.000504	0.963561 0.000489	WLSE
50	$\hat{\theta}$ $MSE(\hat{\theta})$	0.502260 0.008259	0.476724 0.012161	0.485436 0.010117	MLE

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	$\hat{\lambda}$ $MSE(\hat{\lambda})$	-0.474203 0.127038	-0.395571 0.184167	-0.422415 0.157727	**MLE**
	$\hat{R}$ $MSE(\hat{R})$	0.964525 0.000303	0.963312 0.000330	0.963513 0.000310	**MLE**
75	$\hat{\theta}$ $MSE(\hat{\theta})$	0.508128 0.006357	0.487256 0.007757	0.494526 0.006835	**MLE**
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	-0.495283 0.083799	-0.436491 0.104347	-0.457644 0.093870	**MLE**
	$\hat{R}$ $MSE(\hat{R})$	0.964516 0.000213	0.963129 0.000229	0.963532 0.000212	**WLSE**
100	$\hat{\theta}$ $MSE(\hat{\theta})$	0.506079 0.004193	0.491414 0.004608	0.497466 0.003816	**WLSE**
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	-0.530973 0.063682	-0.486711 0.061819	-0.506638 0.052790	**WLSE**
	$\hat{R}$ $MSE(\hat{R})$	0.966394 0.000161	0.964894 0.000165	0.965367 0.000150	**WLSE**

Table3: Estimating values for parameters (θ and λ) and reliability function (R) using (E2, where R=0.8054687)

n	**Estimators**	**Methods**			**Best**
		MLE	**LSE**	**WLSE**	
25	$\hat{\theta}$ $MSE(\hat{\theta})$	1.318670 0.215291	1.578430 0.281668	1.572160 0.291582	**MLE**
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	0.358792 0.410571	0.086479 0.243470	0.089662 0.259049	**LSE**

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	$\hat{R}$ $MSE(\hat{R})$	0.807078 0.002823	0.809417 0.003576	0.810778 0.003363	MLE
50	$\hat{\theta}$ $MSE(\hat{\theta})$	1.232350 0.196155	1.574090 0.178678	1.564700 0.179206	LSE
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	0.466012 0.387549	0.091193 0.149546	0.098263 0.164693	LSE
	$\hat{R}$ $MSE(\hat{R})$	0.803715 0.001145	0.806588 0.001348	0.807430 0.001292	MLE
75	$\hat{\theta}$ $MSE(\hat{\theta})$	1.278910 0.146140	1.483350 0.147616	1.493190 0.136480	WLSE
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	0.367499 0.291592	0.143549 0.131149	0.129583 0.132410	LSE
	$\hat{R}$ $MSE(\hat{R})$	0.807253 0.000895	0.808870 0.00095	0.809296 0.000933	MLE
100	$\hat{\theta}$ $MSE(\hat{\theta})$	1.270020 0.131953	1.493280 0.105157	1.504450 0.098325	WLSE
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	0.381262 0.287476	0.128564 0.115833	0.112301 0.116065	LSE
	$\hat{R}$ $MSE(\hat{R})$	0.807999 0.000679	0.808416 0.000789	0.809227 0.000767	MLE

Table4: Estimating values for parameters (θ and λ) and reliability function (R) using (E3, where R=0.8527667)

n	Estimators	Methods			Best
		MLE	LSE	WLSE	

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25	$\hat{\theta}$ $MSE(\hat{\theta})$	1.120490 0.106047	1.103590 0.167764	1.115970 0.172523	**MLE**
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	0.012773 0.329546	0.067200 0.290015	0.043020 0.298663	**LSE**
	$\hat{R}$ $MSE(\hat{R})$	0.867444 0.002964	0.864645 0.002961	0.866197 0.002810	**WLSE**
50	$\hat{\theta}$ $MSE(\hat{\theta})$	1.093200 0.070063	1.083090 0.101201	1.083610 0.088666	**MLE**
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	0.102911 0.161224	0.161144 0.148526	0.150439 0.150117	**LSE**
	$\hat{R}$ $MSE(\hat{R})$	0.857622 0.000985	0.853498 0.001094	0.854225 0.001046	**MLE**
75	$\hat{\theta}$ $MSE(\hat{\theta})$	1.105460 0.065863	1.093730 0.077258	1.088700 0.078882	**MLE**
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	0.087831 0.144009	0.117106 0.134005	0.122118 0.145041	**LSE**
	$\hat{R}$ $MSE(\hat{R})$	0.858297 0.000643	0.856176 0.000699	0.856848 0.000682	**MLE**
100	$\hat{\theta}$ $MSE(\hat{\theta})$	1.009280 0.048464	1.054380 0.051931	1.038770 0.050048	**MLE**
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	0.230327 0.132205	0.175861 0.110679	0.195843 0.115646	**LSE**
	$\hat{R}$ $MSE(\hat{R})$	0.854455 0.000454	0.853556 0.000537	0.853637 0.000525	**MLE**

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Table5: Estimating values for parameters (θ and λ) and reliability function (R) using (E4, where R= 0.999728)					
n	Estimators	Methods			Best
		MLE	LSE	WLSE	
25	$\hat{\theta}$ $MSE(\hat{\theta})$	0.102993 0.000789	0.104034 0.000240	0.107796 0.000245	LSE
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	-0.698344 0.600129	-0.902530 0.036050	-0.938380 0.021308	WLSE
	$\hat{R}$ $MSE(\hat{R})$	0.997881 0.000029	0.998445 0.000008	0.998836 0.000006	WLSE
50	$\hat{\theta}$ $MSE(\hat{\theta})$	0.101646 0.000643	0.102438 0.000119	0.107032 0.000127	LSE
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	-0.730738 0.521135	-0.928316 0.018934	-0.966916 0.005909	WLSE
	$\hat{R}$ $MSE(\hat{R})$	0.998146 0.000022	0.998782 0.000004	0.999225 0.000002	WLSE
75	$\hat{\theta}$ $MSE(\hat{\theta})$	0.105730 0.000557	0.106015 0.000110	0.109660 0.000126	LSE
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	-0.797629 0.400136	-0.959749 0.007845	-0.981873 0.002959	WLSE
	$\hat{R}$ $MSE(\hat{R})$	0.998535 0.000017	0.999138 0.000002	0.999407 0.000001	WLSE
100	$\hat{\theta}$ $MSE(\hat{\theta})$	0.106660 0.000486	0.106589 0.000085	0.110607 0.000119	LSE
	$\hat{\lambda}$ $MSE(\hat{\lambda})$	-0.814580 0.360371	-0.966511 0.006389	-0.985551 0.001779	WLSE

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	$\hat{R}$ $MSE(\hat{R})$	0.998561 0.000017	0.999214 0.000002	0.999449 0.000001	**WLSE**

For the application, data on the mortality rate of dehydration for COVID-19 patients for 24 days was used, according to source number [8], as shown in the following table:

Table6: rate of dehydration for COVID-19		
ti	**ti**	**ti**
0.22	**0.25**	**0.39**
0.22	**0.28**	**0.47**
0.21	**0.22**	**0.35**
0.23	**0.18**	**0.32**
0.10	**0.12**	**0.29**
0.11	**0.42**	**0.34**
0.34	**0.35**	**0.16**
0.26	**0.30**	**0.14**

The parameters of the TITL and ITL distributions were estimated according to the maximum likelihood method and the results were as follows:

dist	\[Theta]	\[Lambda]
TITL	32.1933	-1
ITL	21.7393	_

The researcher also matched the data with the following distributions:

Goodness of Fit tests	TILT		ILT	
TEST	Statistic	P-Value	Statistic	P-Value
Anderson-Darling	1.80965	0.117602	1.921310	0.105401
Cramér-von Mises	0.32418	0.115722	0.341210	0.119820
Kolmogorov-Smirnov	0.237026	0.113872	0.245221	0.103966

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According to the error criteria, it was found that the TITL distribution is better than the ITL distribution, as shown below:

Dist	AIC	AICc	BIC
TITL	-38.2572	-37.9715	-35.9011
ITL	-32.0347	-31.8529	-30.8567

The following form of the probability density function confirms the result:

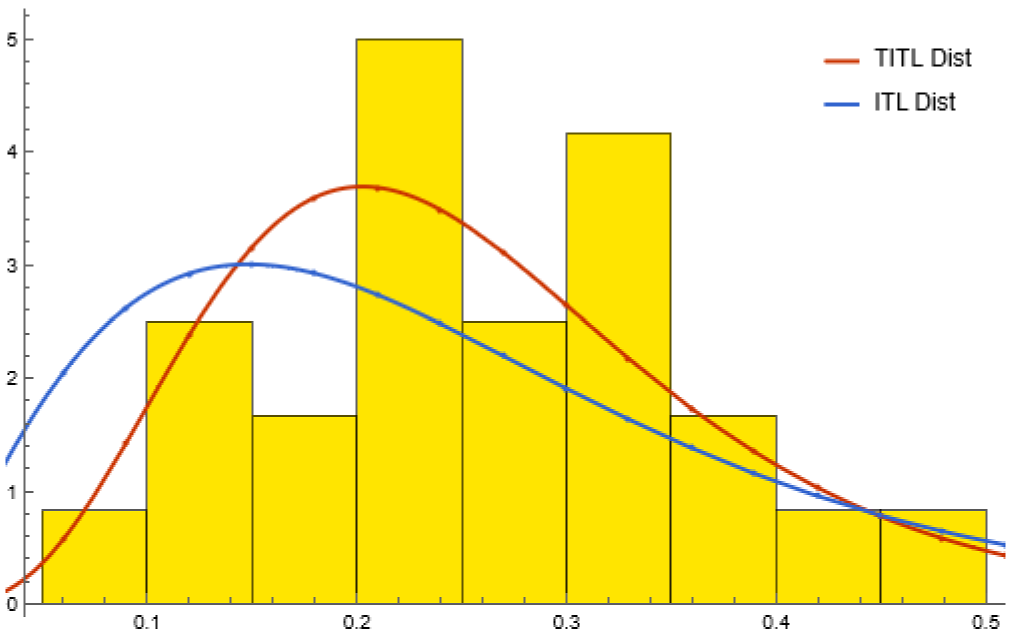

Figure (3) Comparison of the two distributions for fitting the data to the density function

7- Conclusions

The following points represent the main conclusions of this study

- In E1,for parameter θ the (MLE) was the best at (n=25,50,75) but at (n=100)

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the (WLSE) was preferred. For λ the (MLE) was the best at the same sample sizes in θ and also the (WLSE) was the best at (n=100). For estimating reliability the (WLSE) was the best for all sample sizes except (n=50) where the (MLE) it was by far the best.

- In E2,for parameter θ the (MLE) was the best at (n=25) and the (LSE) was the best at (n=50) but at (n=75,100) the (WLSE) was preferred. For λ the (LSE) was the best at all sample sizes. For estimating reliability the (MLE) was the best for all sample sizes.
- In E3,for parameter θ the (MLE) was the best at all sample sizes. For λ the (LSE) was the best at all sample sizes. For estimating reliability the (MLE) was the best for all sample sizes except smaller sample size (n=25) where the (WLSE) was preferred.
- In E4,for parameter θ the (LSE was the best at all sample sizes. For λ the (WLSE) was the best at all sample sizes. For estimating reliability the (WLSE) was the best for all sample sizes.
- The TITL distribution was better at representing the data than the ITL distribution.

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