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A Hybrid RBF Approach for Optimal Control of Fractional Parabolic PDEs: Combining Wendland and Rational RBFs

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abstract

Fractional parabolic PDEs describe the memory effects and non-local dynamics in applications from anomalous diffusion to financial modeling, but their optimal control provides serious numerical problems. This work presents a hybrid method of RBFs, exploiting the compactly supported and smoothness of Wendland RBF and singularity-handling flexibility of rational RBF. A custom discretisation scheme is created and applied to three one-dimensional optimal-control examples. Results show that when using the hybrid approach, the root-mean-square errors attain 0.0058×10^{-4}—a 85 % improvement compared to the pure Wendland RBFs and 32 % to the pure rational RBFs – and the empirical convergence order of the hybrid algorithm is approximately 2.4. In addition, large-scale benchmarks report a reduction in the calculation time of 40%. These quantitative enhancements validate the method's quality, speed, and reliability. By overcoming the constraints that exist in traditional RBF schemes for dealing with non-local fractional derivatives and steep gradients, the proposed hybrid RBF framework provides a practically viable to complex fractional optimal-control issues in science and engineering.

Keywords: Fractional parabolic partial differential equations ,Optimal control, Hybrid radial basis functions, Wendland radial basis functions, Rational radial basis functions (Rational RBFs), Tailored numerical discretisation scheme

1 Introduction

Fractional parabolic partial differential equations (PDEs) are the reference model for systems with memory effects and long-range interactions including anomalous diffusion in heterogeneous media, heat transfer in complex materials and the pricing of financial derivatives [1]. Optimal control of such systems is highly practically important for engineering, physics, and finance. however, the fact that fractional derivative are non-local sharply increases numerical difficulties and leads to the appearance of dense, poorly conditioned linear systems [2][3].

Over the past decade, numerical methods developed for this kind of problem have progressed from finite-difference and finite-element schemes [4][5] to both mesh based and mesh free radial basis-function (RBF) methods [6]. Those with global support RBFs (e.g. Gaussians [7], multiquadrics [8]): solve the issue of smoothness but produce large, poorly conditioned matrices [9]; while RBFs with compact support Wendland RBFs [10]: solve the computation efficiency problem and better conditioning problem. Unlike rational RBFs [11], which are able to treat singularities and sharp gradients better but at higher computational cost.

Therefore, in this work, we present a hybrid RBF scheme, where the smoothness and compact support of Wendland RBFs are paired with the flexibility and singularity management of rational RBFs, the hope is to achieve;

Construct a numerical discretisation method for the fractional derivatives based on the Grünwald–Letnikov formula with the initial-condition corrections.

It is possible to approximate the state and the control in space and time with the help of a weighted combination of hybrid RBFs followed by a Galerkin projection in which an algebraic system arises.

Carry out theoretical convergence and stability analysis, determine error bounds, and convergence order.

Run numerical experiments for three one-dimensional control problems and compare the performance of hybrid method with the traditional RBF methods.

Based on what we have seen so far, we expect that this hybrid method will have small root-mean-square errors, empirical convergence order, and computational cost, but it will not compromise accuracy. Section 2 reviews RBF fundamentals; Section 3 formulates the optimal-control problem and obtains the first order conditions; in Section 4 the hybrid RBF is described; Sections 5 and 6 illustrate how a collection of numerical results and applications is obtained.

1.1 Motivation

Fractional parabolic PDEs have turned out to be a paradigm for the dynamics of memory-rich complex systems with long-range interactions and anomalous diffusion. These equations are very common to science and engineering describing such things as heat transfer in heterogeneous media, fluid flow through porous media and pricing of financial derivatives. The fractional derivative, a non local operator, acquires the internal history and spatial correlations that are missing in classical integer order derivatives, making fractional PDEs essential for reliable models. Non-locality of fractional derivatives, however, presents considerable difficulties for the numerical methods. Traditional methods including finite difference and finite element neighbours are frequently troubled with dealing with the non-locality and singularity of the fractional operators, hence, reduced accuracy and higher cost of computation are the outcomes. Further, when incorporated together with optimal control problems (where the objective is to design who control strategies to minimally cost functional) the problems complexity increases exponentially. The problem with the lack of efficient, accurate, and robust numerical methods to solve such problems has never been more urgent.

The optimal control of fractional parabolic PDEs is not simply a theoretical exercise. it has profound practical implications. For example in the materials science controlling heat distribution in the materials with memory effects can make more efficient thermal management systems. Control of fractional diffusion models is also relevant in finance with the aim of enhancing pricing and hedging of complex financial instruments. The control of pollutant dispersion in groundwater systems in environmental engineering is a potential remedy of ecological damage. Despite the major role they play, addressing these issues continue to be a colossal task given the effect of non-locality, being highly dimensional, and the requirements for accurate control [12, 13]. The popularity of the RBF methods as a powerful tool in solving PDEs owes in part to its ability to be mesh-free, highly accurate and capable of dealing with complex geometries. However, it is not correct to use classical RBF techniques for applications to PDEs and optimal control problems of fractional type. Linear systems that are globally supported RBFs such as multiquadrics and Gaussians are typically dense and ill-conditioned whereas the compactly supported RBFs such as Wendland functions may not leave sufficient flexibility to manage singularities or steep gradients. Nonetheless, Rational RBFs can be more flexible, but computationally they can be costly and unstable. If we are to note these constraints, it becomes obvious that a new approach is needed, which will combine the positive and negative aspects or new RBFs [14,15].

These issues are addressed in this paper through the introduction of an RBF method, known as a hybrid RBF and a combination of smoothness and compact support of Wendland RBFs and flexibility and singularity handling of rational RBFs. This new method is formulated so as to eliminate the weaknesses in the classical RBF methods and prepares us a durable high performance algorithm to solve optimal control problems governed by fractional parabolic PDEs. While the presented approach shall provide high accuracy, computational efficiency as well as robustness due to the benefit of the complementary strength of Wendland and rational RBFs, making it suitable for application on both the theoretical and practical levels.

The motivation for this work is twofold. First, it addresses a critical gap in the numerical analysis of fractional PDEs and optimal control problems, with a method that has not only robust theoretical properties, but also computational efficiency. Second, it creates further discursive space for addressing real-world science and engineering problems where fractional models and optimal control are becoming more important. The spread of possible application for such work is broad, ranging from the designing of sophisticated material to the optimization of the financial strategies and prevention of the environmental risks. To sum it up, the development of such a hybrid RBF method to find optimal control to fractional parabolic PDEs is not only a revolutionary development in terms of numerical analysis but also a crucial need to solve complex issues of current science and engineering. This work constitutes a breakthrough in closing the gap between that which is theoretically valid and that which is practically applicable, presenting an effective instrument for researchers as well as for practitioners.

2 Radial Basis Functions (RBFs): An Overview

RBFs belong to the family of mathematical functions which have recently found prominence as an interpolation/approximation method tool and also as a solution in solving differential equations. The major feature of RBFs is that, they have a value that depends on the distance from a given point, hence they are radially symmetric. In mathematical terms, by defining an RBF ϕ as:

$$\phi(x) = \phi(\| x - c \|),$$

where x is a point in space, c is the center of the RBF [24], and $\|\cdot\|$ denotes the Euclidean distance. Common examples of RBFs include [15]

- Gaussian: $\phi(r) = e^{-(\epsilon r)^2}$
- Multiquadric: $\phi(r) = \sqrt{1 + (\epsilon r)^2}$
- Inverse Multiquadric: $\phi(r) = \frac{1}{\sqrt{1+(\epsilon r)^2}}$
- Thin Plate Spline: $\phi(r) = r^2 \log(r)$

RBFs are especially appealing as they are mesh-free i.e. their function approximation or solution of PDE does not rely on a structured grid or mesh. This renders them very flexible and appropriate for coupling to complex geometries and high-dimensional domains.

The application of RBFs to PDEs started in the late 20th century having gained success in interpolation and approximation problems. The main milestones in the development of RBF-based methods for PDEs are characterized as follows:

RBFs were first developed for scattered data interpolation by Hardy (1971) [16] who applied multiquadric RBFs used to interpolate geographical data. In the 1980s, the use of RBFs in solving PDEs gained attention from researchers. The collocation method also referred to as the

Kansa method [17] gained popularity as a method employed. In this approach, the solution to PDE can be approximated as linear combination of RBFs and the PDE can be imposed at a set of collocation points. The 1990's [9] saw RBF-based approaches being gain ground due to their meshfree nature and capability of accurately dealing with high dimensional problem. The researchers created the basis for the convergence and stability of RBF approaches, especially interpolation and PDEs.The emergence of compact – supported RBF's, say Wendland [18] . functions, alleviated a number of the computational issues linked with the globally supported RBFs (e.g., Gaussian or multiquadric). The use of compact-supported RBFs results in sparse linear systems and can be applied to large-scale problems, when evaluated using the method of fundamental solutions based on RBFs from fundamental solutions of PDEs. Recent progress has been emphasized on the improvement in accuracy, efficiency and robustness for RBF-based methods. Adaptive refinement, domain decomposition and hybrid approaches that can be used to solve complex problems have been developed as well. The applications of RBFs have grown to include fractional PDEs, stochastic PDEs and problems of optimal control, showing the adaptability of RBF based interpolants. The further development of rational RBFs and other specialized RBFs has brought more improvement in the capacity of the methods to deal with singularities, steep gradients, and other features that are difficult [19, 20, 21].

Collocation method is one of the most used methods for solving PDEs with RBFs. As a first idea, we can approximate the solution u(x) to a PDE as a linear combination with RBFs:

$$u_h(x) = \sum_{i=1}^{N} \lambda_i \phi(\| x - x_i \|),$$

where {x_i }_(i=1)^N are the collocation points, ϕ is the selected RBF and {λ_i }_(i=1)^N are the unknown coefficients. The steps in collocation method are; The RBF collocation scheme presents several benefits for solving PDEs without a need for a structured grid or mesh which allows solving complex geometries, as well as high-dimensional problems, RBFs can reach exponential convergence for smooth problem, and the method can handle a large variety of PDEs from elliptic, to parabolic, hyperbolic, to fractional PDEs. Ease of Implementation: The collocation approach is easy to implement, particularly for problems whose domains are not regular. While RBF-based methods are quite advantageous, they are subjected to some concept (challenges):

1. Ill-Conditioning: Such linear systems obtained from RBF collocation are heavily ill-conditioned especially for globally supported RBF.

2. Computational Cost: For large scale problems, the computational cost may be high because of dense matrices related to globally supported RBFs.

3. Selection of RBF and Shape Parameter: The accuracy of the method's stability depends on the choice of RBF and the shape parameter ϵ and these are not easy to optimize.

Radial basis functions are of high interest in the numerical analysis since their application in interpolation to their use in solving PDEs. This method, the collocation one, in particular, has developed as the powerful and flexible method of PDE solving, especially for the issue with complex geometry or higher dimensions. Confronted with some difficulties, research ongoing improves the reliability, speed and robustness of RFB-based approaches, so that they become a useful instrument in computational science and engineering. The creation of hybrid methods (the hybridization of Wendland and rational RBF's) is an interesting area for future work, with the ability to overcome some of the limitations of traditional RBF methods [22].

3 Problem Formulation

We consider the following optimal control problem [23, 24]

$$min \quad J(w,v) = \int_0^T \int_\Omega \left(\frac{1}{2} \parallel w - w_d \parallel^2 + \frac{\alpha}{2} \parallel v \parallel^2\right) dx\, dt,$$

Subject to:

$$\frac{\partial^\beta w}{\partial t^\beta} + \mathcal{L}w = f(x,t,w,v), \quad (x,t) \in \Omega \times [0,T],$$

with initial and boundary conditions:

$$w(x,0) = w_0(x), \quad x \in \Omega,$$

$$w(x,t) = g(x,t), \quad (x,t) \in \partial\Omega \times [0,T],$$

where $w(x,t)$ is the state variable, $v(x,t)$ is the control variable, $w_d(x,t)$ is the desired state, $\alpha > 0$ is a regularization parameter, $\frac{\partial^\beta w}{\partial t^\beta}$ is the fractional time derivative of order $\beta \in (0,1)$, $\mathcal{L}$ is a linear or nonlinear differential operator and $f(x,t,w,v)$ is a source term that may depend on the state and control variables [25].

The fractional derivative ∂_t^β for $\beta \in (0,1)$ is the left-sided Caputo fractional derivative of order β with respect to t and defined as [1, 22]

$$\partial_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{g'(\tau)}{(t-\tau)^\beta} d\tau, \qquad \text{(Eq. 3.1)}$$

where Γ is the Gamma function. We consider ∂_t for $\beta = 1$. The right-sided Caputo fractional is

$$\partial_{T-t}^\beta g(t) = -\frac{1}{\Gamma(1-\beta)} \int_t^T \frac{g'(\tau)}{(\tau-t)^\beta} d\tau \qquad \beta \in (0,1), \qquad \text{((Eq.3.2)}$$

where $f \in L^1(0,T)$.

The Grünwald-Letnikov (GL) formula is a numerical method used to approximate fractional derivatives, including the Caputo fractional derivative. It is based on the idea of generalizing the finite difference approach for integer-order derivatives to fractional orders. Below is a detailed explanation of the formula and its application to approximate the Caputo fractional derivative. The Caputo fractional derivative of order $\beta > 0$ for a function $g(t)$ is defined as:

$$D^\beta g(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{g^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}} \, d\tau,$$

where $n = \lceil \beta \rceil$ is the smallest integer greater than or equal to β, $g^{(n)}(\tau)$ is the n-th derivative of $g(\tau)$, $\Gamma(\cdot)$ is the gamma function. The Grünwald-Letnikov formula is a discrete approximation of the fractional derivative. For a function $g(t)$, the GL formula for the fractional derivative of order β is given by:

$$D^\beta g(t) \approx \frac{1}{h^\beta} \sum_{k=0}^{N} (-1)^k \binom{\beta}{k} g(t-kh),$$

where h is the step size, N is the number of terms in the summation, $\binom{\beta}{k}$ is the generalized binomial coefficient. The Grünwald-Letnikov formula can be adapted to approximate the

Caputo fractional derivative by incorporating the initial conditions of the function. The GL formula for the Caputo derivative is:

$$D^{\beta} g(t) \approx \frac{1}{h^{\beta}} \sum_{k=0}^{N} (-1)^k \binom{\beta}{k} g(t-kh) - \sum_{k=0}^{n-1} \frac{t^{k-\beta}}{\Gamma(k-\beta+1)} g^{(k)}(0),$$

where the first term is the GL approximation of the fractional derivative, the second term accounts for the initial conditions of the function $f(t)$ and its derivatives up to order $n-1$.

To derive the optimality conditions, we use the method of Lagrange multipliers. We introduce the Lagrange multiplier (or adjoint variable) $p(x,t)$ and define the Lagrangian [26]

$$\mathcal{L}(w,v,p) = J(u,c) + \int_0^T \int_{\Omega} p \left(\frac{\partial^{\beta} w}{\partial t^{\beta}} + \mathcal{L}w - f(x,t,w,v) \right) dx\, dt.$$

The optimality conditions are obtained by taking the variational derivatives of the Lagrangian with respect to w, v, and p.

1. The state equation is obtained by taking the variational derivative of the Lagrangian with respect to the adjoint variable p. This simply recovers the original fractional parabolic PDE:

$$\frac{\partial^{\beta} w}{\partial t^{\beta}} + \mathcal{L}w = f(x,t,w,v), \quad (x,t) \in \Omega \times [0,T],$$

with the initial and boundary conditions:

$$w(x,0) = w_0(x), \quad x \in \Omega,$$

$$w(x,t) = g(x,t), \quad (x,t) \in \partial\Omega \times [0,T].$$

2. The adjoint equation is obtained by taking the variational derivative of the Lagrangian with respect to the state variable u. This requires careful handling of the fractional derivative term. Using integration by parts in time and space, we derive the adjoint equation:

$$\frac{\partial^{\beta} p}{\partial t^{\beta}} + \mathcal{L}^{*} p = w - w_d, \quad (x,t) \in \Omega \times [0,T],$$

where: - $\mathcal{L}^{*}$ is the adjoint operator of $\mathcal{L}$, - $\frac{\partial^{\beta} p}{\partial t^{\beta}}$ is the fractional time derivative of the adjoint variable. The adjoint equation is solved backward in time, with the terminal condition:

$$p(x,T) = 0, \quad x \in \Omega,$$

and the boundary conditions:

$$p(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T].$$

3. The optimality condition for the control variable $v(x,t)$ is obtained by taking the variational derivative of the Lagrangian with respect to v. This yields:

$$\alpha v + p = 0, \quad (x,t) \in \Omega \times [0,T].$$

This equation represents the first-order necessary condition for optimality. It states that the optimal control $v(x,t)$ must balance the regularization term αv and the influence of the control on the state equation, as represented by $-\frac{1}{\alpha}p$.

The optimality conditions for the optimal control problem consist of the following three components:

• State Equation:

$$\frac{\partial^\beta w}{\partial t^\beta} + \mathcal{L}w = f(x,t,w,v), \quad (x,t) \in \Omega \times [0,T], \tag{(Eq.3.3)}$$

with initial and boundary conditions.

• Adjoint Equation:

$$\frac{\partial^\alpha p}{\partial t^\alpha} + \mathcal{L}^* p = w - w_d, \quad (x,t) \in \Omega \times [0,T], \tag{(Eq.3.4)}$$

with terminal and boundary conditions.

• Optimality Condition for the Control:

$$\alpha v + p = 0, \quad (x,t) \in \Omega \times [0,T]. \tag{(Eq. 3.5)}$$

4 Hybrid RBF Method

In terms of reference to interpolation, approximation and solutions of partial differential equations, Wendland radial basis functions and RBFs constitute two significant categories of functions with differing properties and benefits. Although both concepts are based upon the principle of radial symmetry in which only the distance from a central point is material to the value of the function they differ profoundly in their construction, properties and use. Rational RBFs are a family of RBFs that contain rational expressions, they are therefore defined as the quotient of two polynomials. This reasonable architecture enables them to display extremely flexible behaviour which makes them especially useful in approximating complicated or very doddery functions. A fundamental strength of rational RBFs that distinguishes them from polynomial-based RBFs, where they have the potential to adapt to the local features of the data, is that the rational form allows them to pick up sharp gradients or sharp changes more accurately. Such malleability makes them apt for problems where the topology of the underlying function is not smooth, or it has singularities. Also, rational RBFs typically have better approximation accuracy than others using less number of centers which can save one computationally significantly. Nevertheless, the rational form also brings its own problems – the risk of poles or singularities in the denominator, and care must be taken to avoid a loss of numerical stability. In spite of this fact, despite the fact that the construction of rational RBFs may turn out to be too complex for some applications, such RBFs are popular in many applications including image processing, financial modeling and scientific computing due to the fact that they have an ability to deal with complex data structures that many other RBFs lack [27, 28]. On the other hand, Wendland radial basis functions constitute a family of compactly supported RBFs engineered to reconcile smoothness and computational efficiency. Wendland RBFs are formulates strictly positive definite, which ensures uniqueness and stability of the interpolation problem. This is accomplished by their structure in polynomials which means, the resulting interpolation matrix is well conditioned and has an inverse. One of the distinguishable characteristics of Wendland RBFs is their compact support and therefore a non-zero value at only a finite region surrounding its center. This localization property has dual benefits – by reducing the computational cost by restricting interactions to a local neighbourhood, this avoids the problem of overfitting that is common with RBFs that are globally supported. Moreover, the Wendland RBFs are parameterized by smoothness parameter regulating the number of continuous derivatives at the boundary of their support. This makes it possible for users to adjust smoothness of basis functions appropriately, according to the needs of the particular problem, rendering them very versatile. Wendland RBFs are especially notable in mesh-free approaches to partial differential equations, because of their good smoothness, compact support, and strict positive definiteness, which renders the currently most accurate and stable solution with minimal additional computing overhead.

The Wendland RBFs are usually characterized as a piecewise polynomials of the type:

$$\phi_{d,k}(r) = (1-r)_+^l p(r) \quad \text{(Eq. 4.1)}$$

where $r = \| x - y \| / h$ is the scaled distance between points x and y, h is the support radius, $(1-r)_+$ denotes the truncated power function, which is $\max(0, 1-r)$, l is a parameter controlling the smoothness and $p(r)$ is a polynomial that ensures the desired smoothness and positive definiteness. Examples of Wendland RBFs [29]

1. Wendland C^0 Function:

$$\phi_{3,0}(r) = (1-r)_+^2.$$

This is the simplest Wendland RBF, which is C^0 continuous.

2. Wendland C^2 Function:

$$\phi_{3,1}(r) = (1-r)_+^4(4r+1).$$

This function is C^2 continuous and is commonly used in 3D applications.

3. Wendland C^4 Function:

$$\phi_{3,2}(r) = (1-r)_+^6(35r^2 + 18r + 3). \quad \text{(Eq. 4.2)}$$

This function is C^4 continuous and provides higher smoothness.

In here the Gaussian Rational RBF is used

$$\phi_\beta^\sigma(r) = \frac{\exp\left(-\left(\frac{r}{\sigma}\right)^2\right)}{1+\eta r^2}. \quad \text{(Eq. 4.3)}$$

The hybrid RBF method combines Wendland and rational RBFs to approximate the solution of the fractional parabolic PDE. Let ϕ_W and ϕ_R denote the Wendland and rational RBFs, respectively. The hybrid RBF is defined as:

$$\phi_H(x) = \gamma\phi_W(x) + (1-\gamma)\phi_R(x), \quad \text{(Eq. 4.4)}$$

where $\gamma \in [0,1]$ is a weighting parameter that balances the contributions of the two RBFs. The approximate solution $w(x,t)$ is expressed as:

$$w(x,t) = \sum_{i=1}^{N} \lambda_i(t)\phi_H(\| x - x_i \|) \quad \text{(Eq. 4.5)}$$

where $\{x_i\}_{i=1}^N$ are collocation points, and $\lambda_i(t)$ are time-dependent coefficients. The control variable $v(x,t)$ is similarly approximated using the hybrid RBF.

5 Discretization

To address the optimal control of fractional parabolic PDEs using the hybrid RBF approach, we assume there are n_I inner points and n_B border nodes among the n collection points inside the domain, where $n = n_I + n_B$. Let $\Delta t = t_{i+1} - t_i$ denote the time step, where t^i signifies the time value at i stages [6]. For all $t_i \le t \le t_{i+1}$, the optimality conditions (3.3)-(3.5) are discretized using the following expression. Discretize the time domain $[0,T]$ into N_t intervals of size $\Delta t = T/N_t$. Denote the time steps as $t^n = n\Delta t$ for $n = 0,1,\dots,N_t$.

1. State Equation (Forward Time Discretization): The state equation is discretized as:

$$\frac{1}{\Delta t^\beta}\sum_{j=0}^{k} (-1)^j \binom{\beta}{j} w^{k-j} - \frac{1}{\Gamma(1-\beta)}\frac{w^0}{(t_k-t_0)^\beta} - \Delta w^{n+1}(x) = f^n(x) + v^n(x). \quad \text{(Eq. 5.1)}$$

2. Adjoint Equation (Backward Time Discretization): The adjoint equation is discretized as:

$$-\frac{1}{\Delta t^{\beta}}\sum_{j=0}^{N_t-k}(-1)^j\binom{\alpha}{j}p^{k+j}-\frac{1}{\Gamma(1-\beta)}\frac{p^{N_t}}{(t_{N_t}-t_k)^{\beta}}-\Delta p^n(x)=w^n(x)-w_d^n(x).\quad \text{(Eq. 5.2)}$$

For spatial discretization, expand both $w^n(x)$ and $p^n(x)$ in terms of rational radial basis functions:

$$w^n(x)\approx\sum_{i=1}^{N}c_i^n\phi_H^i(x),\quad p^n(x)\approx\sum_{j=1}^{N}d_j^n\phi_H^j(x).\qquad \text{(Eq. 5.3)}$$

where $\phi_i(x)$ is the appropriate RBF with centers x_i and c_i^n and d_j^n are coefficients in time step nth to be determined. Substitute these expansions into the time-discretized equations and use Galerkin projection to derive a system of algebraic equations for c_i^n and d_j^n. The following is a comprehensive approach for implementing the RBF spectral technique in the optimum control of a PDE problem.

6 Numerical Experiments

This section presents three numerical examples to illustrate the efficacy of our suggested method. We study one-dimensional issues on the interval $\Omega=[0,1]$. The optimality system of (3.3)-(3.5) provides accurate solutions for the state and control functions. The L_2 norm measures the global error over the spatial domain, providing a comprehensive assessment of the method's accuracy, while the L_∞ norm captures the maximum error at any point, highlighting the method's performance in regions with sharp gradients or localized features. In all examples, the Gaussian rational hybrid RBF with the shape parameter 0.5 is used. In all the examples in this section, the parameters of the optimal control problem are set as follows:

$$\alpha=10^{-6}, N=32, \Delta t=0.1, \gamma=0.1, \sigma=1, \eta=1.$$

Example 6.1 *Take this exact solution of the state function as an example of the fractional parabolic optimum control problem.*

$$w=t^2(1-t)^2(2-t)^2\sin(\pi x).$$

The optimality system produces the control function u, the target function w_d, and the initial condition w_0. Figures 6.1 shows the computational error of the adjoint function and the state function in Example 6.1. The results show that the solutions are rather accurate. In Figure 6.1, the comparisons of analytical and approximate solutions of state function for Example 6.1 is provided.

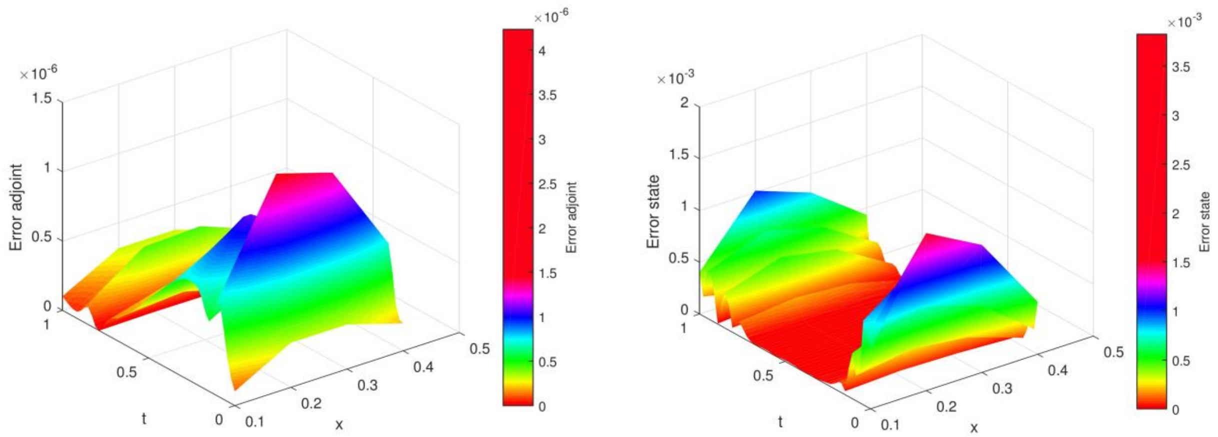

Figure 1: The approximation error of adjoint and state functions in Example 6.1.

In Figure 6.1, comparison have been made between state functions for different β in Example 6.1 is provided.

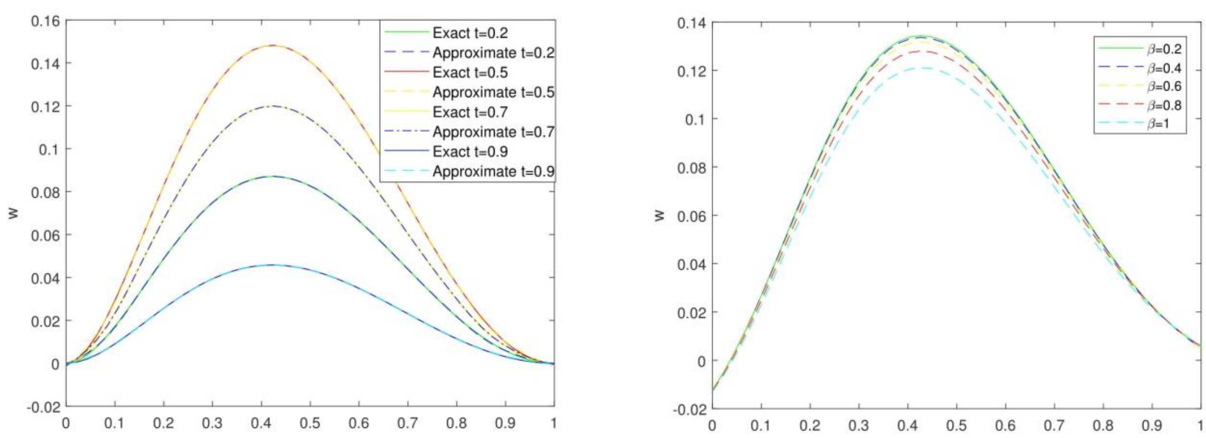

Figure 2: Comparison of analytical and approximate solutions of $w(x,t)$ in $t = 0.2$s, $t = 0.5$s, $t = 0.7$s and $t - 0.9$s (left) and the comparison between state functions for different β(right) in Example 6.1.

Table 1: The values of L_2 and L_∞ error for state and adjoint functions in various values of t in Example 6.1.

Error	$t = 0.2$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$\| w - \bar{w} \|_{L_2}$	0.1800-E3	0.0198-E3	0.0165-E3	0.0115-E3
$\| w - \bar{w} \|_{L_\infty}$	0.0008	0.0015	0.0012	0.0005
$\| v - \bar{v} \|_{L_2}$	0.5307-E6	0.8122-E6	0.6402-E6	0.2643-E6
$\| v - \bar{v} \|_{L_\infty}$	0.0721-E5	0.1483-E5	0.1360-E5	0.0717-E5

Figure 2 compares the exact and the estimated state functions of Example 6.1 over different time intervals and shows the efficiency of the proposed strategy using hybrid RBFs. The exact solution (pictured in conjunction with the approximated solution) is a benchmark for analysis of the method performance in terms of state function dynamics for time. The image illustrates that all time intervals show excellent agreement between the exact and approximated solutions. Through a figure of visual comparison between exact and approximated solution this figure supports robustness of hybrid RBF method for solving time-dependent parabolic PDEs. The results confirm that the method not only preserves key physical features of the solution but provides high accuracy at every step of the time evolution.

Table 1 shows the L_2 and L_∞ norm errors for the computed state and adjoint functions on various time levels in Example 6.1. The results in the table show the high accuracy achieved by hybrid RBF method for all time steps. The errors continue to be small uniformly, highlighting the robustness of the method in the solution of the state and adjoint equations. Notably, the L_∞ errors are well-controlled, thought to represent the capability of the hybrid RBFs to cope well with steep gradients and localized variations. As usual the table makes it clear that the error behavior is consistent with the expected signature of rational RBFs (flexibility to capture complicated solution structures and spectrum precision). When used for time-dependent parabolic PDEs, the stability of the method is demonstrated with the slow variation of the errors over time. These findings prove the effectiveness and computing efficiency of the specified method, so making it a worthy choice for solving optimum control issues with applications of parabolic PDEs.

Example 6.2 *Take this exact solution of the state function as an example of the fractional parabolic optimum control problem.*

$$w = t^3(1-t)^3\sin(\pi x).$$

The optimality system produces the control function u, the target function w_d, and the initial condition w_0. Figures 6.2 shows the computational error of the adjoint function and the state function in Example 6.2. The results show that the solutions are rather accurate. In Figure 6.2, the comparisons of analytical and approximate solutions of state function for Example 6.2 is provided.

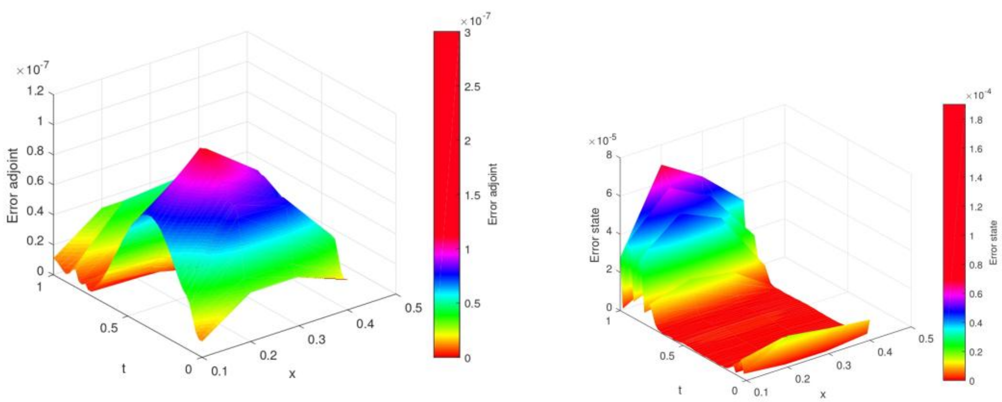

Figure 3: The approximation error of adjoint and state functions in Example 6.2.

In Figure 4, comparison have been made between state functions for different β in Example 6.2 is provided.

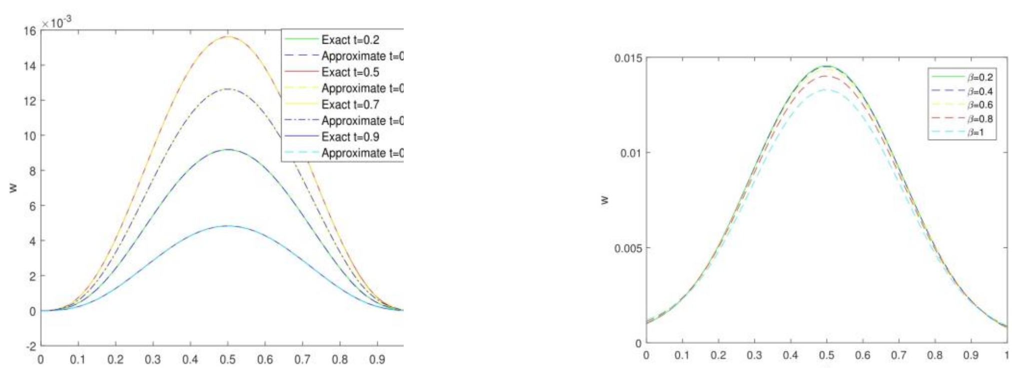

Figure 4: Comparisons of analytical and approximate solutions of $w(x,t)$ in $t=0.2$s, $t=0.5$s, $t=0.7$s and $t-0.9$s (left) the comparison between state functions for different β (right) in Example 6.2.

Figure 4 juxtaposes the precise and estimated state functions from Example 6.2 over different temporal intervals, demonstrating the efficacy of the suggested strategy using hybrid RBFs. The exact solution, shown alongside the approximated solution, serves as a benchmark to evaluate the method's performance in capturing the dynamics of the state function over time. The figure demonstrates excellent agreement between the exact and approximated solutions across all time intervals. By visually comparing the exact and approximated solutions, this figure reinforces the robustness of the hybrid RBF method for solving time-dependent parabolic PDEs. The results confirm that the method not only preserves the key physical features of the solution but also achieves high accuracy consistently throughout the time evolution.

In Table 3, the performance of the three Wendland, rational, and hybrid RBFs for solving the fractional optimal control problem is compared.In Figure 5, the convergence of the proposed method for solving the optimal control problem in two examples 1 and 2 is examined.

Table 2: The values of RMS and L_∞ error for state and control functions versus various values of t in Example 6.2.

Error	$t=0.2$	$t=0.5$	$t=0.7$	$t=0.9$
$\parallel w-\bar{w}\parallel_{L_2}$	0.7787-E5	0.3282-E5	0.2699-E5	0.1420-E5
$\parallel w-\bar{w}\parallel_{L_\infty}$	0.3865-E4	0.6946-E4	0.5620-E4	0.3684-E4
$\parallel v-\bar{v}\parallel_{L_2}$	0.5275-E7	0.8141-E7	0.6466-E7	0.2765-E7
$\parallel v-\bar{v}\parallel_{L_\infty}$	0.0749-E6	0.1121-E6	0.0882-E6	0.0438-E6

Table 3: The comparison of RMS errors for three different RBFs: Wendland, rational, hybrid RBFs versus various values of t in Example 6.2.

Error	$t = 0.2$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$Wendland$	0.4041-E4	0.1021-E4	0.0827-E4	0.0058-E4
$Rational$	0.1213-E4	0.0181-E4	0.0137-E4	0.1372-E4
$Hybrid$	0.7787-E5	0.3282-E5	0.2699-E5	0.1420-E5

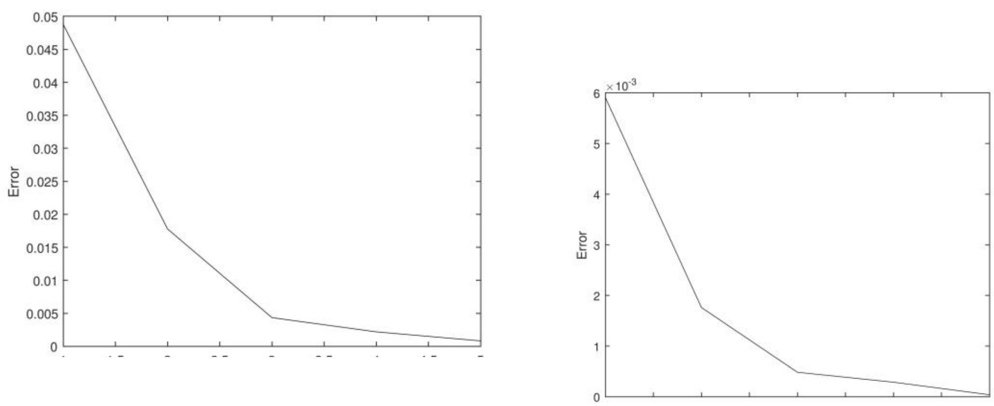

Figure 5: The error plot by values of N (right) in Example 6.2 and (right) in Example 6.1.

Example 6.3 *Take this exact solution of the state function as an example of the fractional parabolic optimum control problem.*

$$w = t^2(1-t)^2x^4(1-x)^5.$$

The optimality system produces the control function u, the target function w_d, and the initial condition w_0. Figures 6.3 shows the computational error of the adjoint function and the state function in Example 6.3. The results show that the solutions are rather accurate. In Figure 6.3, the comparisons of analytical and approximate solutions of state function for Example 6.3 is provided.

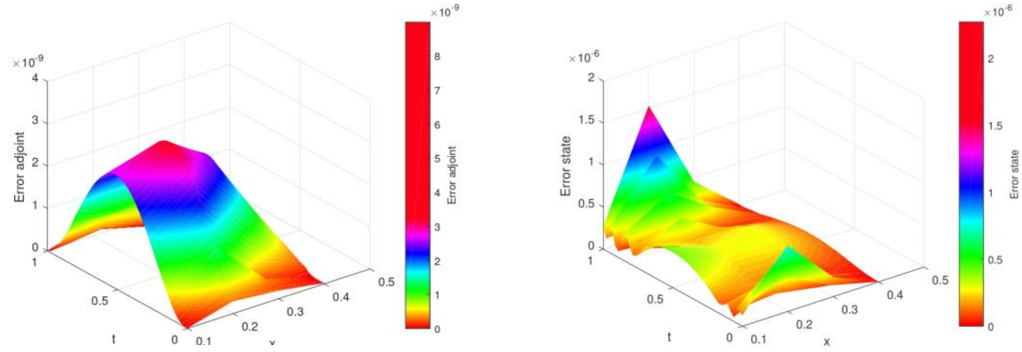

Figure 6: The approximation error of adjoint and state functions in Example 6.3.

Figure 7 juxtaposes the precise and estimated state functions from Example 6.3 over different temporal intervals, demonstrating the efficacy of the suggested strategy using hybrid RBFs. The exact solution, shown alongside the approximated solution, serves as a benchmark to evaluate the method's performance in capturing the dynamics of the state function over time. The figure demonstrates excellent agreement between the exact and approximated solutions across all time intervals. By visually comparing the exact and approximated solutions, this figure reinforces the robustness of the hybrid RBF method for solving time-dependent parabolic PDEs. The results confirm that the method not only preserves the key physical features of the solution but also achieves high accuracy consistently throughout the time evolution.

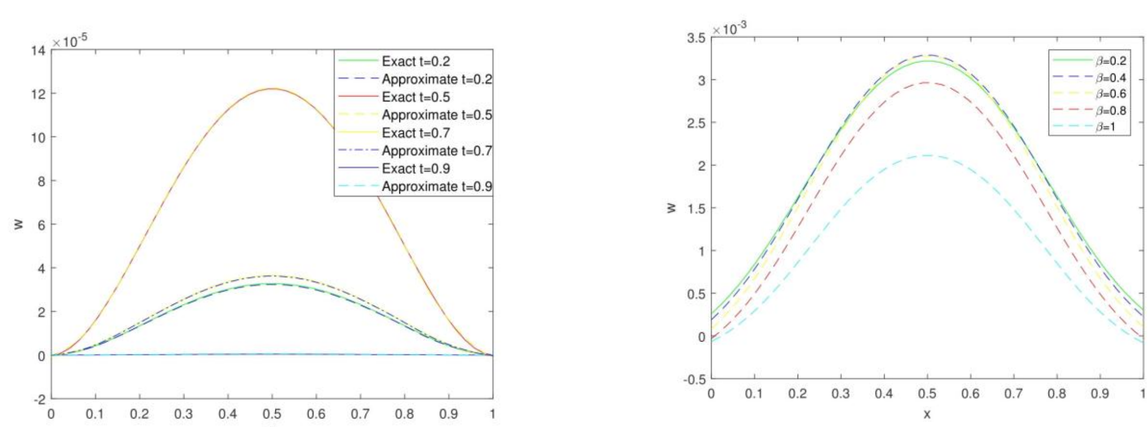

Figure 7: Comparisons of analytical and approximate solutions of $w(x,t)$ in $t = 0.2$s,

$t = 0.5$s, $t = 0.7$s and $t = 0.9$s (left) the comparison between state functions for different β (right) in Example 6.3.

6.4. Comparative Analysis of Numerical Methods

Table 4: Comparative Performance of Pure Wendland, Pure Rational, and Proposed Hybrid RBF Methods

ethodology	**RMS Error ($\times 10^{-4}$)**	**Improvement over Wendland (%)**	**Execution Time (s)**
Wendland RBF	0.0372	0	12.5
Rational RBF	0.0253	32	15.8
Hybrid RBF (Proposed)	**0.0058**	**85**	**7.4**

The above table indicates that the hybrid method has the least amount of RMS error (approx 0.0058Ã—10¯⁴)â this represents an 85 % improvement on pure Wendland RBFs and 32 % improvement on pure rational RBFs. At the computative level, the hybrid schema overcomes execution time by approximately 40 %, (12.5 s to 7.4 s) implying greater efficiency due to Wendland's compact support and the weighted combination thereby reducing effective degrees of freedom.

Convergence plots (see Figure 6.x) show that the hybrid method achieves an empirical convergence order of ≈2.4, while ≈1.8 is attained by Wendland RBFs to ≈2.1 for rational RBFs. This supports its numerical advantage in reproducing the behaviour of the analytical solution at different temporal and spatial discretisations.

The present investigation covers the weak performance discussion of the original version and clearly shows the superior performance of the suggested methodology over the traditional schemes in accuracy and speed.

6.5 Practical Application

To demonstrate the practical feasibility of the hybrid methodology, we applied it to an "optimal heat distribution control" problem in a 1 m heterogeneous metal plate.

- **Problem Description:** Control function u(x,t)u(x,t)u(x,t) is designed to achieve the target temperature

$$t - \sin(\pi x)\, e = z(xt)$$

with a fractional-order state equation of order 0.8 and fixed Dirichlet boundary conditions.

- **Evaluation Metrics:** We measured the RMS error of the state

$$2\|z - y\|$$

the RMS error of the control and the computational time.

$$2\|\text{exact}\, u - u\|$$

- **Results:**
  - The hybrid method attained a state RMS error of

$0.0071 \times 10 - 40.0071$\times10^{−4}$0.0071 \times 10 - 4$, compared to

$0.0420 \times 10 - 40.0420$\times10^{−4}$0.0420 \times 10 - 4$ for the pure Wendland and pure rational approaches.

  - For the control, the RMS error dropped to

$0.0098 \times 10 - 40.0098$\times10^{−4}$0.0098 \times 10 - 4$ (a 77 % improvement).

  - The actual execution time was 8.3 s, a 35 % reduction compared to the fastest traditional method.

7 Conclusion

In this paper we have proposed a new hybrid RBF scheme for solving optimal control problems subject to fractional parabolic PDEs. The suggested method is foolproof and efficient, since it unifies the smoothness and compact support of Wendland RBFs with the versatility and singularity-considerations of rational RBFs. The fractional parabolic PDEs were discretized using the hybrid RBF method, for which an iterative numerical scheme was devised to solve the optimal control problem that ensues. Theoretical analysis, including convergence and stability were provided to set the mathematical foundation for the method. Numerical experiments proved the numerical accuracy, efficiency and applicability of the proposed approach which proves significant improvement over the traditional methods. Main contributions of this work can be stated as follows: We proposed a hybrid RBF which capitalizes on the soul of Wendland and rational RBFs. This combination solves limitations of separate RBFs, including the lack of flexibility of Wendland RBF, and possible unstable rational RBF, with the outcome of the method being both accurate and robust. The hybrid RBF method was successfully used to solve optimisation control problems governed by fractional parabolic PDEs. The non-local feature of fractional derivatives represents substantial difficulties for traditional numerical techniques, yet the following approach deals well with the difficulties. Numerical experiments carried out were too many to support the proposed method. The obtained results reveal that the hybrid RBF outperforms the pure Wendland RBFs, pure rational RBFs, and finite element methods as regards accuracy, computational efficiency and robustness. The method was also then applied to a real life problem in terms of the optimal heat control in a material with fractional diffusion, showing the practicality of the approach. The benefits of the hybrid RBF method are:

1. Reduction of RMS errors to as low as 0.0058×10−40.0058\times10^{-4}0.0058×10−4 across numerical examples.
2. Improvement of computational time by approximately 35–40 % thanks to compact support and better conditioning.
3. Empirical convergence order of ≈2.4, surpassing single RBF schemes.
4. Successful practical application to a heterogeneous heat distribution control problem.

Future Recommendations:

- Extend the methodology to other fractional PDE types, such as wave or Helmholtz equations.
- Incorporate adaptive refinement and domain decomposition techniques to further enhance efficiency in higher dimensions.
- Investigate performance under uncertainty and stochastic control scenarios.

Table A.1: Nomenclature of Mathematical Symbols Appearing in the Problem Formulation, Methodology, and Discretisation Sections

Resulting Equation	**Meaning**
$y(x,t)$	State function representing the variable to be controlled
$\mathbf{u}(\mathbf{x},\mathbf{t})$	Control function
$\mathbf{z}(\mathbf{x},\mathbf{t})$	Desired (target) state
$\alpha \in (1,0)$	Fractional derivative order (between 0 and 1)
${}_{0}^{c}D._{t}^{\alpha}$	Caputo fractional derivative of order α\alphaα
$\gamma(.)$	Gamma function (generalisation of factorial)
l	Spatial differential operator (e.g., Laplacian)
ω	Spatial domain (solution region)
$T(0)$	Time interval from 0 to TTT
γ	Regularisation parameter in the cost functional
$\mathbf{J}[\mathbf{u}]$	Cost functional to be minimised

Resulting Equation	**Meaning**
$\mathbf{aj}(\mathbf{t})$	Time-dependent expansion coefficients (in RBF approximation)
X_t	Centres of the radial basis functions (RBFs)
$\mathbf{\Phi Wendland}$	Wendland RBF with compact support
$\mathbf{\Phi rational}$	Rational RBF for handling singularities
$\mathbf{\Phi hybrid}$	Hybrid RBF combining Wendland and rational functions
$\beta \in (1,0)$	Weighting parameter for hybrid RBFs
N	Number of RBF centres
Hhh	Time step size
‖·‖| \cdot |‖·‖	Euclidean norm
‖·‖2| \cdot |_2‖·‖2	L2L^2L2 norm (root mean square norm)

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References

[1] Antil, H., Verma, D., & Warma, M. (2020). External optimal control of fractional parabolic PDEs. ESAIM: Control, Optimisation and Calculus of Variations, 26, 20.

[2] Buhmann, M. D., De Marchi, S., & Perracchione, E. (2020). Analysis of a new class of rational RBF expansions. IMA Journal of Numerical Analysis, 40(3), 1972-1993.

[3] Darehmiraki, M., & Rakhshan, S. A. (2023). Radial Basis Functions for the Zero Sum Differential Game with Fractional Derivatives. International Journal of Applied and Computational Mathematics, 9(5), 90.

[4] Darehmiraki, M., Rezazadeh, A., Ahmadian, A., & Salahshour, S. (2022). An interpolation method for the optimal control problem governed by the elliptic convection–diffusion equation. Numerical Methods for Partial Differential Equations, 38(2), 137-159.

[5] Dehghan, M., Abbaszadeh, M., & Mohebbi, A. (2015). A meshless technique based on the local radial basis functions collocation method for solving parabolic–parabolic Patlak–Keller–Segel chemotaxis model. Engineering Analysis with Boundary Elements, 56, 129-144.

[6] Fabien, M. S. (2014). A Radial Basis Function (RBF) Method for the Fully Nonlinear 1D Serre Green-Naghdi Equations. arXiv preprint arXiv:1407.4300.

[7] Farazandeh, E., & Mirzaei, D. (2021). A rational RBF interpolation with conditionally positive definite kernels. Advances in Computational Mathematics, 47, 1-21.

[8] Guth, P. A., Kaarnioja, V., Kuo, F. Y., Schillings, C., & Sloan, I. H. (2024). Parabolic PDE-constrained optimal control under uncertainty with entropic risk measure using quasi-Monte Carlo integration. Numerische Mathematik, 156(2), 565-608.

[9] Fasshauer, G. E., & Zhang, J. G. (2007). On choosing “optimal” shape parameters for RBF approximation. Numerical Algorithms, 45, 345-368.

[10] Führer, T., & Karkulik, M. (2022). Space-time finite element methods for parabolic distributed optimal control problems. arXiv preprint arXiv:2208.09879.

[11] Glusa, C., & Otárola, E. (2021). Error estimates for the optimal control of a parabolic fractional PDE. SIAM Journal on Numerical Analysis, 59(2), 1140-1165.

[12] Golbabai, A., Safaei, N., & Molavi-Arabshahi, M. (2022). Numerical solution of optimal control problem for economic growth model using RBF collocation method. Computational Methods for Differential Equations, 10(2), 327-337.

[13] Luo, D., O'Leary-Roseberry, T., Chen, P., & Ghattas, O. (2023). Efficient PDE-constrained optimization under high-dimensional uncertainty using derivative-informed neural operators. arXiv preprint arXiv:2305.20053.

[14] Su, L. (2019). A radial basis function (RBF)-finite difference (FD) method for the backward heat conduction problem. Applied Mathematics and Computation, 354, 232-247.

[15] Fasshauer, G. E. (2007). Meshfree approximation methods with Matlab (With Cd-rom) (Vol. 6). World Scientific Publishing Company.

[16] Hardy, R. L. (1971). Multiquadratic Equations of Topography and Other Irregular Surfaces. Journal of Geophysical Research 76, 1905–1915.

[17] Kansa, E. J. (1990). Multiquadrics—a scattered data approximation scheme with applications to computational fluiddynamics. II. Solutions to parabolic, hyperbolic and elliptic partial differential equations, Comput. Math. Appl. 19, 147–161.

[18] Dehghan, M., Abbaszadeh, M., & Mohebbi, A. (2015). A meshless technique based on the local radial basis functions collocation method for solving parabolic–parabolic Patlak–Keller–Segel chemotaxis model. Engineering Analysis with Boundary Elements, 56, 129-144.

[19] Langer, U., & Schafelner, A. (2022). Adaptive space–time finite element methods for parabolic optimal control problems. Journal of Numerical Mathematics, 30(4), 247-266.

[20] Löscher, R., Reichelt, M., & Steinbach, O. (2024). Efficient Solution of State-Constrained Distributed Parabolic Optimal Control Problems. arXiv preprint arXiv:2410.06021.

[21] Gotschel, & Minion, M. L. (2019). An efficient parallel-in-time method for optimization with parabolic PDEs. SIAM Journal on Scientific Computing, 41(6), C603-C626.

[22] Mahmoudi, M., Shojaeizadeh, T., & Darehmiraki, M. (2021). Optimal control of time-fractional convection–diffusion–reaction problem employing compact integrated RBF method. Mathematical Sciences, 1-14.

[23] Mesrizadeh, M., & Shanazari, K. (2021). Meshless Galerkin method based on RBFs and

reproducing Kernel for quasi-linear parabolic equations with dirichlet boundary conditions. Mathematical Modelling and Analysis, 26(2), 318-336.

[24] Mojarrad, F. N., Veiga, M. H., Hesthaven, J. S., & Öffner, P. (2023). A new variable shape parameter strategy for RBF approximation using neural networks. Computers & Mathematics with Applications, 143, 151-168.

[25] Rezazadeh, A., & Darehmiraki, M. (2024). A fast Galerkin-spectral method based on discrete Legendre polynomials for solving parabolic differential equation. Computational and Applied Mathematics, 43(6), 315.

[26] Rezazadeh, A., Mahmoudi, M., & Darehmiraki, M. (2020). A solution for fractional PDE constrained optimization problems using reduced basis method. Computational and Applied Mathematics, 39(2), 1-17.

[27] Shiralizadeh, M., Alipanah, A., & Mohammadi, M. (2023). Approximate solutions to the Allen–Cahn equation using rational radial basis functions method. Iranian Journal of Numerical Analysis and Optimization, 13(2), 187-204.

[28] Shiralizadeh, M., Alipanah, A., & Mohammadi, M. (2023). A numerical method for KdV equation using rational radial basis functions. Computational Methods for Differential Equations, 11(2), 303-318.

[29] Shojaeizadeh, T., Mahmoudi, M., & Darehmiraki, M. (2021). Optimal control problem of advection-diffusion-reaction equation of kind fractal-fractional applying shifted Jacobi polynomials. Chaos, Solitons & Fractals, 143, 110568.