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On Average Modulus of Smoothness and Comonotony Approximation by Splines

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Abstract:

In this research, some of the properties of the modulus of smoothness and the average modulus of smoothness in weighted space $L_{w,q}(I)$, $0 < q < 1$, $I = [-d, d]$, d =positive integer were discussed. The best approximation of the monotonic function $f \in L_{w,q}(I) \cap \Delta^1(A_s)$ was also found, using the spline S_r , $r = 1 \dots n$, $n \in N$, in the weighted space.

1.Introduction

There is research (see[5],[6],[8])in which the approximation of the function$f$ in space $c[a,b], L_P[a,b]$,(see[3,4]) has been studied, using splines and algebraic polynomials, and numerical applications have been found for it. In this research, some properties of the modulus of smoothness were found $w_m^{\varphi}(f,x,\delta)_{w,q} = \sup_{0<h\leq\delta} \left\| \Delta^m_{\frac{\varphi h}{m}}(f,.) \right\|_{L_{w,q}(I)}$

$$\Delta^m_{\frac{\varphi h}{m}}(f,x)_w = \begin{cases} \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} f\left(x+\frac{i\varphi h}{m}\right) w\left(x+\frac{\varphi h}{m}\right) & , \ if \ \ x+\frac{\varphi h}{m} \ , x \in I \\ 0 & , \quad 0 \cdot W \end{cases}$$

$Q(x) = \sqrt{1-(\frac{x}{d})^2}$, represent m-th symmetric difference.

And finding the properties of average modulus of smoothness were found

$$\tau_m(f,\delta)_{w,q} = \left\| w_m^{\varphi}(f,.,\delta) \right\|_{L_{w,q}(I)}$$

in weighted space $L_{w,q}(I)$ where as

$$L_{w,q}(I) = \{f: f: I \to R: (\int_I |f(x)w(x)|^q \, dx)^{\frac{1}{q}} \le \infty \ \ ,0 < q < 1\}.$$

the function f monotony in weighted space $f \in L_{w,q}(I) \cap \Delta^1(A_s)$

where $A_s = \{a_2, \dots, a_s, -d = a_1 < a_2 < \cdots < a_s < a_{s+1} = d\}$.

Let $\ell_{r,s} \subset I$, where $\ell_{r,s} = [c_{r,s}, c_{r-1,s}]$, $r = 2 \dots n+1$, $s = 1 \dots n$

and
$c_s = \{c_2 \dots c_n , Y_{i+1} = c_1 < c_2 < \cdots < c_n < c_{n+1} = Y_i\}$, $i = 0 \dots n$

The interior points (knods) at the interval $\ell_{r,s}$ can be defined as follows

$c_{2,s} < \tilde{c}_{2,s} < \cdots < c_{n-1,s} < \tilde{c}_{n-1,s} < c_{n,s}$.

Where $\{\tilde{c}_s\}_{r=2}^{n-2}$ represents a sequence of knods in an interval $\tilde{\ell}_{r,s} = [\tilde{c}_{r,s}, \tilde{c}_{r-1,s}]$ $r = 2, \dots, n+1$, $s = 1, \dots, n$.

Let S_r represent the spline which is define in form

$$S_r(x) = \begin{cases} s_0(x) , & x_0 < x \le x_1 \\ s_1(x) , & x_1 < x \le x_2 \\ \vdots & \\ s_n(x) , & x_{n-1} < x \le x_n \end{cases}$$

In this research we will find the best approximation of the function $f \in L_{w,q}(I) \cap \Delta^1(A_s)$ using the spline $s_r \in S_{m-1}(I) \cap \Delta^1(A_s)$, which is comonotony with the function f.

2.Lemma. In the waited space $L_{w,q}(I)$, the modulus of smoothness $w_m^{\varphi}(f,x,\delta)_{w,q}$ has the following properties , if $f \in L_{w,q}(I) \cap \Delta^1(A_s)$,$0 < \delta_1 \le \delta_2$, $\lambda > 0$, then:

1) $w_m^{\varphi}(f,\delta)_w$, is non negative function with

$$\lim_{\delta \to 0} w_m^{\varphi}(f,\delta)_w = 0$$

Proof: Since $w_m^{\varphi}(f,\delta)_w = \underset{|h|\le\delta}{\text{Sup}} \left\|\Delta^m_{\frac{h\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)}$

$$\lim_{\delta\to 0} w_m^{\varphi}(f,\delta)_w = \lim_{\delta\to 0} \underset{|h|\le\delta}{\text{Sup}} \left\|\Delta^m_{\frac{h\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)}$$

$$= \underset{|h|\to 0}{\text{Sup}} \left\|\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} f(x+\frac{h\varphi}{m}) w(x+\frac{h\varphi}{m})\right\|$$

$$= \underset{|h|\to 0}{\text{Sup}} \left\|\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} f(x) w(x)\right\|$$

Hence

$$\lim_{\delta\to 0} w_m^{\varphi}(f,\delta)_w = 0$$

2) $w_m^{\varphi}(f,\delta_1)_{w,q} \le w_m^{\varphi}(f,\delta_2)_{w,q}$ for $\delta_1 \le \delta_2$

Proof: we have

$$w_m^{\varphi}(f,\delta_1)_{w,q} = \underset{|h|\le\delta_1}{\text{Sup}} \left\|\Delta^m_{\frac{h\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)}$$

$$\le \underset{|h|\le\delta_2}{\text{Sup}} \left\|\Delta^m_{\frac{h\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)}$$

$$= w_m^{\varphi}(f,\delta_2)_w$$

Hence

$$w_m^{\varphi}(f,\delta_1)_{w,q} \le w_m^{\varphi}(f,\delta_2)_{w,q} \qquad \text{for } \delta_1 \le \delta_2 .$$

3) $w_m^{\varphi}(f,\lambda\delta)_{w,q} \le \lambda^n w_m^{\varphi}(f,\delta)_{w,q}$

Proof: We have

$$w_m^{\varphi}(f,\lambda\delta)_{w,q} = \underset{h\le\lambda\delta}{\text{Sup}} \left\|\Delta^m_{\frac{h\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)}$$

$$\le \underset{h\le\lambda\delta}{\text{Sup}} \left\|\Delta^m_{\frac{\lambda\delta\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)}$$

$$= \text{Sup} \left\|(\frac{\lambda\delta\,\varphi}{m})^m D^m f\right\|_{L_{w,q}(I)}$$

$$= |\lambda^m| \text{Sup} \left\| (\frac{\delta\varphi}{m})^m D^n f \right\|_{L_{w,q}(I)}$$

$$= |\lambda^m| \underset{h \le \delta}{\text{Sup}} \left\| \Delta^m_{\frac{\delta\varphi}{m}}(f,.) \right\|_{L_{w,q}(I)}$$

$$= |\lambda|^m w^{\varphi}_m(f,\delta)_{w,q} \quad ,\lambda > 0$$

Hence

$$w^{\varphi}_m(f,\lambda\delta)_{w,q} \le \lambda^m w^{\varphi}_m(f,\delta)_{w,q}$$

3.Lemma. Let $f \in L_{w,q}(I) \cap \Delta^1(A_s)$, the average modulus of smoothness $\tau_m(f,\delta_1)_{w,q}$, in the waited space $L_{w,q}(I)$, where w is a waited bounded function , has the following properties

1) $\tau_m(f,\delta_1)_{w,q} \le \tau_m(f,\delta_2)_{w,q} \quad , \quad \delta_1 \le \delta_2$

Proof: Let $\delta = \min_{0 \le i \le n} \{\delta_i\}$ and i = 1,2 , $\delta \le \delta_1$, $\delta \le \delta_2$ and $\delta_1 \le \delta_2$, then

$$\underset{h \le \delta_1}{\text{Sup}} \left\| \Delta^m_{\frac{\varphi h}{m}}(f,x) \right\|_{L_{w,q}(I)} \le \underset{h \le \delta_2}{\text{Sup}} \left\| \Delta^m_{\frac{\varphi h}{m}}(f,x) \right\|_{L_{w,q}(I)}$$

Then we get by definition of the usually modulus of smoothness that

$$w^{\varphi}_m(f,x,\delta_1)_{w,q} \le w^{\varphi}_m(f,x,\delta_2)_{w,q}$$

By take $L_{w,q}(I)$_normd of both sides, we get

$$\left\| w^{\varphi}_m(f,.,\delta_1) \right\|_{L_{w,q}(I)} \le \left\| w^{\varphi}_m(f,.,\delta_2) \right\|_{L_{w,q}(I)}$$

Then by definition of $\tau_m(f,\delta)_{w,q}$ we get

$$\tau_m(f,\delta_1)_{w,q} \le \tau_m(f,\delta_2)_{w,q} \quad , \quad \delta_1 \le \delta_2$$

2) $\tau_m(f_1 + f_2,\delta)_{w,q} \le \tau_m(f_1,\delta)_{w,q} + \tau_m(f_2,\delta)_{w,q}$

Proof: We have by m-th property of symmetric difference

$\Delta^m_{\frac{\varphi h}{m}}(f_1+f_2,x)_{w,q} = \Delta^m_{\frac{\varphi h}{m}}(f_1,x)_{w,q} + \Delta^m_{\frac{\varphi h}{m}}(f_2,x)_{w,q}$, then

$$\left|\Delta^m_{\frac{\varphi h}{m}}(f_1+f_2,x)_{w,q}\right|^q \le \left|\Delta^m_{\frac{\varphi h}{m}}(f_1,x)_{w,q}\right|^q + \left|\Delta^m_{\frac{\varphi h}{m}}(f_2,x)_{w,q}\right|^q$$

By take the integration for $I \subset R$, then

$$\left(\int_I \left|\Delta^m_{\frac{\varphi h}{m}}(f_1+f_2,x)_{w,q}\right|^q dx\right)^{\frac{1}{q}}$$

$$\le \left(\int_I \left|\Delta^m_{\frac{\varphi h}{m}}(f_1,x)_{w,q}\right|^q dx\right)^{\frac{1}{q}} + \left(\int_I \left|\Delta^m_{\frac{\varphi h}{m}}(f_2,x)_{w,q}\right|^q dx\right)^{\frac{1}{q}}$$

$$\sup_{0\le h\le\delta}\left\|\Delta^m_{\frac{\varphi h}{m}}(f_1+f_2,.)\right\|_{L_{w,q}(I)}$$

$$\le \sup_{0\le h\le\delta}\left\|\Delta^m_{\frac{\varphi h}{m}}(f_1,.)\right\|_{L_{w,q}(I)} + \sup_{0\le h\le\delta}\left\|\Delta^m_{\frac{\varphi h}{m}}(f_2,.)\right\|_{L_{w,q}(I)}$$

Then we get by definition of the usually modulus of smoothness that

$$w^{\varphi}_m(f_1+f_2,x,\delta)_{w,q} \le w^{\varphi}_m(f_1,x,\delta)_{w,q} + w^{\varphi}_m(f_2,x,\delta)_{w,q}$$

Now by take the $L_{w,q}(I)$_normd of both sides then we get

$$\left\|w^{\varphi}_m(f_1+f_2,.,\delta)\right\|_{L_{w,q}(I)} \le \left\|w^{\varphi}_m(f_1,.,\delta)\right\|_{L_{w,q}(I)} + \left\|w^{\varphi}_m(f_2,.,\delta)\right\|_{L_{w,q}(I)}$$

By definition of $\tau_m(f,\delta)_{w,q}$, we get

$$\tau_m(f_1+f_2,\delta)_{w,q} \le \tau_m(f_1,\delta)_{w,q} + \tau_m(f_2,\delta)_{w,q}$$

3) $\tau_m(f,\delta)_{w,q} \le \delta^m \tau_{m-1}(f',\delta)_{w,q}$

Proof: We have

$$w^{\varphi}_m(f,\delta)_{w,q} = \delta w^{\varphi}_{m-1}(f',\delta)_{w,q}$$

$$w^{\varphi}_m(f,\delta)_{w,q} = \sup_{0\le h\le\delta}\left\{\left|\Delta^m_{\frac{h\varphi}{m}}(f,x)_w\right| \quad , \; x \in I\right\}$$

$$\Delta^{m}_{\frac{h\varphi}{m}}(f,x)_w = \Delta^{m-1}_{\frac{h\varphi}{m-1}}\left(\Delta^{1}_{\frac{h\varphi}{1}}(f,x)_w\right)_w = \Delta^{m-1}_{\frac{h\varphi}{m-1}}\left(\Delta_{h\varphi}(f,x)_w\right)_w$$

$$= \Delta^{m-1}_{\frac{h\varphi}{m-1}}\left(\sum_{i=0}^{1}(-1)^{1-i}\binom{1}{i}f\left(x+\frac{ih\varphi}{1}\right)w(x+\frac{h\varphi}{1})\right),x+h\varphi\in I$$

$$= \Delta^{m-1}_{\frac{h\varphi}{m-1}}(-f(x)w(x+h\varphi)+f(x+h\varphi)w(x+h\varphi))_w$$

$$= \Delta^{m-1}_{\frac{h\varphi}{m-1}}([f(x+h\varphi)-f(x)]w(x+h\varphi))_w$$

$$= \Delta^{m-1}_{\frac{h\varphi}{m-1}}\left(\int_0^h(f',x+t)_w dt\right)_w$$

$$sup_{0<h\leq\delta}\left|\Delta^{m}_{\frac{h\varphi}{m}}(f,x)_w\right| = sup_{0<h\leq\delta}\left|\Delta^{m-1}_{\frac{h\varphi}{m-1}}\left(\int_0^h(f',x+t)_w dt\right)_w\right|$$

$$\leq \sup_{0\leq h\leq\delta}\left\{\int_0^h\left|\Delta^{m-1}_{\frac{h\varphi}{m-1}}(\acute{f},x+t)_w\right|dt\right\}$$

Since from definition of modulus of smoothness then

$$w^{\varphi}_m(f,\delta)_{w,q} \leq \sup_{0\leq h\leq\delta}\left\{\int_0^h\left|\Delta^{m-1}_{\frac{h\varphi}{m-1}}(\acute{f},x)_w\right|dt\right\}$$

$$w^{\varphi}_m(f,\delta)_{w,q} \leq \int_0^h w^{\varphi}_{m-1}(f',\frac{m}{m-1}\delta)_{w,q}$$

$$= hw^{\varphi}_{m-1}(f',\frac{m}{m-1}\delta)_{w,q} \quad ,h\leq\delta$$

$$\leq \delta w^{\varphi}_{m-1}(f',\frac{m}{m-1}\delta)_{w,q}$$

$$w^{\varphi}_m(f,\delta)_{w,q} \leq \delta w^{\varphi}_{m-1}(f',\delta)_{w,q}$$

Since $w^{\varphi}_m(f,\delta)_{w,q} = \left\|w^{\varphi}_m(f,x,\delta)_{w,q}\right\|_{L_{w,q}(I)}$, and by definition of average modulus of smoothness then we get

$$\tau_m(f,\delta)_{w,q} \leq \delta^m\tau_{m-1}(f',\delta)_{w,q}$$

4) $\tau_m(f,k\delta)_{w,q} \le (2k)^m(2k-1)\tau_m(f,\delta)_{w,q} \quad , \quad k \in N$

Proof: since

$$\Delta^m_{\frac{kh\varphi}{m}}(f,x) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} f(x+\frac{ikh\varphi}{m})w(x+\frac{kh\varphi}{m})$$

$$=\left[(-1)^m\binom{m}{0}f(x)+(-1)^{m-1}\binom{m}{1}f\left(x+\frac{kh\varphi}{m}\right)+(-1)^{m-2}\binom{m}{2}f\left(x+\frac{2kh\varphi}{m}\right)+\cdots+(-1)^{m-m}\binom{m}{m}f(x+\frac{mkh\varphi}{m})\right]w(x+\frac{kh\varphi}{m})$$

$$\left\|\Delta^m_{\frac{kh\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)} \le \left(\left|\binom{m}{0}\right| \|f\|_{L_{w,q}(I)} + \left|\binom{m}{1}\right| \left\|f(.+\frac{kh\varphi}{m}\right\|_{L_{w,q}(I)} + \cdots + \left|\binom{m}{m}\right| \|f(.+kh\varphi\|_{L_{w,q}(I)}\right)$$

$$= \sum_{i=0}^{(2k-1)m} \left|\binom{m}{i}\right| \sum_{j=1}^{2k-1} \left\|f(x+\frac{(k-j)mh\varphi}{2}\right\|_{L_{w,q}(I)}$$

Let $m = 1$

$$\Delta^1_{\frac{kh\varphi}{1}}(f,x)_w = \sum_{i=0}^{1}(-1)^{1-i}\binom{1}{i} f(x+ikh\varphi)w(x+h\varphi)$$

$$= [f(x+kh\varphi)-f(x)]w(x+h\varphi)$$

$$= \sum_{i=0}^{k-1} \Delta^1_{\frac{kh\varphi}{1}}(x+ih\varphi)$$

Hence for m-th symmetric difference

$$\Delta^m_{\frac{kh\varphi}{m}}(f,x)_w = \sum_{i_1=0}^{k-1}\sum_{i_2=0}^{k-1}\cdots\sum_{i_n=0}^{k-1} \Delta^m_{\frac{kh\varphi}{m}}(f,x+i_1h\varphi+\cdots+i_nh\varphi)_w$$

Therefor

$$\left\|\Delta^m_{\frac{kh\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)} \leq \sum_{i=0}^{2(k-1)m} \left|\binom{m}{i}\right| \sum_{j=1}^{2k-1} \left\|f(x-\frac{(k-j)mh\varphi}{2}\right\|_{L_{w,q}(I)}$$

$$\left\|\Delta^m_{\frac{kh\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)} \leq (2k)^m \sum_{j=1}^{2k-1} \left\|f(x-\frac{(k-j)mh\varphi}{2}\right\|_{L_{w,q}(I)}$$

$$\operatorname*{Sup}_{h\leq k\delta} \left\|\Delta^m_{\frac{kh\varphi}{m}}(f,.)\right\|_{L_{w,q}(I)} \leq (2k)^m \sum_{j=1}^{2k-1} \operatorname*{Sup}_{h\leq\delta} \left\|f(x-\frac{(k-j)mh\varphi}{2}\right\|_{L_{w,q}(I)}$$

$$w^{\varphi}_m(f,x,k\delta)_{w,q} \leq (2k)^m \sum_{j=1}^{2k-1} w^{\varphi}_m\ (f,x-\frac{(k-j)mh\varphi}{2},\delta)_{w,q}$$

By take $L_{w,q}(I)$- normed of both sides

$$\|w^{\varphi}_m(f,.,k\delta)_{w,q}\|_{L_{w,q}(I)}$$

$$\leq (2k)^m \sum_{j=1}^{2k-1} \left\|w^{\varphi}_m(f,x-\frac{(k-j)mh\varphi}{2},\delta)\ \right\|_{L_{w,q}(I)}$$

$$= (2k)^m \sum_{j=1}^{2k-1} \left(\int_{-d}^{d} \left|w^{\varphi}_m(f,x-\frac{(k-j)mh\varphi}{2},\delta)\right|^q dx\right)^{\frac{1}{q}}$$

$$= (2k)^m \sum_{j=1}^{2k-1} \left(\int_{-d-\frac{(k-j)mh\varphi}{2}}^{d-\frac{(k-j)mh\varphi}{2}} |w^{\varphi}_m(f,x,\delta)|^q dx\right)^{\frac{1}{q}}$$

$$\leq (2k)^m(2k-1)\left(\int_{-d}^{d} |w^{\varphi}_m(f,x,\delta)|^q dx\right)^{\frac{1}{q}}$$

$$= (2k)^m(2k-1)\ \|w^{\varphi}_m(f,x,\delta)\ \|_{L_{w,q}(I)}$$

By definition of $\tau_m(f,\delta)_{w,q}$ we get

$$\tau_m(f,k\delta)_{w,q} \leq (2k)^m(2k-1)\tau_m(f,\delta)_{w,q}$$

4.Theorem. Let $f \in L_{w,q}(I) \cap \Delta^1(A_s)$ and

$B_r = \{b_1, \dots, b_n: -d = b_1 < b_2 < \cdots < b_{n-1} < b_n = d\}$

be a given knods sequence, such that there are at least two b_i in each (Y_i, Y_{i+1})

$\forall\, i = 0 \dots n$,then there exists a spline $S_r \in S_{m-1}(I) \cap \Delta^1(A_s), 0 < r \leq m-1$ on knods sequence B_r , such that

$$\|f - S_r\|_{L_{w,q}} \leq A w_m^{\varphi}(f,(n+1)^{-1})_{w,q}$$

$A = A(m,d,q,a), m \in N, d \in Z^+, 0 < q < 1, a > 1.$

Proof: Let $I_i = [\, b_i, b_{i+1}] \subset \zeta_i = [\, b_{i-1}, b_{i+2}]$ $, i = 1 \dots n$

the interval I_i is called contaminated (see[2]) if $b_i \leq a_i \leq b_{i+1}$,for some $a_i \in A_s$. we assume that the interval I_i is uncontaminated and lies between the intervals I_j and I_{j+1} , such that

$I_j = \left[\, b_j, b_{j+1}\right] \subset \zeta_i = \left[\, b_{j-1}, b_{j+2}\right], j = 1 \dots s$

Let g_i best a constant approximation of the function f on the interval I_i $, i = 0 \dots n$ in weighted space $L_{w,q}(I_i)$.

Let $\delta = \min\limits_{0\leq i\leq n} |Y_{i+1} - Y_i|$, since $\delta \leq |Y_{i+1} - Y_i|$,and δ it is longer than the interval I_i , and there for the length of the interval I_i of the function f converges to zero faster than the interval $|Y_{i+1} - Y_i|$ $, i = 0 \dots n$.But if the Points Y_i inside the interval I_i ,it is possible that the function f is not the best approximation, so by Whitney's inequality(see[1]) we get :

$$\|f - g_i\|_{L_{w,q}(I_i)} \leq c(m) w_m^{\varphi}(f, |I_i|, I_i)_{w,q} \qquad \dots (4.1$$

If none of the knods b_i $, i = 0 \dots n$ inside the interval I_i, then g_i is comonotony with the function f in an interval I_i.

We will know

$$S_r = \begin{cases} 0 & ; x \in [\, b_j, b_{j+1}), 1 \le j \le s \\ g_i & ; x \in [\, b_i, b_{i+1}), 0 \le i \le n \ , i \ne j \end{cases}$$

It is clear that S_r, the best constant approximation of the function f inaninterval I_i . Suppose that g_{j-1} and g_{j+1} constant polynomials of opposite sign with respect to the function f that is

$g_{j-1} \le 0$ and $g_{j+1} \ge 0$, hence by[7]

$g_{j+1} - g_{j-1} \ge -g_{j-1}$

$|g_{j-1}| \le |g_{j+1} - g_{j-1}|$,

When $I_j \subset \zeta_j$, then we get

$$|g_{j-1}| \le |g_{j+1} - g_{j-1}| \le |\zeta_i|^{-\frac{1}{q}} \|g_{j+1} - g_{j-1}\|_{L_{w,q}(\zeta_j)}$$

$$\le c|\zeta_j|^{-\frac{1}{q}} \left(\|f - g_{j+1}\|_{L_{w,q}(\zeta_j)} + \|f - g_{j-1}\|_{L_{w,q}(\zeta_j)}\right)$$

By (4.1) we get at the interval ζ_j

$$|g_{j-1}| \le c|\zeta_j|^{-\frac{1}{q}} w_m^{\varphi}(f, |\zeta_j|, \zeta_j)_{w,q} \qquad \dots (4.2$$

Now ,when $x \in [\, b_j, b_{j+1})$, we get

$$\|f - S_r\|_{L_{w,q}(I_j)} = \|f\|_{L_{w,q}(I_j)}$$

By (Lemma (2.1) [7]), we have

$$\|f - S_r\|_{L_{w,q}(I_j)} \le c\|f\|_{L_{w,q}(I_j)}$$

$$\le c[\|f - g_{j-1}\|_{L_{w,q}(I_j)} + \|g_{j-1}\|_{L_{w,q}(I_j)}]$$

We have

$$\|g_{j-1}\|_{L_{w,q}(I_j)} = \left(\int_{I_j} |g_{j-1} w(x)|^q dx\right)^{\frac{1}{q}}$$

$$= |g_{j-1}| \left(\int |w(x)|^q dx\right)^{\frac{1}{q}}$$

$$=|g_{j-1}|\|w\|_{L_{w,q}(I_j)}$$

Since w is a bounded function, then there is $k \geq 1$, such that

$$\|g_{j-1}\|_{L_{w,q}(I_j)} \leq k|g_{j-1}|$$

By (4.2) , we get

$$\|g_{j-1}\|_{L_{w,q}(I_j)} \leq k|I_j|^{-\frac{1}{q}} w_m^{\varphi}(f,|I_j|,I_j)_{w,q} \qquad \dots (4.3$$

Now ,by (4.1) and (4.3) , we get

$$\|f-S_r\|_{L_{w,q}(I_j)} \leq c|I_j|^{-\frac{1}{q}} w_m^{\varphi}(f,|I_j|,I_j)_{w,q}+k|I_j|^{-\frac{1}{q}} w_m^{\varphi}(f,|I_j|,I_j)_{w,q}$$

$$=A|I_j|^{-\frac{1}{q}} w_m^{\varphi}(f,|I_j|,I_j)_{w,q} \ , \quad c+k=A$$

Since $I_j \subset \zeta_j$ and $|I_j| \leq |\zeta_j|$, then

$$\|f-S_r\|_{L_{w,q}(I_j)} \leq A|\zeta_j|^{-\frac{1}{q}} w_m^{\varphi}(f,|\zeta_j|,\zeta_j)_{w,q} \qquad \dots (4.4$$

Now ,for $x \in [\, b_i, b_{i+1})$

$\|f-S_r\|_{L_{w,q}(I_i)} = \|f-g_i\|_{L_{w,q}(I_i)}$, By (4.1) we get

$$\|f-S_r\|_{L_{w,q}(I_i)} \leq c w_m^{\varphi}(f,|I_i|,I_i)_{w,q}$$

And by (4.4) we get

$$\|f-S_r\|_{L_{w,q}(I_i)} \leq c\, |\zeta_i|^{-\frac{1}{q}} w_m^{\varphi}(f,|\zeta_i|,\zeta_i)_{w,q}$$

Since

$$\|f-S_r\|_{L_{w,q}(I_i)} \leq c\sum_{i=1}^{n}|\zeta_i|^{-\frac{1}{q}} w_m^{\varphi}(f,|\zeta_i|,\zeta)_{w,q}$$

$$\leq |I|^{-\frac{1}{q}} w_m^{\varphi}(f,\delta)_{w,q}$$

We have

$\frac{c}{\delta} = N \leq n \ \ then \frac{c}{n} < \delta$, that is $\frac{1}{n} < \frac{\delta}{c} \leq \delta$, $c > 0$,since $n < n + 1$ then $\frac{1}{n+1} < \frac{1}{n}$, hence $\frac{1}{n+1} < \frac{1}{n} < \delta$ and $\delta < \frac{a}{n+1}$, $a > 1$

By the property of modulus of smoothness (1), then

$$\|f - S_r\|_{L_{w,q}(I)} \leq c(2d)^{-\frac{1}{q}} w_m^{\varphi}(f, \frac{a}{n+1})_{w,q}$$

$$= A w_m^{\varphi}(f, (n+1)^{-1})_{w,q}$$

$A = A(m, d, q, a), m \in N , d \in Z^+ , 0 < q < 1 , a > 1$.

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