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Using Krall-Laguerre polynomials in numerical solution of integral equations

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Abstract

In this article, using Krall-Laguerre polynomials in compression we develop the class of methods for the numerical solution of Volterra integral equations (VIE) of the third type. This method leads to a linear system of equations that are easily solvable. Two numerical examples were presented to verify the method, applicability and accuracy of the method. Numerical results are provided to illustrate the proposed method and to compared with the methods in [14].

Key words:, Krall-Laguerre polynomials; numerical method; orthogonal functions; Integral equations.

1. Introduction

Different numerical methods are used to solve ordinary differential equations or differential equations with partial derivatives, or integral equations, linear or non-linear, time-dependent or non-time-dependent. Among these methods, finite differences, finite elements and spectral methods can be mentioned. Spectral methods are of particular importance due to their high accuracy and fast convergence. Spectral methods are divided into three main groups: Galerkin, Tau and collocation, each of which has special capabilities.

There are two main methods for solving integral equations numerically, which are considered as direct and iterative methods, respectively. In the direct methods of solving the problem, it leads to solving the system of linear or non-linear equations, but iterative methods do not need to solve the system. Of course, each of these methods has its own advantages and disadvantages. Integral equations have many applications in other sciences. For example, in [1] Fredholm's integral equations are used in plasma physics calculations. Many studies have been done on the numerical solution of these equations, and many types of numerical methods have been developed to quickly and accurately obtain the approximation of y(x). Literature reviews and references of many existing methods are available in [2]. Collocation methods [3], sinc methods [4-6], general spectral methods [7, 8], convolution equation methods [9], Runge-Kutta methods [10-12] and Galerkin methods [14, 15] are several of the many approaches that have already been considered. It also presents a new numerical method for solving integral Volterra functional equations with variable bounds and mixed delays [16].

Many numerical methods have been introduced to solve IE numerically, but each method has its own shortcomings. Most of the numerical methods convert the model equations containing IE into discrete model equations, which appear in the form of a set of linear or nonlinear algebraic equations. In the case of direct solvers for solving this set of algebraic equations, the computational cost of a large system worsens due to the large computational time and memory requirements required for mathematical operations. Therefore, it is still a challenging task to introduce a simple and efficient technique for IEs. In this method, we focus on improving the efficiency of direct solvers by using orthogonal base functions method. Direct solvers are useful for solving problems that iterative solvers struggle with.

The present study proposes a new method for the numerical solution of third type linear integral equations. It should be noted that the given method is based on the approximation of unknown functions using the Krall-Laguerre polynomial.

This article is organized as follows. In section 2, we explain preliminaries such as integral equations of the third type, Krall orthogonal polynomials, Laguerre polynomials and Krall- Laguerre polynomials. Section 3 is devoted to numerical solution of integral equation of the third type. In Section 4 we achieve numerical examples to show the accuracy of the method and the culmination of paper in section 5 is the conclusion.

2. Preliminaries

The aim of this section is to interpret notations and definition of Integral equations of the third type and Krall-Laguerre polynomials that have expressed entirely in [17].

2.1 Integral equations of the third type

In recent years, researchers have conducted extensive scientific studies on the integral. Equations that can significantly help to model and analyze a wide range of problems in mechanics, engineering, chemistry, physics, biology, astronomy, potential theor are called integral equations of the third type.

For example, the integral equation

$$x^{\beta} f(x) = g(x) + \int_0^x (x-t)^{-\alpha} k(x,t) f(t)\,dt, \quad x \in [0,T] \tag{1}$$

where $\alpha \in [0,1)$, $\beta \in \square$, $\beta > 0$, $\alpha + \beta > 0$, and g(x) is a continuous function on the interval *I*. Also, *k(x,t)* is continuous on the set $\Delta = \{(x,t):\ 0 \le t \le x \le T\}$ and in the form $k(x,t) = x^{\alpha+\beta} k_1(x,t)$ and also $k_1 \in c(\Delta)$.

This class of equations, as stated in (4-1), is found correspondingly in the concepts of single integral equations with boundary value problems for complex partial differential equations.

2.2 Krall orthogonal polynomials

Orthogonal Krall polynomials are known as subsets of polynomials with a linear function u obtained from quasi-definite functions (see [19, 20]), thus $u : H \to \square$ representing a complex polynomial space H with complex coefficients.

In the following relation, the Dirac delta function is added and $\tilde{u}$ refers to the linear function.

$$\tilde{u} = u + \sum_{p=1}^{N} A_p\, \delta(x_p)$$

where $A_p \in \square$, $x_1, x_2, \ldots, x_p \in \square$ and $\delta(x_p)$ is the Dirac delta function at the point x_p .

2.3 Laguerre polynomials

A complete orthogonal sequence in $L^2(-\infty, b]$ and $L^2[a, +\infty)$ space can be obtained from the sequence in $L^2[0, +\infty)$ space. In fact, it will be done by changing variables $t = b - s$ and $t = s + a$.

By applying the Gram-Schmidt process to the sequence defined by:

$$e^{-\frac{t}{2}}, \quad t e^{-\frac{t}{2}}, \quad t^2 e^{-\frac{t}{2}}, \quad \ldots.$$

We find an orthogonal sequence of normal $\{e_n\}$.

It can be shown that sequence $\{e_n\}$ is as follows:

$$e_n = e^{-t/2} L_n(t) \ , \quad n = 0,1,2,\ldots$$

that Laguerre polynomials of order n are defined as follows:

$$L_0(t) = 1 \ ,$$

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n}\left(t^n e^{-t}\right) \ , \quad n = 1,2,\ldots$$

in other words:

$$L_n(t) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n}{j} t^j \ ,$$

The first few terms of the Laguerre polynomial are:

$$L_0(t) = 1$$
$$L_1(t) = 1 - t$$
$$L_2(t) = 1 - 2t + \frac{1}{2}t^2$$
$$L_3(t) = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3$$
$$L_4(t) = 1 - 4t + 3t^2 - \frac{2}{3}t^3 + \frac{1}{24}t^4$$

In fact, Laguerre polynomials $L_n(t)$ are the solutions of Laguerre second order differential equation:

$$t\, L_n'' + (1 - t) L_n' + n\, L_n = 0 \ ,$$

2.4 Krall- Laguerre polynomials

Krall - Laguerre polynomials $K_m(x)$ of degree m in the article [17] are defined as follows:

$$K_m(x) = \sum_{i=0}^{m} \frac{(-1)^i}{(i+1)!} \binom{m}{i} \left[i\,(\alpha + m + 1) + \alpha \right] x^i$$

A family of $\{K_m(x)\}_{m=0}^{\infty}$ polynomials is also orthogonal to the measure of ω .

where

$$d\omega = w(x)\,dx$$

Therefore, the weight function is:

$$w(x)=\frac{1}{\alpha}\delta(x)+e^{-x}H(x)$$

Where H(x) is the heavy side step function and measure $\boldsymbol{w}$ refers to the weight of the Laguerre e^{-x} on the interval $[0,\infty)$.
The first six terms of this polynomial are listed as follows:

$K_0(x)=1,$
$K_1(x)=2-3x$
$K_2(x)=2-7x+2x^2,$
$K_3(x)=2-12x+7x^2-\frac{5x^3}{6},$
$K_4(x)=2-18x+16x^2-\frac{23x^3}{6}+\frac{x^4}{4},$
$K_5(x)=2-25x+30x^2-\frac{65x^3}{6}+\frac{17x^4}{12}-\frac{7x^5}{120}$

The Krall-Laguerre polynomials are shown in Figure 1 for different values of m.

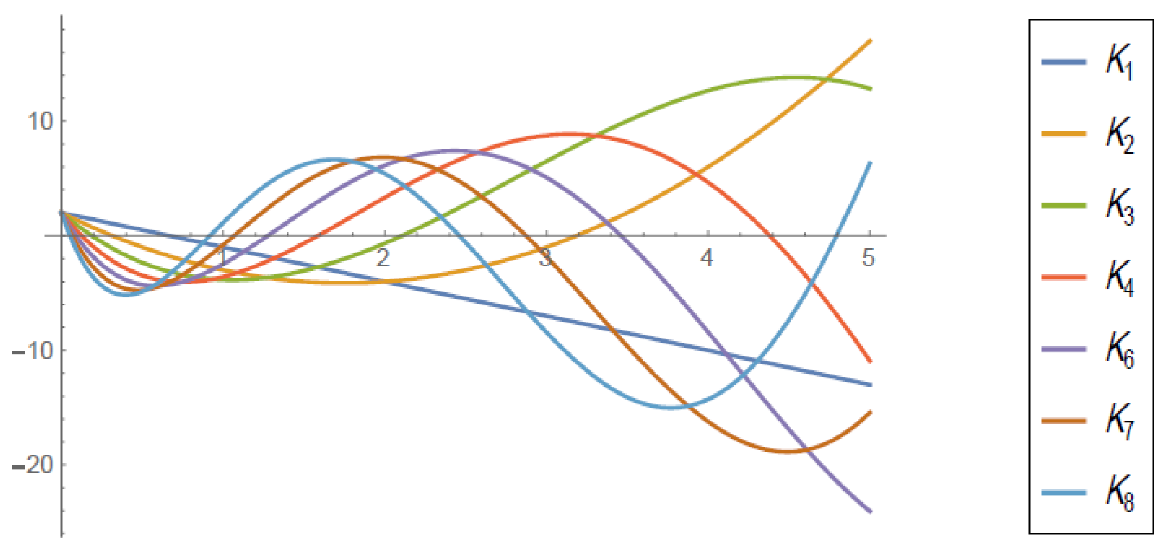

Figure 1.The Krall-Laguerre polynomials

3. Numerical solution of integral equation of the third type

Consider the integral equation of the third type as follows:

$$x^{\beta}f(x)=g(x)+\int_0^x(x-t)^{-\alpha}k(x,t)f(t)dt,\quad x\in[0,T] \tag{2}$$

To numerically solve this type of integral equations, we approximate the unknown function f with equation.

Accordingly, we will have:

$$\sum_{i=0}^{m} f\left(\frac{i}{m}\right)\left(\frac{(-1)^i}{(i+1)!}[i(\alpha+m+1)+\alpha]\binom{m}{i}\left((x^{\beta+i})-\int_0^x (x-t)^{-\alpha}k(x,t)t^i dt\right)\right)=g(x)$$

Accordingly, we will have:

To obtain: $f\left(\frac{i}{m}\right), i=0,\dots,m$, by replacing x with $x_j=\frac{j}{m+\varepsilon}$, $j=0,\dots,m$ and $x_m=1-\varepsilon$ for small values of ε, the above equation becomes a linear system of equations.

After that, the equation can be rewritten as:

$$BX=Y$$

where

$$B=\left[\frac{(-1)^i}{(i+1)!}[i(\alpha+m+1)+\alpha]\binom{m}{i}\left(\left(x_j^{\beta+i}\right)-\int_0^x (x_j-t)^{-\alpha}k(x_j,t)t^i dt\right)\right], i,j=0,1,\dots,m,$$

$$X=\left[f\left(\frac{i}{m}\right)\right]^t, \qquad i=0,\dots,m$$

$$Y=\left[g(x_j)\right]^t, \qquad j=0,\dots,m$$

Therefore, the integral in B formula is calculated numerically.

Here with replacing

$$f\left(\frac{i}{m}\right), i=0,\dots,m, \qquad \text{by} \quad f_m\left(\frac{i}{m}\right), i=0,\dots,m$$

In equation (2), which are here considered as solutions in the nodes, then

$$k_m(f_m)(x_i),\ i=0,\dots,m$$

is obtained that is the solution for integral equation (2).

4. Numerical examples

Now, two different examples are given, which show that the given method can be accurate, applicable and effective. In fact, according to [14], by presenting both examples, the efficiency of the proposed method is evaluated in this study.

Example 1. As the first example, equation (2) considering the values:

$$\alpha=\frac{2}{3}, \beta=\frac{2}{3}, k(t,x)=\frac{\sqrt{3}}{3\pi}x^{1/3}$$

It becomes the following Abel equation:

$$x^{\frac{2}{3}}f(x)=g(x)+\int_0^x \frac{\sqrt{3}}{3\pi}t^{\frac{1}{3}}(x-t)^{-\frac{2}{3}}f(t)\mathrm{d}t, \qquad t\in[0,1]$$

Where

$$g(x)=x^{\frac{47}{12}}\left(1-\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{55}{12}\right)}{\pi\sqrt{3}\Gamma\left(\frac{59}{12}\right)}\right)$$

In other words, the exact solution of the equation in this example is:

$$f(x)=x^{\frac{13}{4}}.$$

First, the above equation was solved through different values of m. Then the numerical outputs were listed in Table 1. In this table, the report of the highest error, the order of convergence and the outputs of the Galerkin method [14] are given.

Notably, the numerical outputs showed that the proposed method has a convergence order of 3.63, while, in the Galerkin method described in [14], these examples had a convergence order of 2.45.

Table 1. Numerical outputs of example 1 and comparison with Galerkin method

m	Proposed method e_m	Proposed method P_m	Galerkin method e_m	Galerkin method P_m
6	1.348×10^{-4}	3.63	3.154×10^{-3}	2.45
12	2.515×10^{-4}	3.94	4.214×10^{-2}	2.94
24	1.747×10^{-4}	3.99	3.910×10^{-3}	3.42
48	5.201×10^{-4}	3.90	3.421×10^{-4}	3.15

Example 2. As a second example, equation (2) considering the values:

$$\beta=1,\ \alpha=0,\ k(t,x)=\frac{1}{2},$$

It becomes the following equation:

$$x\,f(x)=\frac{6}{7}x^{3}\sqrt{x}+\int_{0}^{x}\frac{1}{2}f(t)\,dt, \qquad t\in[0,1]$$

In other words, the exact solution of the equation in this example is:

$$f(x)=x^{\frac{5}{2}}.$$

The above equation is practically used in modeling a number of heat conduction problems with mixed boundary conditions.

Here also, the above equation was solved for different values of m. Then the numerical outputs were listed in Table 2. In this table, the report of the highest error, the order of convergence and the outputs of the Galerkin method [14] are given.

Notably, the numerical outputs showed that the proposed method has a convergence rate of 4.45, while, in the Galerkin method described in [14], these examples had a convergence rate of 1.99.

Table 2. Numerical outputs of example 2 and comparison with Galerkin method

m	Proposed method e_m	Proposed method P_m	Galerkin method e_m	Galerkin method P_m
6	3.012×10^{-5}	4.45	2.543×10^{-3}	3.15
12	6.170×10^{-4}	4.88	3.101×10^{-3}	3.01
24	2.015×10^{-4}	4.53	1.121×10^{-4}	2.41
48	8.124×10^{-5}	4.64	3.311×10^{-4}	1.95

5. Conclusion and further research

The present study proposes a numerical method based on Krall-Laguerre polynomials to solve Volterra integral equations (VIE) of the third type. This method reduces the problem-solving operations, turning it into systems of algebraic equations that are easily solvable. Two numerical examples were presented to verify the method, applicability and accuracy of the method. Numerical results showed that the convergence rate of this method is acceptable. There are several methods for solving integral equations of the third kind. Therefore, other numerical methods can be evaluated with the present study and their results can be compared with each other.

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