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Solving special type of Volterra integral equations using basis polynomials

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Abstract

The present study proposes a numerical method based on genocchi polynomials to solve Volterra integral equations (VIE) of the special type. This method reduces the problem-solving operations, turning it into systems of algebraic equations that are easily solvable. Two numerical examples were presented to verify the method, applicability and accuracy of the method. Numerical results showed that the convergence rate of this method is acceptable.

Key words:, Approximation functions, Genocchi polynomials, Numerical method, Singular kernels, Third-Kind Integral Equations.

1. Introduction

Various numerical methods are used to solve ordinary differential equations or differential equations with partial derivatives or linear or nonlinear integral equations, time-dependent or non-time-dependent. Among these methods, finite differences, finite elements and spectral methods can be mentioned. Spectral methods are of particular importance due to their high accuracy and fast convergence. Spectral methods are divided into three main groups: Galerkin, Tau and collocation, each of which has specific capabilities.

The present study proposes a new method for the numerical solution of the third type linear integral equations. It should be mentioned that the given method is based on the approximation of unknown functions using Genocchi polynomial.

Numerical techniques are best suited for solving various integral problems. Recently, various types of quadrature rules are used in the field of numerical integration for the benefit of science and technology. Real definite integrals are approximated with the anti-Newtonian and anti-Gaussian rule [1]. This mixed method gives the better approximation. A mathematical model for approximate solution of line integral is proposed [2] . Many studies have been done on the numerical solution of integral equations, and many types of numerical methods have been developed to quickly and accurately obtain the approximation of the solution. Also error bounds for numerical integration of functions of lower smoothness and Gauss Legendre quadrature rule has been investigated [3]. A joint quadrature method is based on combining two rules of the same precision level for numerical solution of the double integral is presented [4]. Collocation methods for solving integral equations are obtained [5] . Many practical problems such as vorticity transport equation, heat and advection diffusion equations, whitham-broer-kaup-like equations have been solved numerically in various articles [6-8] . The methods of numerical solution of different types of ordinary and partial differential equations have also been studied in recent years [9-12]. Several numerical methods like block method [13-15]. Finite element and transformation technique [16-18], B-spline collocation [19-21] and higher degree B-spline and block method [22] have also been discussed a lot in various articles.

In this research, we use good and simple functions such as Genocchi polynomials and their matrices to transform the special kind of Volterra integral with singular kernels, as shown below.

$$x^{\propto} y(x) = h(x) + \int_0^x k(x,t)\, f\big(y(t)\big)dt, \quad x \in [0,T] \,.$$

Where $h(x) \in L^2(R),\ 0 \le x\ ,\ t \le T;\ \ k(x,t) = (x-t)^{-\propto} t^{\beta},\ \alpha \in [0,1)\,,\ \beta \in \mathbb{R}\ ,\ \beta > 0\,,\ \alpha+\beta > 0\,,$ and $f(x)$ is a continuous function on the interval $[0,T]$ and $y(x)$ is unknown function.

This class of equations, as stated in (1), is found correspondingly in the concepts of single integral equations with boundary value problems for complex partial differential equations.

All previous studies had been conducted to find an analytical solution method for third kind integral equations and no numerical one had been proposed. Therefore, the present study proposed a new method for numerical solution of the linear integral equations of the third kind. It should be noted that the given method was on the basis of approximating unknown functions using Genocchi polynomials.

2. Genocchi Polynomials and their properties

Genocchi polynomials and Genocchi numbers have been widely used in many branches of mathematics and physics such as analytic and complex number theory, homotopy theory, differential topology and quantum physics.

Genocchi polynomials $G_n(x)$ and Genocchi numbers G_n are usually defined using generating functions $Q(t,x)$, $Q(t)$, respectively, as follows:

$$Q(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \ , \ \left(|t| < \pi \right),$$

$$Q(t,x) = \frac{2t\, e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \ , \ \left(|t| < \pi \right),$$

in which $G_n(x)$ is the Genocchi polynomials are of order *n.*

Also, Genocchi polynomials are obtained as follows:

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_{n-k}\, x^k = 2B_n(x) - 2^{n+1} B_n(x),$$

Which G_{n-k} is obtained from the following relationship:

$$G_n = 2\left(1 - 2^n\right) B_n \ ,$$

and B_n is Bernoulli's number.

Some prime numbers of Genocchi are given in the following table:

n	0	1	2	4	6
G_n	0	1	−1	1	−3

Also, we have

$$G_{2n+1} = 0, \quad n = 1, 2, 3, ...$$

The first few terms of Genocchi polynomials are:

$$G_0(x)=0,$$

$$G_1(x)=1,$$

$$G_2(x)=2x-1,$$

$$G_3(x)=3x^2-3x,$$

$$G_4(x)=4x^3-6x^2+1,$$

$$G_5(x)=5x^4-10x^3+5x,$$

The diagram of Genocchi polynomials is also shown in Figure 1:

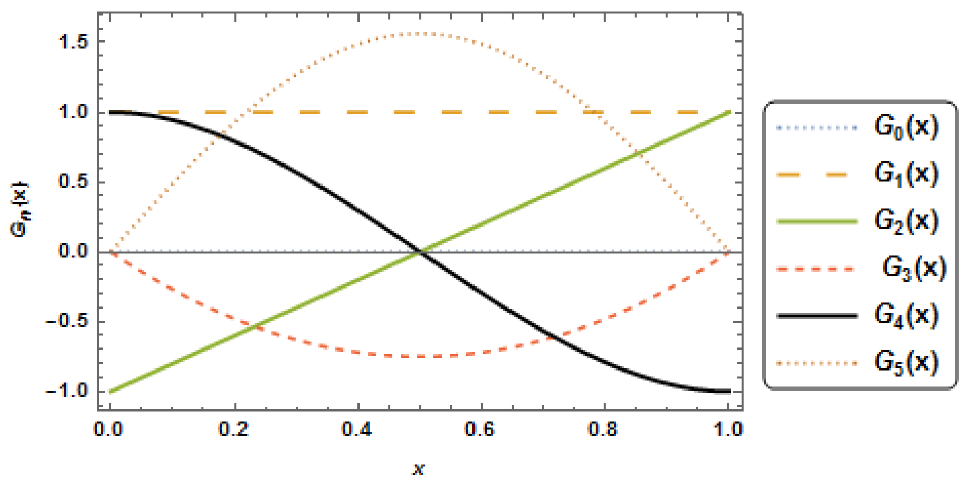

Figure 1. Diagram of Genocchi polynomials

2.1. Approximation of arbitrary function using Genocchi polynomial

Approximation theory plays an important role in solving all kinds of differential equations.

The main goal of this section is to approximate the arbitrary function $f(x)\in L^2[0,1]$ with Genocchi polynomials.

Suppose: $\{G_1(x),G_2(x),\dots,G_N(x)\}\subseteq L^2[0,1]$ be a set of Genocchi polynomials.

also: $P=span\{G_1(x),G_2(x),\dots,G_N(x)\}$

Because P is a finite-dimensional subset of the space $L^2[0,1]$, so $f(x)$ as an arbitrary element of the space $L^2[0,1]$, has the best unique approximation in P, which we call $f^*(x)$ and applies to the following relation:

$$\|f(x)-f^*(x)\|_2\le\|f(x)-y(x)\|_2, \quad \forall y(t)\in P \ .$$

According to the above inequality, the following internal multiplication will also be valid:

$$\left\langle f(x)-f^{*}(x),y(t)\right\rangle=0,\quad \forall\in P\ .$$

Any arbitrary function $f(x)\in L^{2}\left[0,1\right]$ is considered using Genocchi polynomials as follows:

$$f(x)\approx f^{*}(x)=\sum_{n=1}^{N}c_{n}G_{n}(x)=C^{T}G(x)$$

in which *T* is the transduction and *C* is vector coefficients of Genocchi and vector *G(x)* will be as follows:

$$c=\left[c_{1},c_{2},...,c_{N}\right]^{T},\ \ G(x)=\left[G_{1}(x),G_{2}(x),...,G_{N}(x)\right]^{T}$$

Therefore, the coefficient using genocchi polynomials is:

$$c_{n}=\frac{1}{2n!}\left(f^{(n-1)}(0)+f^{(n-1)}(1)\right),\ \ n=1,2,...,N\ .$$

Of course, we must pay attention to the important point that the calculation of the approximation coefficient by the Genocchi polynomial in the above relation is not possible for a function that is not differentiable *n-1* times at distinct points $x=0$, $x=1$.

3. Genocchi method for the integral equation of the third type

Consider the integral equation of the third type as follows:

$$x^{\propto}y(x)=h(x)+\int_{0}^{x}(x-t)^{-\propto}\ t^{\beta}\ f\big(y(t)\big)dt,\quad x\in\left[0,T\right]. \tag{1}$$

To numerically solve this type of integral equations, we approximate the all known and unknown functions of the equation as below,

$$y(x)\approx\sum_{n=1}^{N}c_{n}G_{n}(x)=C^{T}G(x)=C^{T}G\,X_{x}$$

Where

$$\mathrm{C}=\left[c_{1},c_{2},...,c_{N}\right]^{T},$$

is the vector of unknown coefficients.

$$X_{x}=\left[1,x,x^{2},...,x^{n}\right]^{T},$$

$$G(x)=\left[G_{1}(x),G_{2}(x),...,G_{N}(x)\right]^{T}=G\,X_{x}\ ,$$

where G is a $n\times n$ matrix of coefficients that can be approximated by X_{x}.

Therefore, before applying the new approach to solving the nonlinear singular integral equation (1), we must calculate the following integral:

$$\int_0^x (x-t)^{-\alpha} t^j \, dt = \frac{\Gamma(1-\alpha)\Gamma(j+1)}{\Gamma(j-\alpha+2)} x^{(j-\alpha+1)} \ , \ j=0,1,\ldots$$

If we put, $g(x) = f\big(y(x)\big), \quad 0 \le x \le 1$.

According to the previous relations and placing in integral equation (1), we will have,

$$g(x) = f\left(x^{-\alpha}\left(h(x) + \int_0^x (x-t)^{-\alpha} t^{\beta} f\big(y(t)\big) dt\right)\right),$$

will have:

$$C^T G(x) = f\left(x^{-\alpha}\left(h(x) + \int_0^x (x-t)^{-\alpha} t^{\beta} C^T G X_t \, dt\right)\right), \quad 0 \le x \le 1$$

And also,

$$C^T G(x) = f\left(x^{-\alpha} h(x) + C^T G\, x^{-\alpha} \int_0^x (x-t)^{-\alpha} t^{\beta} X_t \, dt\right), \quad 0 \le x \le 1$$

At this stage, we must convert the integral part of the above relationship into matrix form:

Assuming

$$X_t = \left[1, t, t^2, \ldots, t^n\right]^T,$$

can write:

$$x^{-\alpha}\int_0^x (x-t)^{-\alpha} t^{\beta} X_t \, dt = x^{-\alpha}\left[\int_0^x (x-t)^{-\alpha} t^{\beta} \, dt, \int_0^x (x-t)^{-\alpha} t^{\beta}\, t \, dt, \ldots, \int_0^x (x-t)^{-\alpha} t^{\beta}\, t^n \, dt, \ldots\right]^T$$

$$= x^{-\alpha}\left[\int_0^x (x-t)^{-\alpha} t^{\beta} \, dt, \int_0^x (x-t)^{-\alpha} t^{\beta+1} \, dt, \ldots, \int_0^x (x-t)^{-\alpha} t^{\beta+2} \, dt, \ldots\right]^T$$

so

$$x^{-\alpha}\int_0^x (x-t)^{-\alpha} t^{\beta} X_t \, dt = [\frac{\Gamma(1-\alpha)\Gamma(\beta+1)}{\Gamma(\beta-\alpha+2)} x^{(\beta-2\alpha+1)}, \frac{\Gamma(1-\alpha)\Gamma(\beta+2)}{\Gamma(\beta-\alpha+3)} x^{(\beta-2\alpha+2)}, \ldots$$

$$, \frac{\Gamma(1-\alpha)\Gamma(\beta+m+1)}{\Gamma(\beta+m-\alpha+2)} x^{(\beta+m-2\alpha+1)}, \ldots]^T$$

With considering:

$$\rho_{m.m} = \frac{\Gamma(1-\alpha)\Gamma(\beta+m+1)}{\Gamma(\beta+m-\alpha+2)}, \ m = 0,1,2,\ldots$$

The previous relationship is as follows:

$$x^{-\alpha}\int_0^x (x-t)^{-\alpha} t^{\beta} X_t \, dt = \begin{bmatrix} \rho_{0,0} & 0 & 0 & \dots & 0 \\ 0 & \rho_{1,1} & 0 & 0 & 0 \\ 0 & 0 & \rho_{2,2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \rho_{m,m} \end{bmatrix} \begin{bmatrix} x^{(\beta-2\alpha+1)} \\ x^{(\beta-2\alpha+2)} \\ \vdots \\ x^{(\beta+m-2\alpha+1)} \\ \vdots \end{bmatrix} = \mathrm{M}\mathrm{K}$$

where M is an infinite diagonal matrix and K is also:

$$K = \left[x^{(\beta-2\alpha+1)}, x^{(\beta-2\alpha+2)}, \dots, x^{(\beta+m-2\alpha+1)}, \dots\right]^T$$

Each infinite vector domain above can be written as follows in terms of Genocchi polynomials:

$$x^{(\beta+m-2\alpha+1)} = \sum_{i=1}^{\infty} a_{m,i} G_i(x) = \partial_m G X_x \quad , \quad \partial_m = \left[a_{m,1}, a_{m,2}, \dots, a_{m,2}, \dots\right]$$

so

$$K = \left[\partial_1 G X_x, \partial_2 G X_x, \dots, \partial_m G X_x, \dots\right]^T = A G X_x \ , \quad A = \left[\partial_{1\ t}, \partial_2, \dots, \partial_m, \dots\right]^T$$

By placing in the previous relations, we have:

$$x^{-\alpha}\int_0^x (x-t)^{-\alpha} t^{\beta} X_t \, dt = \mathrm{M} A G X_x$$

By combining the two final relations, we will have:

$$C^T G(x) = f\left(x^{-\alpha} h(x) + C^T G \mathrm{M} A G X_x\right), \ \ 0 \le x \le 1$$

We select N nodal points using the Newton-Cots method to find the values of vector C as follows:

$$x_p = \frac{2p-1}{2N}, \ \ p = 1, 2, \dots, N$$

$$C^T G(x_p) = f\left(x_p^{-\alpha} h(x_p) + C^T G \mathrm{M} A G X_{x_p}\right), \ \ p = 1, 2, \dots, N \ .$$

We can solve the above nonlinear system by using Newton's iteration scheme to calculate the unknown vector C. After calculating the unknown vector C by solving the above nonlinear equation, to obtain the approximate solution of equation (1), we will have the following form:

$$y_n(x) = x^{-\alpha} f(x) + C^T G \mathrm{M} A G(x), \quad 0 \le x \le 1 \ ,$$

4. Error analysis and convergence order of the method

In this section, we estimate the error of the approximate solution to find the error bounds of the new numerical approach by applying Genocchi polynomials.

Consider the special kind of Volterra integral equations of the form of equation (1).

Suppose:

$$e_n(t) = y(t) - y_n(t) \ \ ,$$

Be the error function is the integral equation (1) in which $y_n(x),\ y(t)$ are the known function and its approximation, respectively.

we have:

$$\|e_n(t)\|_2^2 = \|y(t) - y_n(t)\|_2 = \int_0^1 |y(t) - y_n(t)|^2\, dt$$
$$= \int_0^1 \left| x^{-\alpha}\left(h(x) + \int_0^x (x-t)^{-\alpha} t^{\beta} f(y(t))dt\right) - x^{-\alpha}\left(h(x) + \int_0^x (x-t)^{-\alpha} t^{\beta} f(y_n(t))dt\right)\right|^2 dx$$
$$= \int_0^1 \left|\int_0^x x^{-\alpha}(x-t)^{-\alpha} t^{\beta}\left(f(y(t)) - f(y_n(t))\right)dt\right|^2 dx$$

Since $f(t)$ is Lipshitz continuous, there exists a number C_1 such that:

$$|f(y(t)) - f(y_n(t))| \le C_1 |y(t) - y_n(t)|$$

So

$$\|e_n(t)\|_2^2 \le \int_0^1 \left|\int_0^x x^{-\alpha}(x-t)^{-\alpha} t^{\beta} C_1 |y(t) - y_n(t)| dt\right|^2 dx$$
$$= \int_0^1 \left|\int_0^x x^{-\alpha}(x-t)^{-\alpha} t^{\beta} C_1 \left|y(t) - \sum_{n=1}^{N} c_n G_n(t)\right| dt\right|^2 dx$$
$$= \int_0^1 \left|\int_0^x x^{-\alpha}(x-t)^{-\alpha} t^{\beta} C_1 \left|\sum_{n=N+1}^{\infty} c_n G_n(t)\right| dt\right|^2 dx$$
$$\le \int_0^1 \left|\int_0^x x^{-\alpha}(x-t)^{-\alpha} t^{\beta} C_1 \sum_{n=N+1}^{\infty} |c_n| |G_n(t)| dt\right|^2 dx$$

And similarly, by placing the formula of Genocchi polynomials in the above relationship, we will have:

$$\|e_n(t)\|_2^2 \le \int_0^1 \left|\int_0^x x^{-\alpha}(x-t)^{-\alpha} t^{\beta} C_1 \sum_{n=N+1}^{\infty} |c_n| \left|\sum_{k=0}^{n} \binom{n}{k} G_{n-k}\, t^k\right| dt\right|^2 dx$$
$$\le \int_0^1 \left(\int_0^x x^{-\alpha}(x-t)^{-\alpha} t^{\beta} C_1 \sum_{n=N+1}^{\infty} |c_n| \left|\sum_{k=0}^{n} \binom{n}{k}\right| |G_{n-k}| t^k dt\right)^2 dx$$
$$= \int_0^1 \left(\sum_{k=0}^{n} \sum_{n=N+1}^{N} C_1 |c_n| \left|\binom{n}{k}\right| |G_{n-k}| \int_0^x x^{-\alpha}(x-t)^{-\alpha} t^{\beta+k} dt\right)^2 dx$$

By defining the term $\gamma(t,\beta,\alpha)$ as follows:

$$\gamma(t,\alpha,\beta) = \int_0^x x^{-\alpha}(x-t)^{-\alpha} t^{\beta} dt = B(1-\alpha, 1+\beta) x^{1-2\alpha+\beta}$$

And the beta function as follows:

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$$

We will have

$$\|e_n(t)\|_2^2 \leq \int_0^1 \left(\sum_{k=0}^{n} \sum_{n=N+1}^{N} C_1 |c_n| \binom{n}{k} |G_{n-k}| B(1-\alpha, 1+\beta+k) x^{1-2\alpha+\beta+k} \right)^2 dx$$

5. Numerical examples

In this section, two numerical examples with tables and graphs are performed to check the accuracy of the proposed method as well as the accuracy and efficiency of Genocchi polynomial scheme.
To reflect on method error, notations would be introduced:

$$e_m = \max_{0 \leq i \leq n} |y(t_i) - y_m(t_i)| \ , \ P_m = \log_2\left(\frac{e_m}{e_{2m}}\right)$$

Example 1. Consider the following special kind of Volterra integral equation,

$$x^{\frac{2}{3}} y(x) = h(x) + \int_0^x \frac{\sqrt{3}}{3\pi} t^{\frac{1}{3}} (x-t)^{\frac{-2}{3}} y(t) dt, \ t \in [0,1]$$

Where

$$h(x) = x^{\frac{47}{12}} \left(1 - \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{55}{12}\right)}{\pi\sqrt{3}\,\Gamma\left(\frac{59}{12}\right)} \right)$$

And the exact solution of the equation is: $y(x) = x^{\frac{13}{4}}$.

First, the above equation was solved with different values of *m*. Then, numerical outputs were listed in Table 1 reporting the highest error, order of convergence, and outputs of the collocation method [11]. Notably, numerical outputs demonstrated that the proposed method had a convergence of 3.76. Whereas, in the collocation method described in [11], these examples had an order of convergence by 2.96.

Table 1. Numerical outputs for example 1

m	Proposed method e_m	Proposed method P_m	Collocatrion method e_m	Collocatrion method P_m
6	1.348×10^{-4}	3.76	8.112×10^{-3}	2.96
12	3.123×10^{-4}	4.12	6.174×10^{-2}	3.01
24	2.045×10^{-4}	4.03	2.210×10^{-3}	3.42
48	1.217×10^{-5}	3.95	3.421×10^{-4}	3.15

Example 2. Consider the following special kind of Volterra integral equation that would be applied for modeling a number of heat conduction problems with the mixed-kind boundary conditions:

$$x\, y(x) = h(x) + \int_0^x \frac{1}{2} y(t)dt,\ t \in [0,1]$$

Where

$$h(x) = \frac{6}{7} x^3 \sqrt{x}$$

And the exact solution of the equation is: $y(x) = x^{\frac{5}{2}}$.

Here also, the above equation was solved for different values of m. Then the numerical outputs were listed in Table 5.2. In this table, the report of the highest error, the order of convergence and the outputs of the collocation method [11] are given.

Notably, the numerical outputs showed that the proposed method has a convergence rate of 4.88, while, in the collocation method described in [11], these examples had a convergence rate of 2.59

. **Table 2. Numerical outputs for example 2**

m	Proposed method e_m	Proposed method P_m	Collocatrion method e_m	Collocatrion method P_m
6	2.741×10^{-5}	4.99	7.541×10^{-3}	3.75
12	2.141×10^{-4}	4.88	5.738×10^{-3}	2.59
24	1.440×10^{-4}	5.03	3.667×10^{-4}	3.48
48	3.277×10^{-5}	5.34	2.213×10^{-5}	3.19

6. Conclusion

The present study proposes a numerical method based on Genocchi polynomials to solve special kind of Volterra integral equations. This method reduces the problem-solving operations, turning it into systems of algebraic equations that are easily solvable. Two numerical examples were presented to verify the method, applicability and accuracy of the method. Numerical results showed that the convergence rate of this method is acceptable.

References

[1] Jena SR, Nayak D, Paul AK, Mishra SC. Mixed anti-Newtonian-Gaussian rule for real definite integrals. Adv Math Sci. J. 2020; 9: 1081-1090. https://doi.org/10.37418/amsj.9.11.115

[2] Jena SR, Singh A. A Mathematical Model for Approximate Solution of Line Integral. J Comp Math Sci. 2019; 10(5): 1163-1172. http://www.compmath-journal.org/

[3] Surulere SA, Oladeji AO. Error bounds for numerical integration of functions of lower smoothness and Gauss Legendre quadrature rule. Aust J Math Anal Appl. 2021; 18(2): 1-21. https://ajmaa.org/searchroot/files/pdf/v18n2/v18i2p19

[4] Jena SR, Nayak D, Paul AK, Mishra SC. Numerical Investigation, Error Analysis and Application of Joint Quadrature Scheme in Physical Sciences, Baghdad Sci J. 2023, 20(5): 1789-1796. https://dx.doi.org/10.21123/bsj.2023.3337

[5] Allaei, SS, Yang ZW, Brunner H. Collocation methods for third-kind VIEs. IMA Journal of Numerical Analysis, 2016. 37(3): p. 1104-1124. https://doi.org/10.1093/imanum/drw033

[6] Rasheed MA, Kadhim SN. Numerical Solutions of Two-Dimensional Vorticity Transport Equation Using Crank-Nicolson Method. Baghdad Sci J. 2022; 19(2): 321-328, https://doi.org/10.21123/bsj.2022.19.2.0321

[7] Jena SR, Gebremedhin GS. Computational technique for heat and advection–diffusion equations. Soft Comput. 2021; 25: 11139–11150. https://doi.org/10.1007/s00500-021-05859-2

[8] Al-BotaniZM,Al-RozbayaniAM. SolvingWhitham-Broer-Kaup-LikeEquations Numerically by using Hybrid Differential Transform Method and Finite Differences Method. Baghdad Sci J. 2022; 19(1): 64-70.: http://dx.doi.org/10.21123/bsj.2022.19.1.0064

[9] Gebremedhin GS, Jena SR. Approximate Solution of Ordinary Differential Equations via Hybrid Block Approach. Int J Emerg Technol. 2019; 10(4): 210-211, www.researchtrend.net

[10] Jena SR, Mohanty M, Mishra SK. Ninth step block method for numerical solution of fourth order ordinary differential equation. Adv Model Anal. 2018; 55:45-56, https://doi.org/10.18280/ama_a.550202.

[11] Gebremedhin GS, Jena SR. Approximate solution of a fourth order ordinary differential equation via tenth step block method. Int J Comput Sci Math. 2020; 11: 253-2,10.1504/IJCSM.2020.10028216. https://doi.org/10.1504/IJCSM.2020.106695

[12] Kazem S, Dehghan M. Application of finite difference method of lines on the heat equation. Numerical Methods for Partial Differential Equations . 2018 Mar; 34(2):626-660. https://doi.org/10.1002/num.22218

[13] Jena SR, Gebremedhin GS. Approximate solution of fifth order ordinary differential equations with block method. Int J Comput Sci Math. 2020; 12(4): 413-426. https://doi.org/10.1504/IJCSM.2020.112652

[14] Jena SR, Senapati A, Gebremedhin GS. Numerical study of solutions in BFRK scheme. Int J Mech Control. 2020; 21: 163-170, www.jomac.it .

[15] Jena SR, Moohanty M, Numerical treatment for ODE (Fifth order). Int J Emerg Technol. 2019; 10(4): 191–196. www.researchtrend.net

[16] Rasheed MA, Kadhim SN. Numerical Solutions of Two-Dimensional Vorticity Transport Equation Using Crank- Nicolson Method. Baghdad Sci J. 2022; 19(2): 321-328, https://doi.org/10.21123/bsj.2022.19.2.0321

[17] Mohanty M, Jena SR, Mishra SK. Mathematical Modelling in Engineering with Integral Transforms viaModified Adomian Decomposition Method. Math Model Eng Probl. 2021; 8: 409-417, https://doi.org/10.18280/mmep.080310

[18] Jana SR, Senapati A, Gebremedhin GS.Approximate solution of MRLW equation in B-spline environment.Math Sci. 2020; 14: 345-357. https://doi.org/10.1007/s40096-020-00345-6.

[19] Jena SR, Gebremedhin GS. Numerical treatment of Kuramoto -Sivashinsky equation on B-spline collocation. Arab J Basic Appl Sci. 2021; 28: 283-291, https://doi.org/10.1080/25765299.2021.1949

[20] Senapati A, Jena SR. A computational scheme for fifth order boundary value problems. Int J Inf Technol. 2022: 1-8. https://dx.doi.org/10.1007/s41870-022-00871-7.

[21] Jena SR, Sahoo M. Mohanty PK, Misra SK,Mishra BB. Application and convergence analysis of mixed quadrature rule for approximate solution of product integral. Math Eng Sci Aero. 2022; 13(4):1119-1129. www.journalmesa.com

[22] Beni MR. Legendre wavelet method combined with the Gauss quadrature rule for numerical solution offractional integro-differential equations. Iran J Numer Anal Optim. 2022; 12(1): 229249, https://dx.doi.org/10.22067/IJNAO.2021.73189.1070

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