Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: 10.29350/jops

Al-Qadisiyah Journal of Pure Science

Utilizing The Numerical Methods or Finding The Optimal Solution The Linear Modeling Problems

ali abbas obied

A.BSc: College of Pure Science Department of Mathematics, University of Babylon, Babylon, Iraq

Master Degree: College of Education for Pure Sciences, Islamic Azad University, Asfahan, Iran

Corresponding email: ali.abbas.obied.11@gmail.com

Abstract

The purpose of the study is to find an optimal solution to linear programming problems instead of solving them using linear programming methods, such as the simplex method, for example. Because it is lengthy in finding the basic initial solutions and iterative and mathematical operations until the optimal solution its reached, and these methods only solve problems in which the number of constraints equals the number of variables. Here we try to use numerical methods to solve these problems. It is easier and faster than the simplex method, and it does not require additional (imaginary) variables. In addition, numerical analysis contains many ready-made, fast programs and simple numerical arithmetic methods to reach the optimal solution, which are rare in linear programming problems. Among these numerical methods for finding numerical solutions to systems of linear equations in this research are (Gaussian Elimination method and Gauss-Jordan method). The research also includes a number of conclusions.

Keyword: Gauss elimination Gauss-Jordan, Simplex method, linear programming,

1. Introduction:

Linear programming is a mathematical technique used to achieve the best possible outcome such as maximum profit or minimum cost given a given set of constraints. This usually done using a mathematical model that describes the relationship between goals and constraints. They are used in a wide range of applications in business and industry, including production, transportation, finance, and urban planning.

A typical mathematical problem consists of one function that represents either the profit to be increase to the maximum, or the cost to be reduce to the minimum, in addition to a set of constraints that constitute a limit to the decision variables.

In the case of linear programming, both the function and constraints are linear functions of the decision variables.

Linear programming is a widely used model that can solve decision-making problems with many variables. The values of possible decisions can be limit by a set of mathematically described constraints that are compare using a function of the decision variables.

When a problem has only two variables, a graphical method can be use in the solution process. Actually, most linear programming problems are simple and can be solve graphically. However, larger problems with many constraints consume a significant amount of time to solve.

Making any decision has two main stages: the first is formulating the issue according to mathematical relationships called the mathematical model, and the second is solving the mathematical model and searching for the best solutions and applying them to the real problem.

2. The general formula for linear programming

It is possible to develop a fixed formulation for the linear program that includes the objective function Z and the restrictions that it can take ≥, =, ≤

It is common knowledge that all variables $\boldsymbol{X_j}$

What is required to make a decision on them are non-negative because they are variables that relate to reality, so negative results become unreal quantities

Maximize or Minimize
$Z = c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n$ Objective function
Subject to:

$$
\begin{aligned}
& a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \, (\le, =, \ge) b_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \, (\le, =, \ge) b_2 \\
& a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n \, (\le, =, \ge) b_3 \\
& \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n \, (\le, =, \ge) b_m
\end{aligned} \tag{1}
$$

$x_1, x_2, x_3, \boldsymbol{K}, x_n \ge 0$ Restrictions

Where;

a_{ij}, b_i, c_j The constants are determined from the context of the problem

$i = 1,2,3, K, m$

$j = 1,2,3, K, n$

x_{ij} The variables on which a decision must be make.

We notice from the formulation that the sign of the variables x_i is restricted by the non-negative condition, and this condition is necessary for developing a method for solving the *Lp* model.

In general,

If

b_i It represents the amount of limited resources required to be programme to achieve a specific goal,

Then

a_{ij} It represents the amount of limited resources of type i that are required to be allocated to each single unit of activity or effectiveness j,

The value of this for each unit is expressed as c_j.

Where c_j it represents profit or cost.

Remark: The general formula can put in the following abbreviated form, using the sum sign first and using matrices and vectors second:

Maximize or Minimize
$Z = \sum_{j=1}^{n} c_j x_j$ Objective function
Such that:

$$Z = \sum_{j=1}^{n} a_{ij} x_j (\leq, =, \geq) b_i \quad ,i = 1,2,3, K, m \qquad (2)$$

$x_j \geq 0 \quad , j = 1,2,3, K, n$ Restrictions

Or

Maximize or Minimize
$Z = CX$

Subject to:

$AX(\leq, =, \geq)B$

$X \geq 0$

Where: ***C*** represents a row vector with n elements, ***X*** represents a columnar vector with n elements

A is an **n×m** matrix,

B is a perpendicular vector with m elements

The next main steps after formulating the linear programming model are to analyze the model mathematically, but due to the differences in linear programming formulas, it is necessary to modify these formulas to determine a suitable solution model.

There are two formats suitable for this purpose: the Canonical form and the Standard form.

First: the general form

It is possible to put the general formula of linear programming defined above in the following form, which we express in the legal form:

Maximize
$Z = \sum_{j=1}^{n} c_j x_j$ Objective function
Subject to:

$Z = \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad ,i = 1,2,3, K, m$

$x_j \geq 0 \quad , j = 1,2,3, K, n$ Restrictions

The properties of this formula are:

● All variables x_j are sign bound.

● All constraints, the number of which are m, are of the type less than or equal to ≤.

● Maximum objective function.

It is possible to develop any formula for linear programming, in a legal or general form, using elementary transformation operations, which we will briefly review below:

1. Minimize Z= Max –Z

3. $a_1x_1 + a_2x_2 \geq b \quad = to \quad -a_1x_1 - a_2x_2 \leq -b$

4. The equality constraint is transformed into two inequalities that are opposite in direction, that’s mean one $\geq$ and the other $\leq$ then transformed the $\geq$ into $\leq$ this is after multiplying by (-1) as shown in the following example:

$a_1x_1 + a_2x_2 = b$

= to

$a_1x_1 + a_2x_2 \leq b$

$a_1x_1 + a_2x_2 \geq b \quad = to \quad -a_1x_1 - a_2x_2 \leq -b$

As for the constraint consisting of an absolute value on the left side of it, it is converte into two inequalities as shown in the following example:

$|a_1x_1 + a_2x_2| \leq b$

= to

Either

$a_1x_1 + a_2x_2 \leq b$

Or

$-(a_1x_1 + a_2x_2) \leq b$

Second: The Standard form

This formula is considered better than the previous one because it is use in the general method adopted in analyzing linear programs, i.e. the Simplex Method. The most important characteristics of this formula are as follows:

1. All constraints in the problem are coefficients, except for the constraint on the sign of variables.

2. The elements of the right side of each constraint are ≥ 0 this mean $b_i \geq 0$

3. All variables are greater than or equal to zero, $x_i \geq 0$ meaning they are restricted.

4. The objective function is of the Maximum or Minimum type.

Remark: Variable constraints are converted into equality “equations” by adding or subtracting dummy variables, Slack Variable ($s_i \geq 0$), to the left side of each constraint. These variables are add to constraints of the smallest type or equal to ($\leq$) and subtracted from constraints of the type greater or equal to ($\geq$).

As shown in the following example:

$a_1x_1 + a_2x_2 \geq b$

$\rightarrow$ to

$$a_1x_1 + a_2x_2 - s_1 = b_1 \quad (3)$$

Where $s_1 \geq 0$ a dummy variable does not affect the solution

$a_1x_1 + a_2x_2 \leq b$

$\rightarrow$ to

$a_1x_1 + a_2x_2 + s_1 = b_1$

The standard form plays an important role in solving linear programming problems, in general, if you have a problem L_p as follows:

Maximize
$Z = \sum_{j=1}^{n} c_j x_j$ Objective function
Subject to:

$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad ,b_i \geq$ (4)

$x_j \geq 0$ Restrictions

The above formula is represent in standard form as follows:

Maximize or Minimize
$Z = \sum_{j=1}^{n} c_j x_j$ Objective function
Subject to:

$\sum_{j=1}^{n} a_{ij} + s_i = b_i \quad ,i = 1,2,K,m$ (5)

$x_j \geq 0$

$s_i \geq 0$

3. The general method for analyzing linear programs: Simplex Method

The simplex method is consider a highly efficient mathematical method for finding solutions to linear programming problems.

This method was developed by the American mathematician George Dantberg in 1947, and its principle is based on starting with a specific solution, everything that is known to be acceptable, then we continue in a periodic iterative manner in developing this solution until we obtain, after a specific number of steps, the optimal solution.

It is worth noting to emphasize the issue of the existence of an acceptable basic solution and then test this solution to reach the optimal solution.

The simplex method of calculation requires that there be at most *m* positive variables $X_i > 0$, where *m* represents the number of constraints contained in the problem, at any stage of the cycle. If the number of variables at any stage becomes less than the number of constraints, then the solution is degenerated.

Examples of linear programming:

Example (1):

Max
$\boldsymbol{Z} = \mathbf{8}x_1 + \mathbf{6}x_2$ Objective function
Such that:

$\mathbf{4}x_1 + \mathbf{2}x_2 \leq \mathbf{60}$

$\mathbf{2}x_1 + \mathbf{4}x_2 \leq \mathbf{48}$

$x_1, x_2 \geq 0$ Restrictions

The optimal solution to this linear programming problem is:

$$x_1 = \mathbf{12}, \qquad x_2 = \mathbf{6}, \qquad \boldsymbol{Z} = \mathbf{132}$$

Example (2):

Min
$\boldsymbol{Z} = -\mathbf{5}x_1 - \mathbf{3}x_2 + \mathbf{2}x_3$ Objective function
Such that:

$x_1 + x_2 + x_2 \leq \mathbf{5}$

$x_1 - \mathbf{2}x_2 - \mathbf{2}x_3 \leq \mathbf{4}$

$\mathbf{3}x_1 + \mathbf{3}x_2 + \mathbf{2}x_3 \leq \mathbf{15}$

$x_1, x_2, x_3 \geq 0$ Restrictions

The optimal solution to this linear programming problem is:

$x_1 = \frac{\mathbf{14}}{\mathbf{3}}, \quad x_2 = \frac{\mathbf{1}}{\mathbf{3}}, \quad x_3 = \mathbf{0}, \quad \boldsymbol{Z} = -\frac{\mathbf{73}}{\mathbf{3}}$

Example (3):

Max
$\boldsymbol{Z} = \mathbf{2}x_1 + \mathbf{3}x_3 + \mathbf{6}x_5 + x_7 + \mathbf{2}x_8$ Objective function
Such that:

$\mathbf{2}x_1 + \mathbf{3}x_4 + \mathbf{9}x_5 + \mathbf{8}x_7 \leq \mathbf{18}$ Objective function
$x_4 + x_5 + \mathbf{16}x_6 + x_8 \leq \mathbf{90}$

$\mathbf{2}x_3 + \mathbf{11}x_4 + \mathbf{2}x_6 + x_2 \leq \mathbf{20}$

$\mathbf{8}x_1 + \mathbf{20}x_3 + \mathbf{2}x_5 + \mathbf{5}x_6 + x_8 \leq \mathbf{50}$

$\mathbf{9}x_2 + \mathbf{3}x_4 - x_7 + \mathbf{11}\, x_8 \leq \mathbf{40}$

$\mathbf{2}x_2 + \mathbf{10}x_3 + \mathbf{9}x_7 \leq \mathbf{22}$

$-x_5 + \mathbf{7}x_6 \leq \mathbf{18}$

$\mathbf{3}x_2 + \mathbf{8}x_3 + \mathbf{9}x_5 - x_7 \leq \mathbf{36}$

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$ Restrictions

The optimal solution to this linear programming problem is:

$x_1 = -\mathbf{21.1068}$,

$x_2 = -\mathbf{28.4028}$,

$x_3 = \mathbf{8.1484}$,

$x_4 = \mathbf{2.2904}$,

$x_5 = \mathbf{6.1915}$,

$x_6 = \mathbf{3.4559}$,

$x_7 = -\mathbf{0.2976}$,

$x_8 = \mathbf{26.2233}$,

$\boldsymbol{Z} = \mathbf{71.5296}$

So x_1, x_2 *and* x_7 Are not restricted by signal

4. Numerical Solution of Linear System

In numerical solutions for systems of linear equations, there are two types of methods. The first is the Direct Methods type, whereby performing a series of calculations once, an approximate value for the required solution is reached. We say the approximate solution, not the exact one, because the results of the calculations contain some errors. Circularity, the amount of which depends on several factors.

The second type of numerical methods is the iterative method, in which we reach the desired solution by calculating successive approximations to it. That is, we start with an approximate solution to the system of equations, then a series of calculations are performed that lead to us obtaining a better approximate solution, that is, more accurate. This, in turn, is used again in the same series of operations to produce another, more accurate numerical solution, and so on.

The iterative method is consider convergent if successive solutions increase in accuracy and approach a fixed limit. Conversely, the method is consider divergent.

It is worth noting that it is possible to combine the two types of methods to obtain better solutions.

By first using the direct method to obtain an approximate solution with a small number of exact decimal places, and this in turn is used in an iterative method that gives other exact decimal places, and thus we obtain a more accurate solution with effort less.

This is the general formula:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

Or

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad ,i = 1,2,3,K,n$$

In the following methods, we will solve the previous linear programming examples "Simplex Method":

First: Gaussian Elimination Method

One of the simplest direct methods for solving systems of linear equations is the Chaos elimination method, which is a structured form of the elimination method known in mathematics textbooks.

Second: Gauss-Jordan Method

The Gauss-Gordon method for solving the system of equations AX=B is similar to the Gaussian elimination method, except that the matrix A reduces to a diagonal matrix in the Gauss-Gordon method and not a triangular matrix as in the Gauss method.

We solve the Simplex Method linear programming system

- solving the first example using numerical methods:

$$\begin{bmatrix} 4 & 2 & 60 \\ 2 & 4 & 48 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 15 \\ 2 & 3 & 18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 15 \\ 0 & 3 & 18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 15 \\ 0 & 1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 6 \end{bmatrix}$$

The results are as follows:

$x_1 = \mathbf{12}, \quad x_2 = \mathbf{6} \quad \boldsymbol{and} \quad \boldsymbol{Z} = \mathbf{132}$

● solving the second example using numerical methods:

$$\begin{matrix} 1 & 1 & 1 & 5 \\ 1 & -2 & -2 & 4 \\ 3 & 3 & 2 & 15 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 0 & 14/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{matrix}$$

The results are as follows:

$x_1 = \frac{\mathbf{14}}{\mathbf{3}}, \quad x_2 = \frac{\mathbf{1}}{\mathbf{3}}, \quad x_3 = \mathbf{0} \quad \boldsymbol{and} \quad \boldsymbol{Z} = -\frac{\mathbf{73}}{\mathbf{3}}$

● solving the Simplex Method linear programming system in the third example using numerical methods:

$$\left|\begin{matrix} 2 & 0 & 0 & 3 & 9 & 0 & 8 & 0 & 18 \\ 0 & 0 & 0 & 1 & 1 & 16 & 0 & 1 & 90 \\ 0 & 1 & 2 & 11 & 0 & 2 & 0 & 0 & 20 \\ 8 & 0 & 20 & 0 & 2 & 5 & 0 & 1 & 50 \\ 0 & 9 & 0 & 3 & 0 & 0 & -1 & 11 & 40 \\ 0 & 2 & 10 & 0 & 0 & 0 & 9 & 0 & 22 \\ 0 & 0 & 0 & 0 & -1 & 7 & 0 & 0 & 18 \\ 0 & 3 & 8 & 0 & 9 & 0 & -1 & 0 & 36 \end{matrix}\right| \rightarrow \left|\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21.1068 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -28.4028 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 8.1484 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2.2904 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6.1915 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3.4559 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.2976 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 26.2233 \end{matrix}\right|$$

The results are as follows:

$x_1 = \mathbf{-21.1068}, \; x_2 = \mathbf{-28.4028},$

$x_3 = \mathbf{8.1484}, \quad x_4 = \mathbf{2.2904},$

$x_5 = \mathbf{6.1915}, \quad x_6 = \mathbf{3.4559},$

$x_7 = \mathbf{-0.2976}, \; x_8 = \mathbf{26.2233} \; \boldsymbol{and} \; \boldsymbol{Z} = \mathbf{71.5296}$

The method can be generalized to all dimensional linear programming problems according to the formula shown previously, meaning that the solution using numerical methods gives the same results as the Simplex method.

5. Conclusion

1. Linear equation solving methods are effective and good for solving linear programming problems in which the number of constraints is equal to the number of variables.

2. Numerical methods for solving linear equations are easier and faster than the simplex method.

3. Numerical methods for solving linear equations do not require additional (imaginary) variables.

References

[1] Render, B., Stair R. M., Quantitative Analysis for Management , 7th ed., Prentice Hall, New York, 2000.
[2] William, H"Model building In Mathematical programming" 3ed Wiley new York 1990.
[3] Nemhauser, G and L.Wolsey Integer and Combinatorial optimization Wiley new York 1998.
[4] Anderson, D . R. Sweeney D. G., Williams , T. A. "Introduction to Management Science A Quantitive Approach to Decision Making with CD- Roum, New York, 2001.
[5] Michael, Wasilewski (2004), "Parallel Gaussian Elimination", University of Waterloo
[6] Al-Alusi, Dr. Ahmed Saleh and Adel Zainal Al-Bayati (1989): “Introduction to Numerical Analysis”, Higher Education Press in Mosul
[7] Frederick S. Heller and Gerald J. Lieberman, “Introduction to Operations Research,” Part I, 1987.
[8] Jaber, Adnan Shamkhi and Dhawiya Salman Hassan, “Introduction to Operations Research,” College of Administration and Economics/University of Baghdad, 1988.
[9] Jazaa, Abd Dhiyab, “Operations Research,” Baghdad 1985.
[10] Dr. Ali Muhammad Sadiq Saifi and Dr. Ibtisam Kamal al-Din, “Principles of Numerical Analysis,” 1986.
[11] J. R. Loh, C. Phang, and K. G. Tay, “New method for solving fractional partial integro-differential equations by combination of Laplace transform and resolvent kernel method,” Chinese Journal of Physics, vol. 67, pp. 666–680, 2020..