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Improving the Analytical Method of Convergence Using Runge-Kutta Optimization Algorithm for Solving Fractional-Order Nonlinear Differential Equations

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Abstract

Solving nonlinear ordinary differential equations of fractional orders:
Differential equations are equations that relate a function to one or more of its derivatives. Nonlinear means that the equation is not directly proportional to the function and its derivatives. In other words, its variables cannot be separated simply. The dependent variable and its derivatives are exponential, i.e. not of first order. Fractional order indicates that the derivatives in the equation are not necessarily of integer order (such as the first derivative or the second) and can even be a fraction, meaning that the order is a fractional number. To solve nonlinear ordinary differential equations with fractional derivation, the homotopy analytical method was used because solving these equations is not easy using the usual and well-known methods. However, the results of the homotopy method were not as accurate as required, so algorithms were used to improve the results of the homotopy method. Among these algorithms is the bird flock algorithm (HAM-PSO). In this research, the Runge-Kutta algorithm (HAM-OBE) was used to improve the results of the homotopy method. Through the examples in the diagram, it is noted that the bird flock algorithm (HAM-PSO) improved the results of the homotopy method by an error of 4.17e-02 As for the Runge-Kutta algorithm (HAM-OBE), it improved the results by an error of 4.6835e-07 By comparing the amount of errors with the Runge-Kutta algorithm (HAM-OBE), the results were improved by 99.99% compared to the exact solution, where the root mean square error (RMSE) was used as a fitness function to know the amount of improvement compared to the exact solution. This improvement is useful because nonlinear differential equations with spherical debt are useful in physics, chemistry, medical industries, body motion, and artificial intelligence. It has life applications such as studying sound waves and the movement of objects through fractional derivatives.

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1. Introduction

Fractional-order differential equations, called FDEs, generalize ordinary differential equations to any fractional (non-integer) order. The development of fractional differential operators is documented in [3], and the wide range of applications of FDEs in science and engineering has led to significant growth in research in this field, with a great deal of interest in creating numerical systems to solve them. Among these techniques are Fourier transforms [7], power series method [8], fractional differential transform method (FDTM) [9], homotopy analysis method [10], Adomian decomposition method [11], and particle swarm optimization (PSO) [12]. Ordinary, partial, and nonlinear fractional equations can be solved using HAM [13]. This method was first discovered in a PhD thesis in 1992 by Liao Shijun at Shanghai University.

We show the steps for solving them using the homotopy analytical method with the Runge -Kutta algorithm for their optimization. I took numerical examples and solved them approximately with the sequence that we got from collecting the iterations of the homotopy method and then taking the final sequence and finding the average square of the error and considering it as a fitness function in the Runge -Kutta algorithm for optimization and finding the best value of the auxiliary parameter h (Which is a numerical value that the Runge-Kutta algorithm reaches and thus.

The parameter h ensures that the approximate Eckel series converges to the exact solution with high accuracy). in the virtual homotopy method and takes the fewest steps for solving nonlinear differential equations with fractional ranks using (HAM-OBE) This algorithm is a process of optimization of the results of the homotopy analytical method using a method (Runge -Kutta optimization algorithm)) the method can be summarized in five steps.

1 – We take a nonlinear Differential Equation of fractional rank and calculate the initial guess using the equation $y_0(x) = \sum_{k=0}^{n-1} y^{(k)}\ (0^+)\frac{x^k}{k!}$

2 – Finding subsequent iterations using the equation $y_m(x) = \chi_m y_{m-1}(x) + hI^{\alpha}\left(D_x^{\alpha} y_{m-1}(x) + cy_{m-1}(x)\right)$ Using the first iteration as a basis for finding a solution to a nonlinear differential equation with fractional ranks by the homotopy analytical method (HAM) to arrive at the solution series according to specific iterations.

3 – The last iteration of the last Q4 is taken, which is the product of summing the resulting iterations $Q_4 = y_0 + y_1 + y_2 + y_3$ (we will take four iterations in this search)

4 – The Runge-Kutta optimization algorithm (HAM-OBE) is used to find the best value of the parameter (h) to improve the results of the homotopy analytical method, for solving nonlinear differential equations with fractional ranks. We used the mean squared error as an appropriate function in the algorithm and a criterion to determine the improvement of results, such as the exact solution.

5 – The final results of the methods (HAM-OBE), (HAM-PSO) and homotopy (HAM) are compared with the exact solution using the error ratio (RMSE).

In this research, we solve fractional-order initial value problems of the (Riccati) type (Because the Riccati equation was solved by the homotopy method and the results of the homotopy method were improved by the bird flock algorithm (HAM - PSO), therefore the Riccati equation was improved by Range-Kutta algorithm) using the homotopy Analysis Method (HAM) with the Runge-Kutta Optimization Algorithm (HAM-OBE). The (HAM-OBE) selects the best value for the parameter (h), thereby improving the (HAM) and (HAM-PSO) results.

We have proven this by taking two examples and comparing the mean squared error results of the (HAM), (HAM-PSO) with the (HAM-OBE). It has also been shown that the (HAM-OBE) method is reliable, efficient, and better than previous methods through plotting using MATLAB software.

2. General Concepts

This section will we will present the basic definitions and concepts of the current research topic, as well as the algorithm for solving the homotopy method and the (HAM-OBE) algorithm.

2.1 Definition 1 [14]. The Gamma function is given in the following form:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}\ e^{-t}dt \quad (1)$$

2.2 Definition 2 [15]. Fractional integral operator I^{α} of order α for the Riemann-Liouvills is given as

$$\boldsymbol{I^{\alpha}\, f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1} f(u)du\,,\ \ x > 0, \alpha > 0} \quad (2)$$

The operator $\boldsymbol{I^{\alpha}}$ has the following properties.

1. $\boldsymbol{I^{\alpha}x^n = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\, x^{n+\alpha}}$ (3)
2. $\boldsymbol{I^n I^{\alpha} f(x) = I^{\alpha+n} f(x)}$ (4)
3. $\boldsymbol{I^n I^{\alpha} f(x) = I^{\alpha} I^n f(x)}$ (5)

2.3 Definition 3 [16]. The following formula determines the maximum absolute errors MSE.

MSE=Max {|$\boldsymbol{exact}$ − Q|}, Q is a numerical solution [16] (6)

2.4 Definition 4 [17]. The fractional derivative for the Caputo definition is given as:

$$\boldsymbol{D^{\alpha} f(x) = I^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-u)^{n-\alpha-1} f^n(u)du} \quad (7)$$

Where $\boldsymbol{n-1} < \boldsymbol{\alpha} \le \boldsymbol{n}$, $\boldsymbol{n} \in \boldsymbol{N}$, $\boldsymbol{x} > \boldsymbol{0}$, It has main properties

$$\boldsymbol{D^{\alpha}} I^{\alpha} f(x) = f(x) \quad (8)$$

$$\boldsymbol{I^{\alpha}} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{x^k}{k!} \quad (9)$$

$$D^{\alpha}x^{r}=\frac{\Gamma(r+1)}{(r+1-\alpha)}x^{r-\alpha} \quad , \tag{10}$$

$$D^{\alpha}x^{r}=0 \quad , \quad r<\alpha \tag{11}$$

2.5 Definition 5. [18]. RMSE It is the root of the square of the difference between the exact solution and the approximate solution divided by the number of points used, and the formula is as follows:

$$\text{RMSE}=\sqrt{\frac{\sum_{i=1}^{n}(exact-Q)^2}{n}}, \ Q \text{ is the approximate solution}, n \text{ is the number of points} \tag{12}$$

3. Steps for solving nonlinear differential equations of fractional orders using the homotopy method:

Let us consider the following initial value problem

$$D_x^{\alpha}y+cy=f(x) \quad , x>0 \, , n-1<\alpha<n \tag{13}$$

With the initial conditions

$$y^{(k)}(x)=b_k \ , k=0,1,\dots,n-1$$

Where D^{α} , $(n\in N), n-1<\alpha\le n$, denotes the Caputo fractional derivative, C is a positive constant, $f(x)$ is a known continuous and differentiable function.

Step one: We integrate both sides of equation (**13**) using the Riemann-Liouville integral and substituting the initial conditions; we get:

$$y(x)=\sum_{k=0}^{n-1}y^{(k)}(0^{+})\frac{x^k}{k!}-I^{\alpha}(cy)+I^{\alpha}f(x) \tag{14}$$

Then we choose the initial approximation (initial guess) to be

$$y_0(x)=\sum_{k=0}^{n-1}y^{(k)}(0^{+})\frac{x^k}{k!} \tag{15}$$

Step two: Now we use the m-th order deformation equation

$$D_x^{\alpha}[y_m(x)-\chi_m y_{m-1}(x)]=hR_{m-1}(\vec{y}_{m-1},x) \tag{16}$$

With conditions $y_m(0)=0$ since $n\epsilon N$, $n-1<\alpha\le n$ we get the following iteration formula

$$D_x^{\alpha}y_m(x)=\chi_m D^{\alpha}y_{m-1}(x)+h\,R_{m-1}(\vec{y}_{m-1}(x)) \tag{17}$$

When

$$R_{m-1}(\vec{y}_{m-1}(x))=D_x^{\alpha}y_{m-1}(x)+Cy_{m-1}(x)-f(x)(1-x_m) \tag{18}$$

And also $\chi_m=\begin{cases}0, & m\le 1\\ 1, & othrewise\end{cases}$ (19)

To find the approximate iterations $\dots y_1(x), y_2(x), y_3(x), \dots$ we integrate both sides of equation (17) using the Riemann-Liouville integral. We obtain:

$$y_m(x)=\chi_m y_{m-1}(x)+hI^{\alpha}\left[R_{m-1}(\vec{y}_{m-1}(x))\right] \tag{20}$$

Now, by substituting the values m=1,2, 3, In to equation (20), we obtain the approximate iteration as follows:

$$y_1(x)=h\,I^{\alpha}(D_x^{\alpha}y_0(x)+Cy_0(x)-f(x)(1-\chi_m))$$

$$y_2(x)=\chi_2 y_1(x)+hI^{\alpha}(D_x^{\alpha}y_1(x)+Cy_1(x))$$

$$y_3(x)=\chi_3 y_2(x)+hI^{\alpha}(D_x^{\alpha}y_2(x)+Cy_2(x)$$

$$\vdots$$

$$y_m(x)=\chi_m y_{m-1}(x)+hI^{\alpha}\left(D_x^{\alpha}y_{m-1}(x)+Cy_{m-1}(x)\right) \tag{21}$$

Step three: finally, to obtain an approximate solution for equation (13), We sum the approximate frequencies $y_0(x), y_1(x), y_2(x), y_3(x), \dots, y_m(t)$,in the following form:

$$Q_{m-1}(x)=\sum_{s=0}^{m}y_s(x)$$

4. Runge-Kutta algorithm for improvement (HAM-OBE):

It is a scientific algorithm designed to improve the ability to solve nonlinear fractional-order differential equations and produce results that are very close or sometimes identical to the exact solution. Moreover, it is one of the latest developments in solving fractional-order differential equations.

5. **Numerical Examples**:

Example 1: Let the fractional-order nonlinear differential equation be

(Ali etal.m.,2017)

$$D_x^{\alpha} y(x) + y(x) - y^2(x) = 0\,, \qquad 0 < \alpha \le 1 \tag{22}$$

Let the initial condition be $y(0) = \frac{1}{2}$, and the exact solution is $y(x) = \frac{e^{-x}}{(1+e^{-x})}$

Solution:

We choose the initial guess according to equation (15) and we get:

$y(0) = \frac{1}{2}$

To find the three approximate iterations following the initial guess we use the deformation equation (20) of order m

$$y_m(x) = \chi_m y_{m-1}(x) + hI^{\alpha}[R_{M-1}(\vec{y}_{m-1}(x))] \tag{23}$$

Where

$$R_{m-1}(\vec{y}_{m-1}(x)) = D_x^{\alpha} y(x) + y(x) - \sum_{i=0}^{n} y_i(x) y_{n-i}(x), n \ge i, \tag{24}$$

Substituting m = 1 in the equation (23) we get

$$y_1(x) = \chi_1 y_0(x) + hI^{\alpha}[D_x^{\alpha} y_0(x) + y_0(x) - y_0^2(x)]\ , \qquad x_1 = 0$$

$$y_1(x) = hI^{\alpha}[\, D_x^{\alpha} \frac{1}{2} + \frac{1}{2} - \left(\frac{1}{2}\right)^2]$$

$$y_1(x) = \frac{hx^{\alpha}}{4\Gamma(\alpha+1)}$$

When m=2,3 the equation (23) becomes as follows:

$$y_m(x) = \chi_m y_{m-1}(x) + h[y_{m-1}(x) + I^{\alpha} y_{m-1}(x) - I^{\alpha} \sum_{i=0}^{n} y_i(x) y_{n-i}(x)], \tag{25}$$

Where m = 1,2, 3, …

Substituting m = 2 into equation (25) and using the properties of the Riemann–Liouville integral we get

$$y_2(x) = \chi_2 y_1(x) + h[y_1(x) + I^{\alpha} y_1(x) - I^{\alpha} y_1^2(x)] \quad , \qquad \chi_2 = 1$$

$$y_2(x) = y_1(x) + h[y_1(x) + I^{\alpha} y_1(x) - I^{\alpha}(2y_0(x) y_1(x)]$$

$$y_2(x) = \frac{hx^{\alpha}}{4\Gamma(\alpha+1)} + \frac{h^2 x^{\alpha}}{4\Gamma(\alpha+1)}$$

$$y_2(x) = (1+h)\frac{hx^{\alpha}}{4\Gamma(\alpha+1)}$$

Substituting m = 3 into equation (25) and using the properties of the Riemann-Liouville integral we get

$$y_3(x) = \chi_3 y_2(x) + h[y_2(x) + I^{\alpha} y_2(x) - I^{\alpha} y_2^2(x)] \quad , \chi_3 = 1$$

$$y_3(x) = y_2(x) + h[y_2(x) + I^{\alpha} y_2(x) - I^{\alpha}(2y_0(x) y_2(x) + y_1^2(x))]$$

$$y_3(x) = (1+h)^2 \frac{hx^{\alpha}}{4\Gamma(\alpha+1)} - \frac{h^3 x^{3\alpha}\Gamma(2\alpha+1)}{16\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}$$

We add the four approximate iterations $y_0(x), y_1(x), y_2(x), y_3(x)$ and get the solution

Approximate

$$Q_{m+1}(x) = \sum_{s=0}^{m} y_s(x)$$

$$= = \frac{1}{2} + \frac{hx^{\alpha}}{4\Gamma(\alpha+1)} + (1+h)\frac{hx^{\alpha}}{4\Gamma(\alpha+1)} + (1+h)^2\frac{hx^{\alpha}}{4\Gamma(\alpha+1)} - \frac{h^3 x^{3\alpha}\Gamma(2\alpha+1)}{16(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}$$

$$= \frac{1}{2} + (h^2 + 3h + 3)\frac{hx^{\alpha}}{4\Gamma(\alpha+1)} - \frac{h^3 x^{3\alpha}\Gamma(2\alpha+1)}{16(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} \tag{25}$$

We find the mean square error (RMSE) for equation (22) where RMSE=$\sqrt{\frac{\sum_{i=1}^{n}(exact-Q)^2}{n}}$

(Q) It is the approximate solution and (n) is the number of points and the square error is considered as a fitness function in the Runge-kutta algorithm. To improve, we then apply a set of algorithms to get the best value for the parameter h.
We substitute the best value for h and different values for α according to a specific period as follows:
Table (1) shows us When α= 0.75, $h_1 = -0.4921$, $h_2 = -0.40205$ for the two algorithms (HAM-OBE), (HAM-PSO) and by taking different values for x and substituting them in the solution we get the following table for example (1)

Table (1)

x	Exact solution	HAM h =-1	HAM-PS0 h_1 =-0.4921	HAM-OBE h_2=-0.40205
0	0.5000	0.5000	o.5000	0.5000
0.1	0.4750	0.4518	0.4580	0.4620
0.2	0.4502	0.4197	0.4294	0.4361
0.3	0.4256	0.3923	0.4045	0.4135
0.4	0.4013	0.3681	0.3817	0.3928
0.5	0.3775	0.3464	0.3604	0.3734
0.6	0.3543	0.3268	0.3403	0.3550
0.7	0.3318	0.3091	0.3212	0.3375
0.8	0.3100	0.2933	0.3028	0.3206
0.9	0.2891	0.2791	0.2852	0.3044
1.0	0.2689	0.2666	0.2682	0.2886

Table (2) shows the improvement of the root mean square error ratio using the Runge- Kutta(HAM-OBE) algorithm to improve the results of the homotopy analytical method and the (HAM-PSO) algorithm to solve the Riccati equation for example (1) when α=0.75

Table (2)

	RMSE
h in Classical Method	7.29e-02
Optimal h by HAM-PSO (-0.4921)	4.17e-02
Optimal h by HAM-OBE (-0.40205)	4.6835e-07

Table (3) shows us When α= 0.85, $h_1 = -0.5176$, $h_2 = -0.46999$ for the two methods (HAM-OBE), (HAM-PSO) and by taking different values for x and substituting them in the solution we get the following table for example (1)

Table (3)

x	Exact solution	HAM h =-1	HAM-PS0 h_1=-0.5176	HAM-OBE h_2=-0.46999
0	0.5000	0.5000	o.5000	0.5000
0.1	0.4750	0.4627	0.4669	0.4682
0.2	0.4502	0.4332	0.4403	0.4428
0.3	0.4256	0.4064	0.4159	0.4193

x	Exact solution	HAM h =-1	HAM-PS0 h_1=-0.5176	HAM-OBE h_2=-0.46999
0.4	0.4013	0.3816	0.3927	0.3970
0.5	0.3775	0.3586	0.3705	0.3757
0.6	0.3543	0.3371	0.3491	0.3551
0.7	0.3318	0.3171	0.3284	0.3351
0.8	0.3100	0.2987	0.3083	0.3157
0.9	0.2891	0.2818	0.2887	0.2967
1.0	0.2689	0.2664	0.2696	0.2782

Table (4) shows the improvement of the root mean square error ratio using the Runge -Kutta algorithm to improve the results of the homotopy analytical method and the (HAM-PSO) algorithm to solve the Riccati equation for example (1) when α=0.85

Table (4)

	RMSE
h in Classical Method	4.43e-02
Optimal h by HAM-PSO (-0.5176)	1.69e-02
Optimal h by HAM-OBE (-0.46999)	9.6897e-07

Table (5) shows us when α= 0.95, $h_1 = -0.5904$ $h_2 = -0.5826$ for the two algorithms (HAM-OBE), (HAM-PSO) and by taking different values for x and substituting them in the solution we get the following table for example (1)

Table (5)

x	Exact solution	HAM h =-1	HAM-PS0 h_1=-0.5904	HAM-OBE h_2=-0.5826
0	0.5000	0.5000	o.5000	0.5000
0.1	0.4750	0.4714	0.4733	0.4735
0.2	0.4502	0.4449	0.4485	0.4488
0.3	0.4256	0.4195	0.4245	0.4248
0.4	0.4013	0.3949	0.4009	0.4013
0.5	0.3775	0.3712	0.3777	0.3782
0.6	0.3543	0.3485	0.3549	0.3555
0.7	0.3318	0.3268	0.3325	0.3331
0.8	0.3100	0.3062	0.3104	0.3111
0.9	0.2891	0.2868	0.2887	0.2894
1.0	0.2689	0.2687	0.2673	0.2681

Table (6) shows the improvement of the root mean square error ratio using the Runge-Kutta (HAM-OBE) algorithm to improve the results of the homotopy analytical method and the (HAM-PSO) algorithm to solve the Riccati equation for example (1) when α=0.95

Table (6)

	RMSE
h in Classical Method	1.41e-02
Optimal h by HAM-PSO (-0.5904)	1.6e-03
Optimal h by HAM-OBE (-0.5826)	8.0481e-07

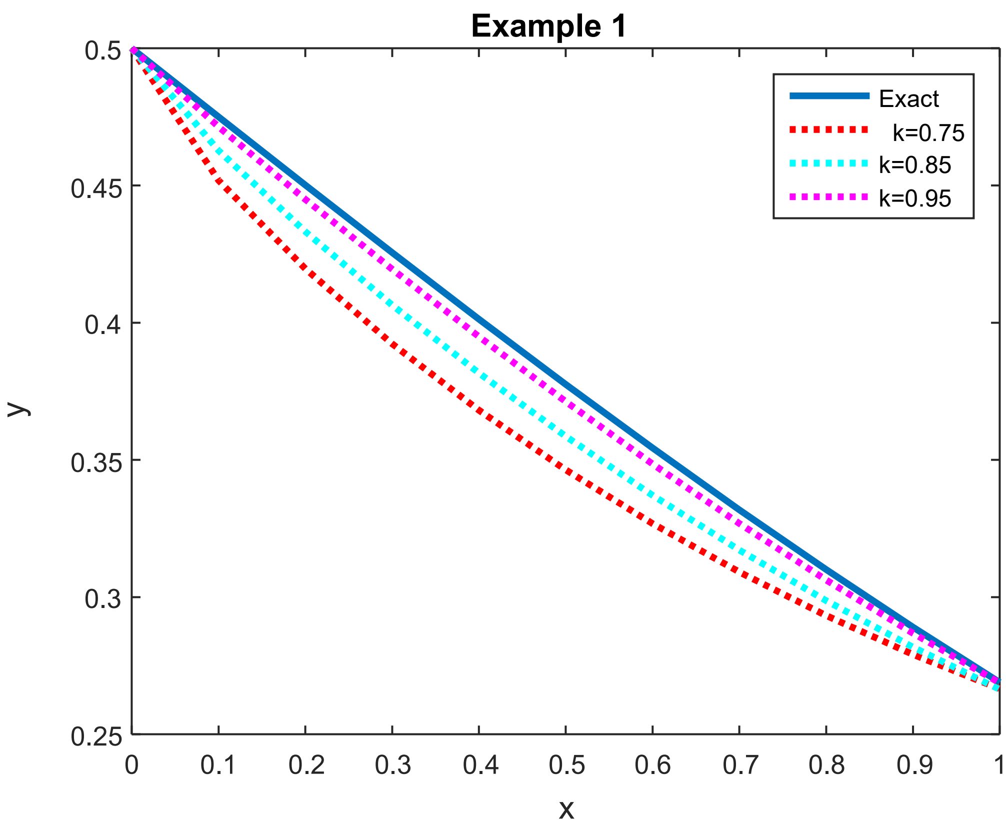

Figure (1) shows the convergence of the solution using the homotopy analytical method for the exact solution with a change in the value of k, where k is the α fractional order of the nonlinear differential equation for example (1).

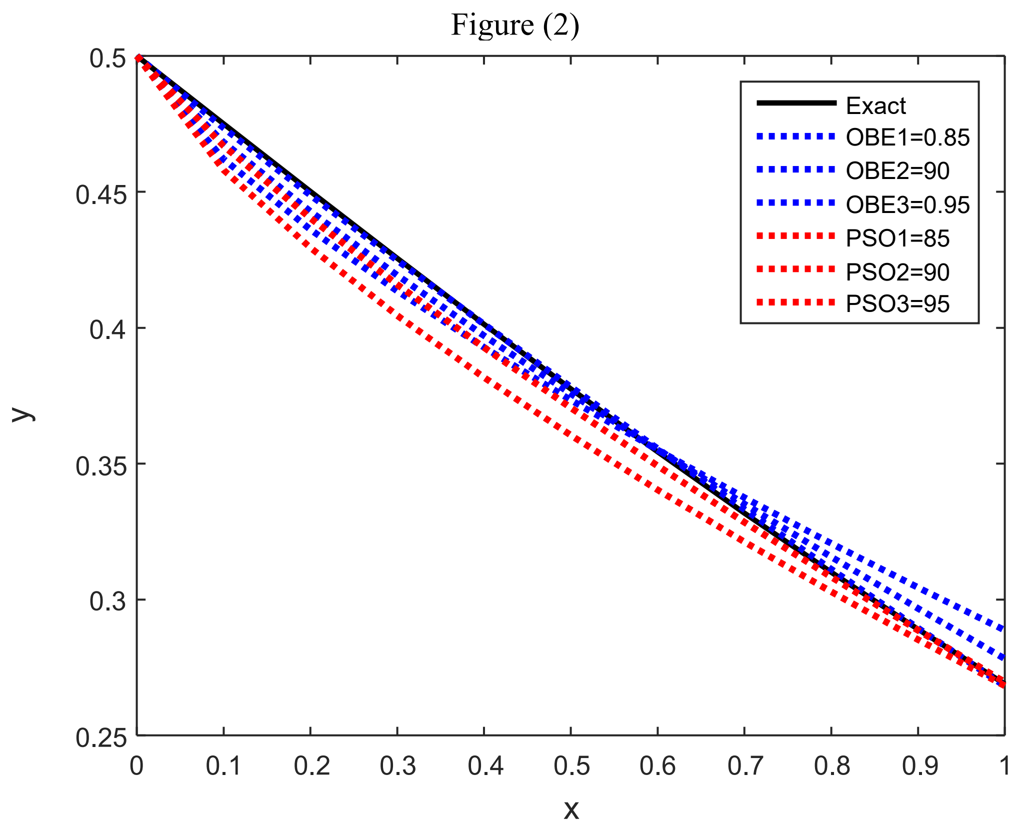

Figure (2) shows the convergence of the approximate solution using the Runge-Kutta optimization algorithm (HAM-OBE) and (HAM-PSO) to the exact solution with the change in the value of k, which is α in the differential equation for example (1)

Example 2: Let the fractional-order nonlinear differential equation be
(Yang and Hou,2013)

$$D^{\alpha}y(x) = 1 + y^2(x) \ , \quad n-1 < \alpha < n \quad , n \in N \tag{26}$$

Let the initial condition be $y^{(r)}(0) = 0$, $r = 0,1,2,m-1$, and the exact solution is
$y(x) = \tan(x)$

Solution:

We choose the initial guess according to equation (15) and we get:

$$y_0(x) = 0 \tag{27}$$

To find the three approximate iterations following the initial guess we use the deformation equation (20) of order m

$$y_m(x) = \chi_m y_{m-1}(x) + hI^{\alpha}[R_{M-1}(\vec{y}_{m-1}(x))] \tag{28}$$

Where

$$R_{m-1}(\vec{y}_{m-1}(x)) = D_x^{\alpha} y_{m-1}(x) - y_{m-1}^2(x) - 1(1-\chi_m) \tag{29}$$

Substituting m = 1 in the equation (28) we get

$$y_1(x) = \chi_1 y_0(x) + hI^{\alpha}[D_x^{\alpha} y_0(x) - y_0^2(x) - 1] \ , \qquad \chi_1 = 0$$

$$y_1(x) = hI^{\alpha}[\,D_x^{\alpha}(0) - (0)^2 - 1]$$

$$y_1(x) = \frac{-hx^{\alpha}}{\Gamma(\alpha+1)}$$

When m=2,3 the equation (28) becomes as follows:

$$y_m(x) = \chi_m y_{m-1}(x) + h[y_{m-1}(x) - I^{\alpha} y_{m-1}^2(x)] \tag{30}$$

Substituting m = 2 into equation (30) and using the properties of the Riemann–Liouville integral we get

$$y_2(x) = \chi_2 y_1(x) + h[y_1(x) - I^{\alpha} y_1^2(x)] \quad , \quad \chi_2 = 1$$

$$y_2(x) = y_1(x) + h[y_1(x) - I^{\alpha}(2y_0(x)y_1(x))]$$

$y_2(x) = y_1(x) + h[y_1(x) - I^\alpha(2(0)y_1(x))]$

$y_2(x) = -(1+h)\dfrac{hx^\alpha}{\Gamma(\alpha+1)}$

Substituting m = 3 into equation (30) and using the properties of the Riemann–Liouville integral we get

$y_3(x) = \chi_3 y_2(x) + h[y_2(x) - I^\alpha y_2^2(x)] \quad ,\chi_3 = 1$

$y_3(x) = y_2(x) + h[y_2(x) - I^\alpha(2y_0(x)y_2(x) + y_1^2(x))]$

$y_3(x) = \frac{-(1+h)^2 h\, x^\alpha}{\Gamma(\alpha+1)} - \frac{h^3\Gamma(2\alpha+1)x^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}$

We add the four approximate iterations $y_0(x), y_1(x), y_2(x), y_3(x)$ and get the solution Approximate

$$Q_{m+1}(x) = \sum_{s=0}^{m} y_s(x)$$

$$=0 - \frac{hx^\alpha}{\Gamma(\alpha+1)} - (1+h)\frac{hx^\alpha}{\Gamma(\alpha+1)} - (1+h)^2\frac{hx^\alpha}{\Gamma(\alpha+1)} - \frac{h^3\Gamma(2\alpha+1)x^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}$$

=

$$= -(h^2+3h+3)\frac{hx^\alpha}{\Gamma(\alpha+1)} - \frac{h^3\Gamma(2\alpha+1)x^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} \qquad (30)$$

We find the mean square error (RMSE) for equation (26) where RMSE=$\sqrt{\frac{\sum_{i=1}^{n}(exact-Q)^2}{n}}$

(Q) It is the approximate solution and (n) is the number of points and the square error is considered as a fitness function in the Runge-kutta algorithm. To improve, we then apply a set of algorithms to get the best value for the parameter h.

We substitute the best value for h and different values for α according to a specific period as follows:

Table (7) shows us when α= 0.85, $h_1 = -0.3840$, $h_2 = -0.38742$ for the two algorithms (HAM-OBE), (HAM-PSO) and by taking different values for x and substituting them in the solution we get the following table for example (2)

Table (7)

x	Exact solution	HAM h =-1	HAM-PS0 h_1 =-0.3840	HAM-OBE h_2=-0.38742
0	0	0	0	0
0.1	0.1003	0.1508	0.1292	0.1297
0.2	0.2027	0.2774	0.2382	0.2392
0.3	0.3093	0.4029	0.3471	0.3487
0.4	0.4228	0.5329	0.4607	0.4632
0.5	0.5463	0.6707	0.5822	0.5858
0.6	0.6841	0.8187	0.7137	0.7186
0.7	0.8423	0.9790	0.8517	0.8637
0.8	1.0269	1.1532	1.0140	1.0225
0.9	1.2602	1.3428	1.1857	1.1965
1.0	1.5574	1.5492	103736	1.3871

Table (8) shows the improvement of the root mean square error ratio using the Runge- Kutta algorithm (HAM-OBE) to improve the results of the homotopy analytical method and the (HAM-PSO) algorithm to solve the Riccati equation for example (2) when α=0.85

Table (8)

	RMSE
h in Classical Method	2.916e-01
Optimal h by HAM-PSO (-0.3840)	1.69e-02
Optimal h by HAM-OBE (-0.38742)	3.6210e-06

Table (9) shows us when α= 0.90, $h_1 = -0.3841, h_2 = -0.38408$ for the two algorithms(HAM- OBE), (HA-MPSO) and by taking different values for x and substituting them in the solution we get the following table for example (2)

Table (9)

x	Exact solution	HAM h =-1	HAM-PS0 h_1=-0.3841	HAM-OBE h_2=-0.38408
0	0	0	0	0
0.1	0.1003	0.1318	0.1132	0.1132
0.2	0.2027	0.2499	0.2165	0.2165
0.3	0.3093	0.3687	0.3232	0.3232
0.4	0.4228	0.4924	0.4380	0.4380
0.5	0.5463	0.6241	0.5643	0.5643
0.6	0.6841	0.7660	0.7049	0.7049
0.7	0.8423	0.9201	0.8623	0.8623
0.8	1.0296	1.0885	1.0388	1.0388
0.9	1.2602	1.2727	1.2366	1.2365
1.0	1.5574	1.4743	1.4576	1.4575

Table (10) shows the improvement of the root mean square error ratio using the Runge -Kutta algorithm (HAM-OBE) to improve the results of the homotopy analytical method and the (HAM-PSO) algorithm to solve the Riccati equation for example (2) when α=0.90

Table (10)

	RMSE
h in Classical Method	1.370e-01
Optimal h by HAM-PSO (-0.3841)	1.1084e-04
Optimal h by HAM- OBE (-0.38408)	3.0259e-06

Table (11) shows us when α= 0.95, h_1=-0.3769 , $h_2 = -0.37688$ for the two algorithms (HAM-OBE), (HAM-PSO) and by taking different values for x and substituting them in the solution we get the following table for example (2)

Table (11)

x	Exact solution	HAM h =-1	HAM-PS0 h_1=-0.3769	HAM-OBE h_2=-0.37688
0	0	0	0	0
0.1	0.1003	0.1150	0.0982	0.0982
0.2	0.2027	0.2251	0.1947	0.1946
0.3	0.3093	0.3375	0.2977	0.2977
0.4	0.4228	0.4554	0.4119	0.4118
0.5	0.5463	0.5812	0.5410	0.5410
0.6	0.6841	0.7172	0.6888	0.6888
0.7	0.8423	0.8654	0.8587	0.8586
0.8	1.0296	1.0277	1.0539	1.0538
0.9	1.2602	1.2060	1.2776	1.2775
1.0	1.5574	1.4023	1.5330	1.5329

Table (12) shows the improvement of the root mean square error ratio using the Runge -Kutta algorithm (HAM-OBE) to improve the results of the homotopy analytical method and the (HAM-PSO) algorithm to solve the Riccati equation for example (2) when α=0.95

Table (12)

	RMSE
h in Classical Method	7.1e-03
Optimal h by HAM-PSO (-0.3769)	9.4974e-05
Optimal h by HAM-OBE (-0.37688)	2.4390e-05

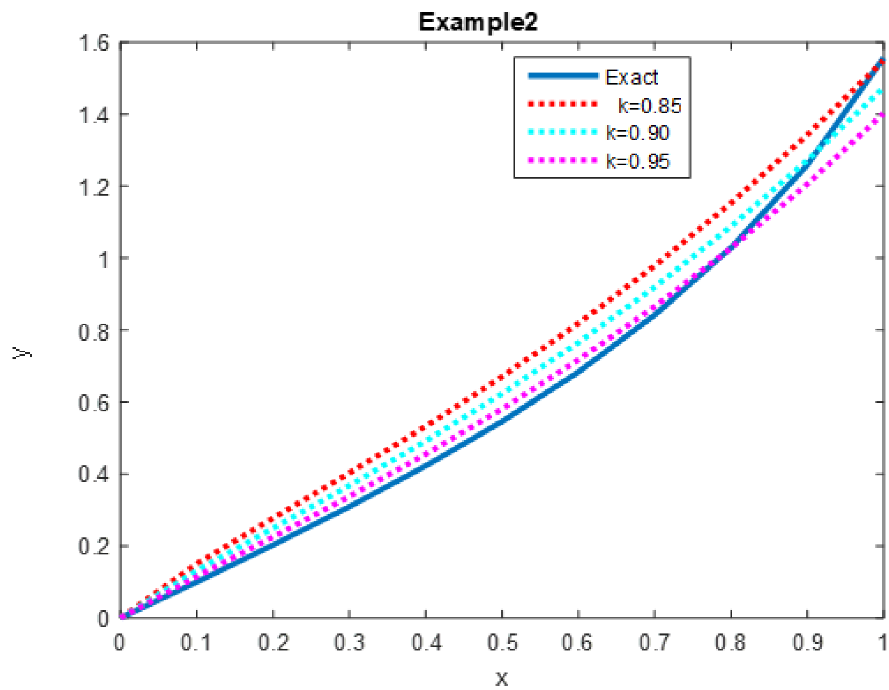

Figure (3) shows the convergence of the solution using the homotopy analytical method for the exact solution with a change in the value of k, where k is the α fractional order of the nonlinear differential equation for example (2).

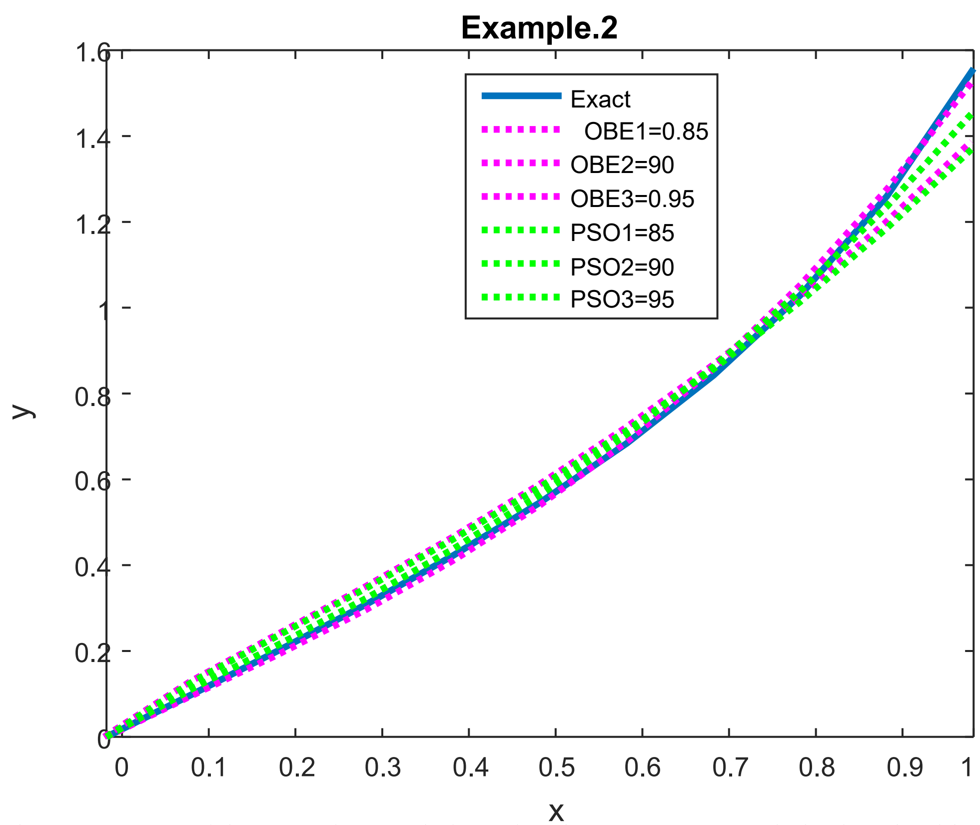

Figure (4) shows the convergence of the approximate solution using the Runge-Kutta optimization algorithm (HAM-OBE) and (HAM-PSO) to the exact solution with the change in the value of k, which is α in the differential equation for example (2)

6. Conclusion

In this research, the analytical method of homotopy was presented to find a series of approximate solutions that give the solution with acceptable accuracy and an error rate that must be reduced due to its importance in scientific calculations that require the solution of nonlinear ordinary differential equations with fractional orders. Therefore, an algorithm was presented to reduce the error rate and improve the results, which is the second-order Runge-Kutta algorithm to improve the solution (HAM-OBE). Examples were solved and the root mean square error (RMSE) was used as a measure to compare the approximate solution of the homotopy method (HAM), the particle swarm optimization algorithm (HAM-PSO), and the second-order Runge-Kutta algorithm (HAM-OBE) with the exact solution. It was shown that the (HAM-OBE) algorithm reaches the best value for the parameter affecting the accuracy of the solution, and with this new value of the parameter h, the accuracy of the solution increased by reducing the error rate. Thus, we can conclude. The homotopy analytical method gave an acceptable solution, but many mathematical, physical and other problems need more accuracy as a result of the progress of science. Many algorithms were proposed to improve the solution, including the (HAM-OBE) algorithm, which made the solution more accurate and closer to the exact solution, as in the following table, which summarizes the results of improving the solutions of the homotopy method and the bird flock algorithm (HAM-PSO) by the Runge-Kutta algorithm (HAM-OBE)
In Table (13), we have noticed that the Rangi-Kuna algorithm has been improved.
The error rate compared to the bird flock algorithm is about 95% as in Table (13).

Table (13)

		RMSE		
Examples	**Value of α**	***h* in Classical Method**	**Optimal *h* by HAM-PSO**	**Optimal *h* by HAM-OBE**
1	α=0.75	7.29e-02	4.17e-02	4.6835e-07
	α=0.85	4.43e-02	1.69e-02	9.6897e-07
	α=0.95	1.41e-02	1.6e-03	8.0481e-07
2	α=0.85	2.916e-01	1.69e-02	3.6210e-06
	α=0.90	1.370e-01	1.1084e-04	3.0259e-06
	α=0.95	7.1e-03	9.4974e-05	2.4390e-05

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9. Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication and/or funding of this manuscript.

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تحسين الطريقة التحليلية للتقارب باستخدام خوارزمية تحسين رونج-كوتا لحل المعادلات التفاضلية غير الخطية من الدرجة الكسرية

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المستخلص:

المعادلات التفاضلية هي معادلات تربط دالة بواحدة أو أكثر من مشتقاتها. غير خطية تعني أن المعادلة ليست متناسبة طرديا مع الدالة ومشتقاتها. بمعنى آخر، لا يمكن فصل متغيراتها ببساطة. المتغير التابع ومشتقاته أسّي، أي ليس من الدرجة الأولى. يشير الترتيب الكسري إلى أن المشتقات في المعادلة ليست بالضرورة من رتبة عدد صحيح (مثل المشتقة الأولى أو الثانية) يمكن أن تكون كسرية، أي أن الترتيب هو عدد كسري. لحل المعادلات التفاضلية الاعتيادية غير الخطية ذات الرتب الكسرية استخدمت طريقة هوموتوبي التحليلية لان حل هذه المعادلات ليس سهلا بالطرائق الاعتيادية المتعارف عليها .ولكن نتائج طريقة هوموتوبي لم تكن بالدقة المطلوبة لذلك استخدمت خوارزميات لتحسين نتائج طريقة هوموتوبي ومن هذه الخوارزميات خوارزمية سرب الطيور (HAM-PSO) وفي هذا البحث تم استخدام خوارزمية رونجي-كوتا (HAM-OBE) لتحسين نتائج طريقة هوموتوبي ومن خلال اول مثال في البحث نلاحظ ان خوارزمية سرب الطيور (HAM-PSO) حسنت نتائج طريقة هوموتوبي بمقدار خطاء يساوي 4.17e-02 اما خوارزمية رونجي-كوتا (HAM-OBE) حسنت النتائج بمقدار خطاء 4.6835e-07 وبمقارنة مقدار الخطائين نجد ان خوارزمية رونجي-كوتا (HAM-OBE)حسنت النتائج

بمقدار 99.99% قياسا للحل المضبوط. حيث استخدم الجذر التربيعي لمربع متوسط الخطاء (RMSE) كدالة لياقة لمعرفة مقدار التحسين قياسا للحل المضبوط وهذا التحسين مفيد في الحياة لان المعادلات التفاضلية غير الخطية ذات الرتب الكسرية مفيدة في علوم الفيزياء والكيمياء والصناعات الطبية وحركة الاجسام والذكاء الصناعي ولها تطبيقات حياتية مثل دراسة الموجات الصوتية وحركة الاجسام من خلال المشتقات ذات الرتب الكسرية.