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RESEARCH ARTICLE

Numerical Solution of Distributed Order Fractional Differential Equations Using Spectral Mittag-Leffler Weight Function Based on Chelyshkov Polynomials

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ABSTRACT

In many studies, spectral methods based on one of the orthogonal polynomials and weighted residual methods (WRMs) have been used to convert the distributed-order fractional differential equation (DOFDEs) into a system of linear or nonlinear algebraic equations, and then solve this system to obtain the approximate solution. In this paper, a numerical method is presented for solving DOFDEs. The approximate solution is imposed as an orthogonal Chelyshkov polynomial with unknown coefficients. The required coefficients are obtained using WRMs, which transform the DOFDEs into a system of algebraic coefficients. The Mittag-Leffler function is proposed as a suitable weight function. The method has been applied to several numerical examples, such as oscillatory mathematical model, and the distributed order fractional Bagley-Torvik equation. Acceptable results were obtained in most tests. The proposed Mittag-Leffler weight method is compared with the WRMs such as Galerkin method, Petrov-Galerkin method and least square method, and the proposed weighted function showed more accurate results than the previous methods in most tests. The study showed that the effect of the test polynomials such as Chebyshev, Jacobi, Legendre, Gegenbauer, Hermite, Taylor, Mittag-Leffler, and Bernstein polynomials has a small impact on most tests. In addition, the impact of the distributed order on the accuracy of the solution was studied, and the results show that the distributed order has a strong impact on the accuracy of the solution, as its impact is direct on the non-homogeneous part, which leads to more complex equations than in cases where the orders are fixed.

Keywords: Chelyshkov polynomials, Distributed order fractional Bagley-Torvik equation, Distributed order fractional derivative, Mittag-Leffler weight method, Spectral method, Weighted residual method

Introduction

Distributed order fractional differential equations (DOFDEs) were first introduced in 1995 by Caputo in his study describing dissipative elastic dynamics. This concept has gained widespread due to its nature, which is characterized by presenting models that are more general than the models presented in fixed-order differentiation. One of the most important applications of DOFDEs is the study of dissipation and decay within viscoelastic solids that have multiple relaxation times, as it explains the amount of deformation within the established models of stress and strain.[1,2]

The numerical solution is considered one of the most important alternative methods to obtain a solution that is close to the analytical solution. One of the numerical methods is to reduce the residual error function to zero. These methods are known as spectral-weighted residual methods (WRMs). The WRMs are depending on the calculus of variation in finding the minimum approximation.[3] Orthogonal

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polynomials are often chosen due to two properties: the first is that they are analytical in wide fields and the second is that the orthogonality property reduces and facilitates mathematical operations.[4]

A spectral collocation method (SCM) based on Jacobi, Chebyshev, Legendre, and shifted Legendre polynomials is presented to solve DOFDEs.[5–7] An operational matrix based on Legendre wavelets is provided to solve linear and nonlinear DOFDEs.[8] A SCM based on fractional Chelyshkov wavelets is presented to solve DOFDEs.[9] A method based on rational Fibonacci functions is proposed to solve DOFDEs.[10] A numerical method based on hybrid Hahn functions is presented to solve the Black-Scholes option pricing partial differential equation with distributed time order.[11]

In this work, a group of WRMs will be applied instead of the clustering method, such as the Galerkin method, the least squares method, the subdomain method, and the momentum method. Moreover, a comparison between these methods is made. The Mittag-Leffler weight function is proposed and compared to other weighted methods, which will be important in future research in this field.

In addition, Chelyshkov polynomials are used, and the results obtained are compared with the polynomials most used by researchers, whether they are orthogonal or non-orthogonal polynomials, such as Jacobi, Legendre, Chebyshev of the first and second kind, Hermite, and Gegenbauer, Laguerre, Bernstein, Taylor, and Mittag-Leffler. Due to the nature of the initial and boundary conditions in DOFDEs, which contain integer orders, it is appropriate to use the Caputo derivative within the distributed order fractional derivative.

In Section 2, the basic concepts of the research are presented. In section 3, the proposed method is presented. Section 4 involved a set of numerical examples and finally the conclusions.

Fundamental concept

In this section, the concept of distributed-order differentiation is presented, followed by the Chelyshkov polynomial, and finally, a simplified concept of weighted residual methods is given.

Fractional and distributed order fractional derivatives

Definition 1: Let $u(t)$ be a continuous function and differentiable n times on $[a,b]$, and suppose $\alpha > 0$, the Caputo fractional derivative is defined as:[12]

$$ {}_{0}^{C}D_{t}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} u(s)\, ds \tag{1} $$

where $n-1 < \alpha < n$. If $u(t) = t^p$ then,

$$ {}_{0}^{C}D_{t}^{\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, \quad p \geq n \tag{2} $$

In Caputo's concept, the initial conditions in the FDEs must be of the integer order. Furthermore, the derivative of the constant function is always zero.[13]

Definition 2: The direct definition of distributed order fractional derivative in the sense of Caputo is presented as:[14]

$$ {}_{k_1,k_2}^{\quad C}\mathbb{D}^{p(\alpha)} u(t) = \int_{k_1}^{k_2} p(\alpha)\, {}_{0}^{C}D_{t}^{\alpha}\, u(t) d\alpha, \tag{3} $$

where $p(\alpha)$ is the distributed order defined on the support $[k_1, k_2]$.

Suppose $p(\alpha) = \sum_{i=1}^{\infty} a_i \delta(\alpha - \alpha_i)$ then Eq. (3) is reduced to an infinite multi-term fractional derivative i.e.

$$ {}_{k_1,k_2}^{\quad C}\mathbb{D}^{p(\alpha)} u(t) = \sum_{i=1}^{\infty} a_i\, {}_{a}^{C}D_{t}^{\alpha_i}\, u(t), \tag{4} $$

where $k_1 \leq \alpha_i \leq k_2, a_i$ are constants for all $i = 1, 2, \ldots, N$, also this derivative can be written in discrete form and approximated as follows:

$$ {}_{k_1,k_2}^{\quad C}\mathbb{D}^{p(\alpha)} u(t) \cong \sum_{i=1}^{N} w_i\, p(\alpha_i)\, {}_{a}^{C}D_{t}^{\alpha_i}\, u(t), \tag{5} $$

where, w_i is the weights which are obtained from numerical integration.

For special cases the distributed order fractional derivative of the function $u(t) = t^p$ is computed as follows using Eq. (2) and Eq. (5):

$$ {}_{k_1,k_2}^{\quad C}\mathbb{D}^{p(\alpha)} u(t) = \sum_{i=1}^{N} w_i\, p(\alpha_i) \frac{\Gamma(p+1)}{\Gamma(p-\alpha_i+1)} t^{p-\alpha_i}, \quad p \geq \lceil k_2 \rceil, \tag{6} $$

Chelyshkov polynomials and approximation

Definition 3: The Chelyshkov polynomials (CPs) are defined by:[15]

$$ C_{Nn}(t) = \sum_{k=n}^{N} \gamma_{Nnk} t^k, \; n \in I^N \tag{7} $$

where $t \in I = [0, 1]$, $I^N = \{0, 1, \ldots, N\}$, and $\gamma_{Nnk} = (-1)^{k-n} \binom{N-n}{k-n} \binom{N+k+1}{N-n}$.

Eq. (7) represents CPs of Degree N. The interval I can be generalized *to Ib* = $[0, b]$ according to the following relationship:[16]

$$C_{Nn}(t) = \sum_{k=n}^{N} \gamma_{Nnk} \left(\frac{t}{b}\right)^{k}, n \in I^N \quad (8)$$

The following is obtained by applying the distributed order fractional derivative represented by Eqs. (6) and (7):

$$ {}_{k_1,k_2}^{\ \ C}\mathbb{D}^{p(\alpha)} C_{Nn}(t) = \left\{ \begin{array}{ll} \sum_{k=n}^{N} \gamma_{Nnk} \sum_{i=1}^{N} w_i\, p(\alpha_i) \frac{\Gamma(k+1)}{\Gamma(k-\alpha_i+1)} t^{k-\alpha_i}, & k \geq \lceil \alpha_i \rceil \\ 0, & k < \lceil \alpha_i \rceil \end{array} \right\} \quad (9)$$

where, w_i is the weight of quadratic numerical integration.

One of the most important properties that characterize these functions is the orthogonality property with weighting $w(x) = 1$, which is given according to the following relationship:

$$\int_0^b C_{Nn}(t) C_{Nk}(t) \mathrm{w}(t) dx = \left\{ \begin{array}{ll} \frac{b}{(k+n+1)}, & n = k \\ 0, & n \neq k \end{array} \right\}, \quad n, k \in I^N \quad (10)$$

Consider the weighted space $L_w^2(I)$, which is defined by:[15]

$$L_w^2(I) = \left\{ f : I \to \mathbb{R};\ f \text{ is measurable on } I,\ \int_0^1 |f(t)|^2 dx < \infty \right\} \quad (11)$$

The inner product and the norm are provided by:

$$\langle f, g \rangle_{w_v} = \int_0^1 f(t) g(t) dx,\ \ \|f\|_{\mathrm{w}} = \langle f, f \rangle^{\frac{1}{2}}{}_{\mathrm{w}} \quad (12)$$

Suppose $S_N = span\{C_{N0}(t), C_{N1}(t), \ldots, C_{NN}(t)\}$ a finite-dimensional base and a subspace of $L_w^2(I)$. For any function $u(t) \in L_{\mathrm{w}}^2(I)$ there exists a unique approximation $u_N(t) \in S_N$ and satisfied the following conditions:

$$\|u - u_N\|_w \leq \|u - U\|_{\mathrm{w}}\ \forall\ U \in S_N \quad (13)$$

Furthermore, it can be expanded $u_N(t)$ by a Chelyshkov polynomials as:

$$u_N(t) = \sum_{k=0}^{N} a_k C_{Nk}(t) \quad (14)$$

Multiply both sides of relation 14 by $C_{N_n}(t)\mathrm{w}(t)$ and integrated the results from 0 to b:

$$\int_0^b u_N(t) C_{Nn}(t) \mathrm{w}(t)\, dt = \sum_{k=0}^{N} a_k \int_0^b C_{Nk}(t) C_{Nn}(t) \mathrm{w}(t)\, dt$$

The following result is obtained by applying the orthogonality property in Eq. (10)

$$a_n = \frac{(2n+1)}{b} \int_0^b u_N(t) C_{Nn}(t) \mathrm{w}(t)\, dt,\ n \in I^N \quad (15)$$

that is:

$$a_n = \frac{\langle u, C_{Nn} \rangle_{\mathrm{w}}}{\langle C_{Nn}, C_{Nn} \rangle_{\mathrm{w}}},\ n \in I^N \quad (16)$$

Theorem 1: *Suppose $D^n u(t) \in C[0,1],\ n \in I^{N+1}$ and let $u_N(t)$ be the best approximate to the function $u(t)$ then the bound of error is given by:*[17]

$$\|u - u_N\|_{\mathrm{w}} \leq \frac{Q}{\Gamma(1 + (N+1))} \frac{1}{\sqrt{(2N+3)}} \quad \text{where } Q = \sup_{0 < x \leq 1} \left\{ \left| D^{(N+1)} u(t) \right| \right\} \quad (17)$$

Weighted residual methods (WRMs)

Let us assume the following DOFDEs:[3]

$$Au(t) = f(t), \quad (18)$$

where A is a distributed order fractional operator and u is an unknown function defined on the Hilbert space $H(a, b)$, first of all, this function is approximated by writing it as a linear combination of an expansion functions $\phi_k(t)$ as follows:

$$u(t) \approx \mathrm{u_N}(t) = \sum_{k=0}^{N} c_k \phi_k(t) \quad (19)$$

Substitute $u_N(t)$ in Eq. (18) to get the following residual function:

$$R(t) = Au_N(t) - f(\mathrm{t}) \neq 0 \quad (20)$$

The following theorem is a fundamental theorem to reduce $R(t)$ to the minimum.

Theorem 2: *Suppose $R(t)$ be a function defined on $H(a, b)$, let the value of the following integral be verified* *for any given positive function $w(t)$ defined on $H(a,b)$* [18] *i.e.*

$$\int_a^b R(t)\, w(t)\, dt = 0, \tag{21}$$

then,

$$R(t) = 0,\ \forall\, x \in (a,b)$$

*These methods are divided into several types, such as the Galerkin method **(GLM)**, the Petrov Galerkin method (PGM), and the least squares method (LSM). In PGM methods, the expansion function is not relied upon, but other functions analytical in $H(a,\ b)$ are used. Several methods fall under this method, for instance, the collocation method (COM), subdomain method (SDM), and momentum method (MNM).* [3]

Methodology

In this section, the proposed spectral Mittag-Leffler weight method based on Chelyshkov polynomials (SMLWM-CPs) is presented.

New Mittag-Leffler weight method (MLWM)

Definition 4: The Mittag-Leffler function of two parameters α *and* β is defined by: [19]

$$E_{\alpha,\beta}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j+\beta)} \tag{22}$$

In this research, the truncated Mittag- Leffler function is suggested to be the weight function; which is defined by:

$$M_k(t) = \sum_{j=0}^{k} \frac{t^j}{\Gamma(j+1)} \tag{23}$$

Proposition 1: If the weight function is defined as $\mathrm{w_k(t)} = M_k(t)$ in the integral represented by Eq. (21), then the residual $R(t)$ is vanished.

Proof: For all $k = 1, 2, \ldots, N$, the functions $M_k(t)$ are analytic on $H(0,\ b)$. Suppose $R(t) \neq 0$, then either $R(t) > 0$, or $R(t) < 0$, if it is positive for some subinterval $[a_1, b_1]$ in $[0,b]$. Since $M_k(t) > 0,\ \forall\, k$, in addition, the value of $M_k(t) = 0$ only if $t = 0$. But the value of the following integral is positive and not equal to zero,

$$\int_0^b R(t) M_k(t) dt = \int_{a_1}^{b_1} R(t) \sum_{j=0}^{k} \frac{t^j}{\Gamma(j+1)} dt > 0, \tag{24}$$

which is a contradiction with the hypotheses. Therefore $R(t) = 0$.

The proposed (SMLWM-CPs) method

Consider the following DOFDEs of order $\lceil k_2 \rceil$: [1]

$${}_{k_1,k_2}^{\ C}\mathbb{D}^{p(\alpha)}\{Lu(t)\} + {}_{k_1,k_2}^{\ C}\mathbb{D}^{p(\alpha)}\{Gu(t)\} = f(t)\ \ t \in [0,b] \tag{25}$$

with initial or boundary conditions

$$L_0 u(0) = a_i,\ \ i = 1, 2, \ldots, m \tag{26}$$

$$L_0 u(0) = a_i,\ \ L_1 u(b) = b_j,\ \ i + j = m, \tag{27}$$

where L,L_0 and L_1 are linear operators, G is a nonlinear operator, m is the number of initial or boundary conditions and

$${}_{k_1,k_2}^{\ C}\mathbb{D}^{p(\alpha)}\{Gu(t)\} = \int_{k_1}^{k_2} G\left\{p(\alpha)\, {}_0^C D_t^{\alpha}\, u(t)\right\} d\alpha \tag{28}$$

Suppose the approximate solution of Eq. (25) can be written as spectral Chelyshkov polynomials as follows:

$$u_N(t) = \sum_{j=0}^{N} q_j C_{Nj}(t) \tag{29}$$

Substituting Eq. (29) in Eq. (25) to compute the residual error:

$$\sum_{j=0}^{N} q_j\, {}_{k_1,k_2}^{\ C}\mathbb{D}^{p(\alpha)}\{LC_{Nj}(t)\} + {}_{k_1,k_2}^{\ C}\mathbb{D}^{p(\alpha)} G\left\{\sum_{j=0}^{N} q_j C_{Nj}(t)\right\} = f(t) \tag{30}$$

To get the best unknown coefficients q_j, Eq. (30) are Multiplied by the Mittag-Leffler weights $M_i(x),\ i = 1, 2 \ldots, N - m$ and integrate the result:

$$\int_0^b \left[\sum_{j=0}^{N} q_j\, {}_{k_1,k_2}^{\ C}\mathbb{D}^{p(\alpha)}\{LC_{Nj}(t)\} + {}_{k_1,k_2}^{\ C}\mathbb{D}^{p(\alpha)} \times G\left\{\sum_{j=0}^{N} q_j C_{Nj}(t)\right\}\right] M_i(t) dt = \int_0^b f(t) M_i(t) dt \tag{31}$$

Which leads to $N - m$ of a nonlinear system of equations. To eliminate the computational efforts resulting from performing direct integration, the integrals are calculated approximately. First, in Eq. (31), the distributed order fractional derivative is calculated approximately using Eq. (6), so it can obtain:

$$\int_0^b \left[\sum_{j=0}^{N} q_j \sum_{n=1}^{N_s} w1_n\, p(\alpha_n) {}_0^C D_t^{\alpha_n} \{LC_{Nj}(t)\} + \sum_{n=1}^{N_s} w2_n G \left\{ p(\alpha_n) {}_0^C D_t^{\alpha_n} \sum_{j=0}^{N} q_j C_{Nj}(t) \right\} \right] M_i(t)\, dt = \int_0^b f(t) M_i(t) dt \tag{32}$$

Finally, the fully discrete is finally, full scheme of the N-m nonlinear algebraic equation is obtained by calculating the integral again approximately in Eq. (32), thus it can obtain:

$$\sum_{h=1}^{N_h} v1_h \left[\sum_{j=0}^{N} q_j \sum_{n=1}^{N_s} w1_n\, p(\alpha_n) {}_0^C D_t^{\alpha_n} \{LC_{Nj}(t_h)\} + \sum_{n=1}^{N_s} w2_n G \left\{ p(\alpha_n) {}_0^C D_t^{\alpha_n} \sum_{j=0}^{N} q_j C_{Nj}(t_h) \right\} - f(t_n) \right] M_i(t_h) = 0, \tag{33}$$

where $N_s = \frac{k_2 - k_1}{s}$, $N_h = \frac{b}{h}$, $v1_h, w1_n$ and $w2_n$ are the weights of the quadrature Simpson integration, s and h are the step size of the numerical integration.

By applying the initial or boundary conditions, it can obtain:

$$L_0 \sum_{k=0}^{N} c_k C_{Nk}(0) = a_i,\ i = 1, 2, \ldots, m \tag{34}$$

$$L_0 \sum_{k=0}^{N} c_k C_{Nk}(0) = a_i,\ L_1 \sum_{k=0}^{N} c_k C_{Nk}(b) = b_j,\ i + j = m. \tag{35}$$

The approximate solution is then evaluated by solving the previous system of N nonlinear algebraic equation using the Newton method. If the nonlinear part G is equal to zero, then the system of N linear algebraic equation is computed using Gauss-Jordan Elimination. The MATLAB is used to implement the proposed method.

Results and discussion

In this section, numerical examples are presented to illustrate the efficiency of the proposed method in solving DOFDEs, a numerical comparison is made between the well-known WRMs and the proposed Mittag-Leffler weighted function. In addition, a numerical comparison is made using the method that relies on the Chelyshkov polynomial and the methods that rely on famous polynomials such as Jacobi, Legendre, etc. The effect of the support function on the accuracy of the approximate solution is also clarified, and the comparison between the approximate solution resulting from distributed order and fixed order is presented. Finally, an applied example of the oscillator mathematical system described in [20] is given. The integral was calculated using the Simpson-1/3 method with a fixed step size of 0.05. The approximate solution $u_N(t)$ generated by the proposed method is compared with the analytical solution of a given problem using the following root mean square (RMS) criteria: [21]

$$RMS = \sqrt{\sum_{i=1}^{M} \frac{(u(t_i) - u_N(t_i))^2}{M}} \tag{36}$$

where $t_i \in [0, b]$, $\forall i$, u the exact solution.

If the analytical solution is not obtained, then the error function $E(t)$ is computed by substituting the approximate solution to the given DOFDE. Finally, the area under the square error is evaluated as follows: [22]

$$LSE = \int_0^b [E(t)]^2\, dt = \int_0^b [{}_{k_1,k_2}^{\ C}\mathbb{D}^{p(\alpha)} L u_N(t) - f(t)]^2\, dt, \tag{37}$$

which represents a good quantitative criterion for knowing the accuracy of the approximate solution. Clearly, as N tends to ∞ then LSE tends to zero.

Example 1: *Consider the following generalized Bagley-Torvik DOFDE:* [23]

$$u'' + p_1\, {}_{k_1,k_2}^{\ C}\mathbb{D}^{p(\alpha)} u + p_2 u = f(t), \quad k_1 < \alpha < k_2 \tag{38}$$

where, p_1, p_2 are constants, with initial conditions:

$$u(0) = u'(0) = 0, \tag{39}$$

In this example, three different cases of the distributed-order generalized Bagley-Torvik problem are taken as follows:

Case 1. $p_1 = p_2 = 1, k_1 = 0, k_2 = 1, p(\alpha) = \delta(\alpha - 0.5)$, where $\delta(.)$ is the Dirac delta function, $f(t) = 6t + t^3 + \frac{3.2t^{2.5}}{\Gamma(0.5)}$, and the exact solution is $u(t) = t^3$.

When $N = 3$ the numerical solution is close to the following solution: $u_3(x) = x^3 + 1.514 \times 10^{-17}x^2$, with $RMS = 1.5611 \times 10^{-18}$, and $LSE = 1.1864 \times 10^{-34}$. It should be noted that since the Dirac delta function is equal to zero when $\alpha \neq 0.5$, then this distributed order is reduced to the following differential equation:

$$u'' + p_1\, u^{(0.5)} + p_2 u = f(t),$$

which is of a constant order.

Case 2. $p_1 = 0.5, p_2 = 1.5, k_1 = 0, k_2 = 1, p(\alpha) = 6\alpha(1-\alpha)$, and $f(t) = 8$. The approximate solution when $N = 5$ is computed as: $u_5(t) = 0.1644t^5 - 0.7817t^4 - 0.1631t^3 + 4.0055t^2$ with errors LSE $= 1.44 \times 10^{-06}$.

Case 3. $p_1 = 2, p_2 = 0.5, k_1 = 0, k_2 = 1, p(\alpha) = \Gamma(4-\alpha)$, and $f(t) = 2 - 3t^3$. The numerical solution when $N = 5$ is evaluated by: $u_5(t) = 0.0573t^5 - 0.0008t^4 - 0.8988t^3 + 1.0888t^2$ and LSE $= 9.8123 \times 10^{-04}$.

In Table 1, the effect of the degree of polynomials from a practical perspective on the accuracy of the solution was studied. It was found that choosing the appropriate degree of N may reduce large arithmetic operations and may have high arithmetic accuracy. This is because the proposed method relies on creating a square matrix with capacity $N + 1$, which requires calculating its inverse once to get the approximate solution. In addition, the effect of the type of polynomials used and the WRMs on the accuracy of the experimental solution was studied, and it was found that the WRMs have a strong effect on the speed of convergence of the solution and its accuracy, in contrast to the effect of polynomials, as shown in Tables 2 and 3 and Fig. 1(a) to (c).

To know the effect of the distribution function on the accuracy of the experimental solution and to compare it with cases in which the ranks are a fixed value, a numerical comparison is made by taking several possibilities for the distribution function and comparing them with the fixed cases of the fractional derivatives, which are shown in Table 4 and Fig. 2.

Table 1. Explains the convergent of the proposed method. In Ex1.

	RMS		LSE	
Degree of polynomials	Case 1	Case 1	Case 2	Case 3
N = 3	1.56×10^{-18}	1.19×10^{-34}	0.1264	0.0121
N = 5	1.59×10^{-18}	3.32×10^{-36}	1.44×10^{-6}	9.8123×10^{-4}
N = 7	2.18×10^{-18}	4.94×10^{-36}	9.72×10^{-7}	5.8797×10^{-6}
N = 9	3.60×10^{-17}	2.25×10^{-36}	1.25×10^{-7}	8.4637×10^{-7}

Table 2. Explains the effect of WRMs on the approximate solution. In Ex1.

	RMS		LSE	
(WRM)	Case 1 when N = 3	Case 1 when N = 3	Case 2 when N = 5	Case 3 N = 5
GLM-CPs	1.56×10^{-18}	2.49×10^{-33}	0.0013	4.4456
COM-CPs	1.62×10^{-18}	9.22×10^{-34}	8.07×10^{-06}	0.0214
SDM-CPs	1.56×10^{-18}	9.71×10^{-35}	1.75×10^{-06}	6.3301×10^{-04}
MNM-CPs	1.62×10^{-18}	2.97×10^{-34}	1.65×10^{-06}	0.0037
MLWM-CPs	1.56×10^{-18}	1.19×10^{-34}	1.44×10^{-06}	9.8123×10^{-04}
LSM-CPs	1.56×10^{-18}	9.62×10^{-35}	1.27×10^{-06}	5.2613×10^{-04}

Table 3. Shows the effect of the expansion function on the approximate solution of Ex1 using SMLWM.

	RMS	LSE	
Polynomials	Case 1, with $N = 3$	Case 2, with $N = 5$	Case 3, with $N = 5$
Chelyshkov	1.5611×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}
Chebyshev first kind [24]	1.5760×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}
Chebyshev second kind [25]	1.5611×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}
Bernstein [26]	3.1061×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}
Legendre [27]	1.5763×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}
Jacobi [28]	1.5763×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}
Gegenbauer [29]	1.5611×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}
Hermite [30]	2.2171×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}
Laguerre [31]	1.0116×10^{-17}	1.4425×10^{-06}	9.8123×10^{-04}
Taylor [32]	1.5611×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}
Mittag-Leffler [19]	1.5611×10^{-18}	1.4425×10^{-06}	9.8123×10^{-04}

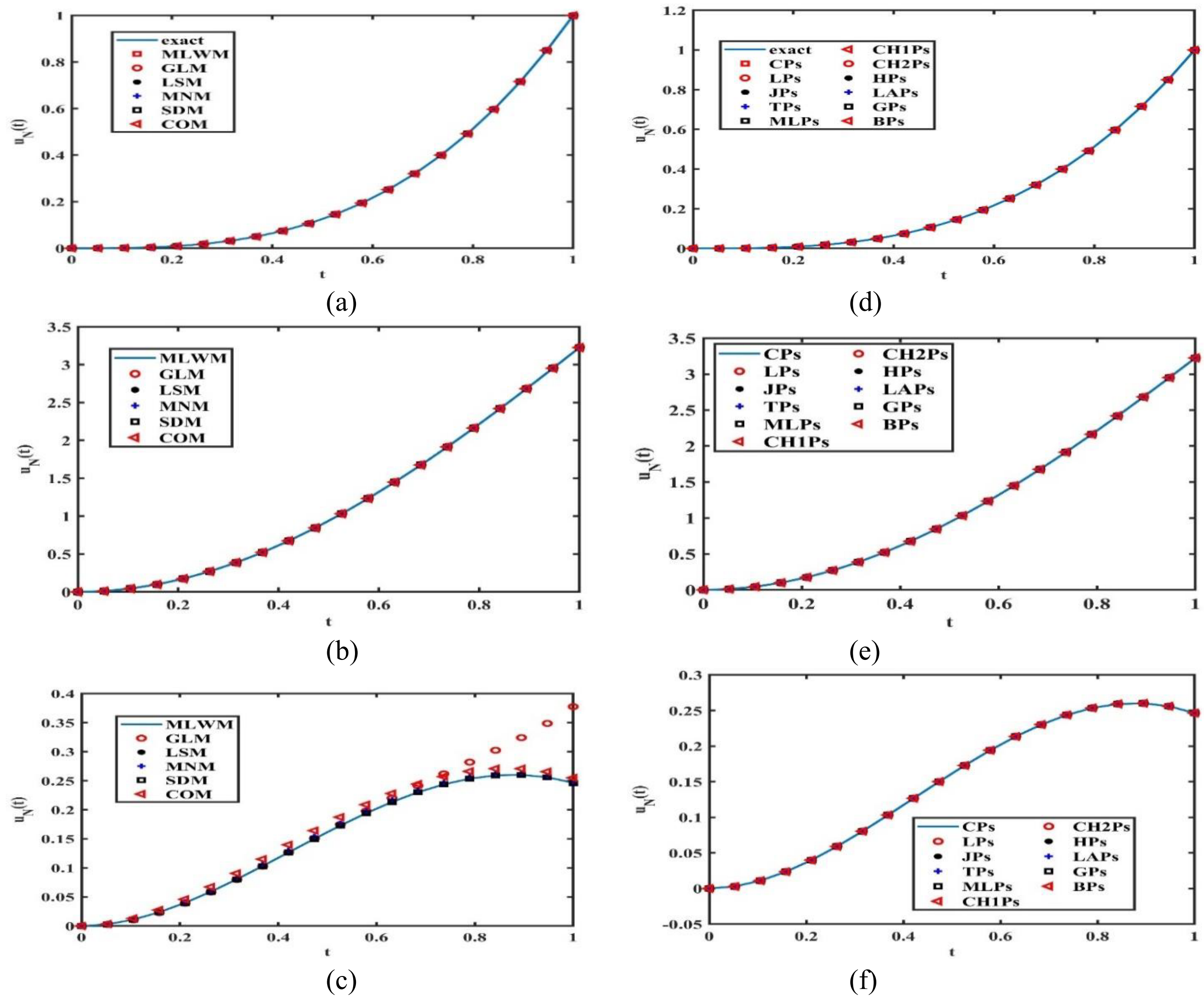

Fig. 1. (a) to (c) illustrate the comparison between weighted weight methods and the efficiency of the Mittag-Leffler method in Ex1. cases 1 to 3 respectively, while (d) to (f) illustrate the effect of polynomials on the approximate solution in Ex1 cases 1 to 3 respectively.

Table 4. A numerical comparison showing the effect of distributed order on the accuracy of the approximate solution. In Ex1 case 2 when N = 5; using the proposed method.

$p(\alpha)$	$6\alpha(1-\alpha)$	$\Gamma(3-\alpha)$	$\sin(\alpha)$	$\cos(\alpha\pi)$
RMS	1.44×10^{-06}	2.94×10^{-06}	3.13×10^{-06}	4.30×10^{-05}

Example 2: *Suppose the following non-linear DOFDE:*[4]

$$ {}^{C}_{0,2}\mathbb{D}^{p(\alpha)}u^2 = \int_0^2 \left[p(\alpha)\,{}^{C}_{a}D^{\alpha}_{t}\,u(t)\right]^2 d\alpha = f(t), \quad 0 < t < 1, \tag{40}$$

with initial conditions:

$$u(0) = u'(0) = 0, \tag{41}$$

where $f:(t) = 18t^2[\frac{t^4-1}{lnt}]$, $p(\alpha) = \Gamma(4-\alpha)$, *and the exact solution is* $u(t) = t^3$. *The approximate solution when* $N = 3$ *is:* $u_3(t) = t^3 - 3.24 \times 10^{-07}t^2$. *with* $RMS = 1.8023 \times 10^{-08}$. *This solution is shown in Fig. 3(a) and (e).*

The values of RMS when using Chelyshkov polynomials with the Galerkin method, momentum method, and collocation method are 9.7913×10^{-07}, 1.5093×10^{-07}, 4.5723×10^{-07} respectively

Example 3: *Suppose the following linear DOFDE:*[4]

$$ {}_{0,2}^{C}\mathbb{D}^{p(\alpha)}u = f(t), \quad 0 < t < 1, \tag{42}$$

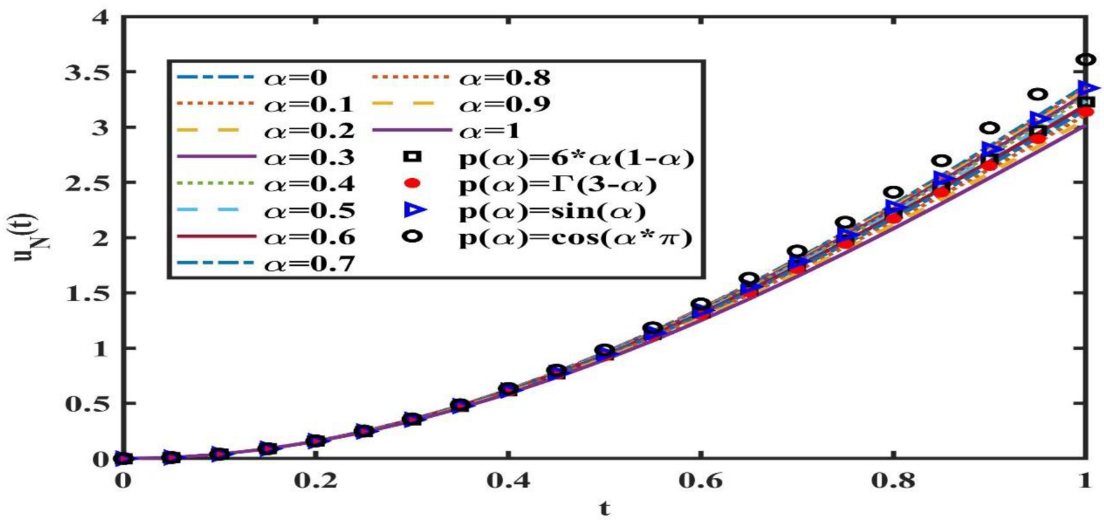

Fig. 2. Show the effect of changing the distributed order on approximate solutions and compare them with cases in which the orders are fixed.

with boundary conditions:

$$u(0) = 0,\ u(1) = 1, \tag{43}$$

Case 1. $f(t) = 6t[\frac{t^2-\cosh(2)-\sinh(2)\ln(t)}{(lnt)^2-1}]$, $p(\alpha) = \sinh(\alpha)\Gamma(4-\alpha)$, and the exact solution is $u(t) = t^3$. The approximate solution when $N = 3$ is: $u_3(t) = t^3 - 9.97\times 10^{-07}t^2 + 5.03\times 10^{-7}t$. with $RMS = 1.4264\times 10^{-08}$. (See Fig. 3(b) and (f)).

Case 2. $f(t) = [\frac{t^5-t^3}{lnt}]$, $p(\alpha) = \frac{\Gamma(6-\alpha)}{5!}$, The exact solution is $u(t) = t^5$. The approximate solution using $N = 5$ is $u_5(t) = t^5 + 2.25\times 10^{-08}t^4 + 7.44\times 10^{-09}t^3 - 2.14\times 10^{-08}t^2 + 6.72\times 10^{-09}t + 3.27\times 10^{-17}$. with $RMS = 9.8307\times 10^{-11}$. (See Fig. 3(c) and (g)).

Example 4: *Consider the following system of oscillator DOFDE:*[7]

$$u'' + \omega^2 u + v(t) = f(t),\ 0 < t < \pi, \tag{44}$$

$${}_{0,1}^{C}\mathbb{D}^{p_1(\alpha)}v(t) - \mu\,{}_{0,1}^{C}\mathbb{D}^{p_2(\alpha)}u(t) = 0,$$

with initial conditions:

$$u(0) = u'(0) = 0, \tag{45}$$

where ω, μ are constant, $u(t)$ represent the displacement, $v(t)$ is the dissipation force and $f(t)$ is the external force.

The previous system can be rewritten as a single DOFDE as follows:

$${}_{0,1}^{C}\mathbb{D}^{p_1(\alpha)}u'' + \omega^2\,{}_{0,1}^{C}\mathbb{D}^{p_1(\alpha)}u + {}_{0,1}^{C}\mathbb{D}^{p_1(\alpha)}v(t) = {}_{0,1}^{C}\mathbb{D}^{p_1(\alpha)}f(t) \tag{46}$$

which implies the following linear equation of two distributed order terms:

$${}_{0,1}^{C}\mathbb{D}^{p_1(\alpha)}u'' + \omega^2\,{}_{0,1}^{C}\mathbb{D}^{p_1(\alpha)}u + \mu\,{}_{0,1}^{C}\mathbb{D}^{p_2(\alpha)}u(t) = {}_{0,1}^{C}\mathbb{D}^{p_1(\alpha)}f(t) \tag{47}$$

Atanakovic and others studied the characteristics of the analytical solution to this problem in the case $p_1(\alpha) = a^\alpha$, $p_2(\alpha) = b^\alpha$ and found the relationship between the functions u and v as follows:[15]

$$v(t) = \mu\mathcal{L}^{-1}\left\{\frac{\ln(a)+\ln(s)}{\ln(b)+\ln(s)}\right\} * \mathcal{L}^{-1}\left\{\frac{bs-1}{as-1}\right\} * u(t) \tag{48}$$

where (*) is the convolution integral. As a special case if we take $a = b = 0.1$, then. Eq. (47) reduces to the following FDE:

$$u'' + (\omega^2 + \mu)u = f(t) \tag{49}$$

Consider $f(t) = \sin(\sigma t)$, then the exact solution is $u(t) = \frac{1}{\omega^2+1-\sigma^2}[sin(\sigma t) - \frac{\sigma}{\sqrt{\omega^2+1}}\sin(\sqrt{\omega^2+1}t)]$.

The following approximate solution is calculated by applying the proposed method to solve Eq. (48) with $\omega = 3$, $\sigma = 3.6$, $\mu = 1$, and $N = 20$; which illustrated in Fig. 3(d) and (h).

$$\begin{aligned} u_{20}(t) = &-0.0003t^{16} + 0.0015t^{15} - 0.0054t^{14} \\ &+ 0.0149t^{13} - 0.0354t^{12} + 0.0720t^{11} \\ &- 0.0923t^{10} + 0.0457t^9 - 0.0984t^8 \end{aligned}$$

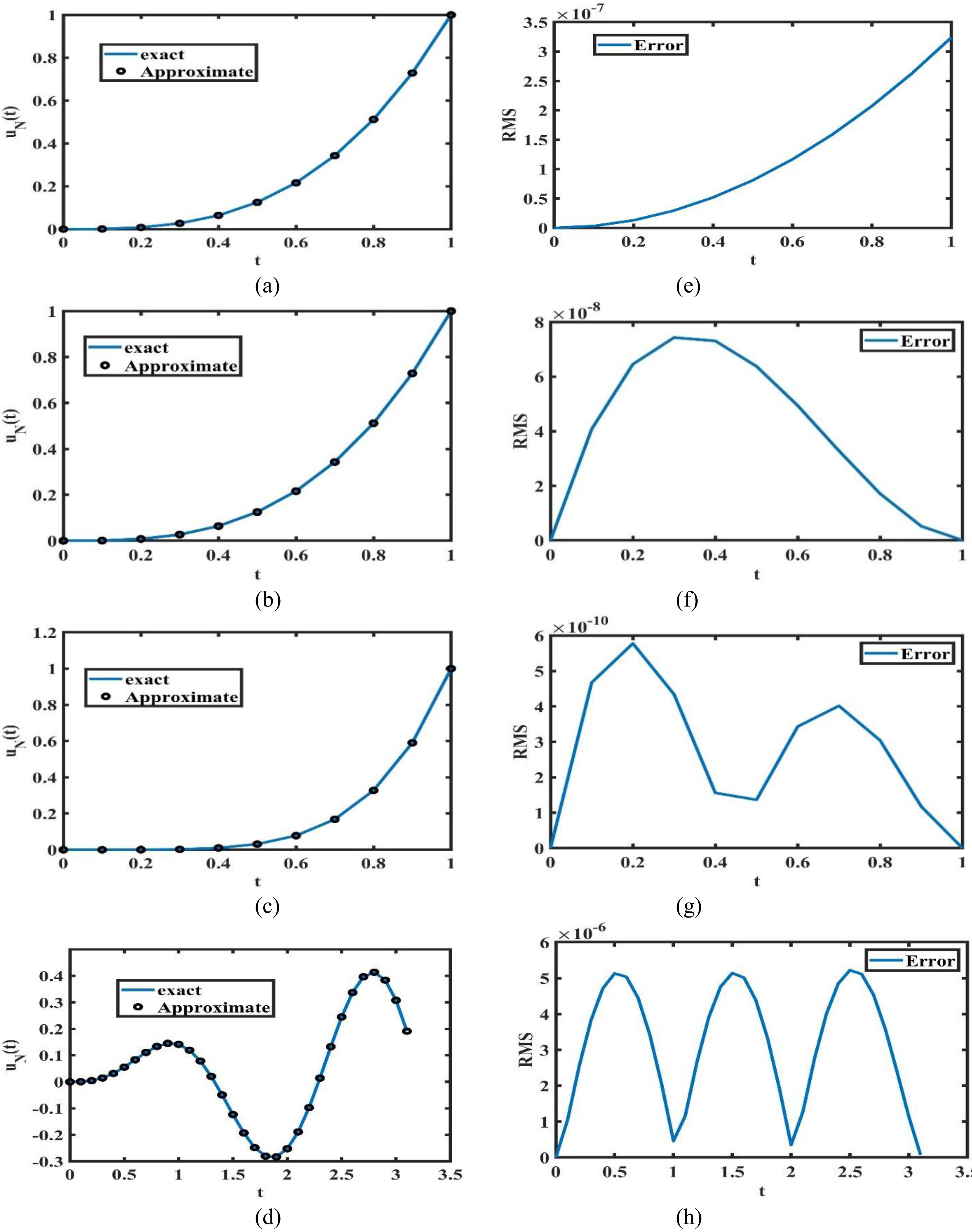

Fig. 3. (a) to (d) illustrate the comparison between exact and numerical solution (a), (b) in Ex2 case 1 and 2, (c) in Ex3 and (d) in Ex4, while (e) to (g) illustrate the absolute error of the approximate solution (e), (f) in Ex2 case 1 and 2, (g) in Ex3 and (h) in Ex4.

$$+ 0.3557t^7 - 0.0408t^6 - 0.6711t^5$$
$$- 0.0057t^4 + 0.6013t^3 - 0.0002t^2$$

With $RMS = 6.3026 \times 10^{-07}$.

Conclusion

In this work, a method based on Chelyshkov polynomials is presented and the Mittag-Leffler function is proposed as the weight function. The proposed method converts DOFDEs into a system of algebraic equations. This system is solved to obtain the desired approximate solution. The proposed method was applied to solve linear and nonlinear DOFDE with initial or boundary conditions, and the method showed acceptable results in most tests. The effect of WRMs on the solution was also studied and compared to the proposed method, showing the strength of the effect of these methods on the approximate solution, while a comparison was made between polynomials, and the results showed that the effect of polynomials is almost minimal. This method has proven effective for solving this type of equation. In the future, researchers suggest applying the method to solve fractional ordinary differential equations of variable order.

Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Mosul.

Authors' contribution statement

The working idea and method proposal were determined in collaboration between all authors. A.T.A. carried out the work, wrote the manuscript, and analyzed and interpreted the results. E.S.A. edited the manuscript and revised it with revised ideas. All authors read and approved the final manuscript.

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الحل العددي للمعادلات التفاضلية الكسرية ذات الترتيب الموزع باستخدام دالة الوزن الطيفية Mittag-Leffler بالاعتماد على متعددات حدود تشيليشكوف

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الخلاصة

في العديد من الدراسات، تم استخدام الطرق الطيفية المعتمدة على إحدى متعددات الحدود المتعامدة والطرق المتبقية الموزونة (WRMs) لتحويل المعادلة التفاضلية الكسرية ذات الترتيب الموزع (DOFDEs) إلى نظام من المعادلات الجبرية الخطية أو غير الخطية، ومن ثم حل هذا النظام للحصول على الحل التقريبي. في هذا البحث تم تقديم طريقة عددية لحل DOFDEs. تم فرض الحل التقريبي على شكل متعددات حدود تشيليشكوف المتعامدة ذات معاملات غير معروفة. يتم الحصول على المعاملات المطلوبة باستخدام WRMs، والتي تحول DOFDEs إلى نظام من المعاملات الجبرية. تم تقديم دالة Mittag-Leffler كدالة مناسبة للوزن. وقد تم تطبيق الطريقة على عدة أمثلة عددية، مثل النموذج الرياضي التذبذبي، ومعادلة باجلي-تورفيك الكسرية ذات الترتيب الموزع. تم الحصول على نتائج مقبولة في معظم الاختبارات. تمت مقارنة طريقة ميتاج-ليفلر الوزنية المقترحة مع WRMs مثل طريقة جاليركين وطريقة بيتروف-جاليركين وطريقة المربعات الصغرى، وأظهرت الدالة الموزونة المقترحة نتائج أكثر دقة من الطرق السابقة في معظم الاختبارات. أظهرت الدراسة أن تأثير متعددات حدود الاساسية مثل متعددات حدود تشيبيشيف، جاكوبي، ليجيندر، جيجنباور، هيرميت، تايلور، ميتاج ليفلر، وبرنشتاين لها تأثير ضئيل في معظم الاختبارات. بالإضافة إلى ذلك تمت دراسة تأثير الترتيب الموزع على دقة الحل، وأظهرت النتائج أن الترتيب الموزع له تأثير قوي على دقة الحل، حيث أن تأثيره يكون مباشراً على الجزء غير المتجانس، والذي يؤدي إلى معادلات أكثر تعقيدًا مما كانت عليه في الحالات التي تكون فيها الرتب ثابتة.

الكلمات المفتاحية: مشتقة الترتيب الموزع الكسرية، الطريقة المتبقية الموزونة، الطريقة الطيفية، متعددات حدود تشيليشكوف، طريقة الوزن ميتاج-ليفلر، معادلة باجلي-تورفيك الكسرية ذات الترتيب الموزع.