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Hameeda O. Al-Humedi
Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq, hameeda.mezban@uobasrah.edu.iq

Zahraa Adnan Jameel
Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq, zahraa.adnan@uobasrah.edu.iq

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RESEARCH ARTICLE

Combining the Least-Squares Method with Touchard Polynomials for Solving Mixed Integro-Differential Equation

Hameeda O. Al-Humedi, Zahraa Adnan Jameel *

Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq

ABSTRACT

Using the well-known least-squares weighted residual method (LSM) in coupling with various degrees of Touchard polynomials (TPs), found the numerical solutions of Volterra–Fredholm integro-differential equations (VFIDEs) and mixed Volterra–Fredholm integro-differential equations (MIDEs) of the second kind. There exist many approaches that have evaluated the approximate solution of the integro-differential equations (VFIDEs (MIDEs)) like the Adomian decomposition method and modified Adomian decomposition method, Homotopy analysis method, Taylor polynomials, power series expansion and cubic Legendre spline collocation method. In this work, we presented a method based on combining LSM with TPs is an essential component of the suggested approach. By implementing such a method, a system of algebraic equations can be generated that can be solved by employing well-known linear algebraic methods.

Several VFIDEs (MIDEs) examples were solved with a comparatively minimal number of reiterations to show the accuracy and effectiveness of the presented approach when comparing the current method with other methods already accessible in the scientific literature, as well as from the approximate solutions of each of these situations, researchers found that there was an apparent agreement with the exact solutions for some examples. The applicability of the proposed method was proven and the convergence analysis was discussed.

Keywords: Approximate solutions, Exact solutions, Least-squares method, Mixed integro-differential equation, Touchard polynomial

Introduction

In recent years, interest in integro-differential equations (IDEs), a significant area of contemporary mathematics, has grown. It frequently occurs in many practical areas, including mechanics, the theory of elasticity, mathematical physics, potential, electrostatics, and engineering.

By Maturi and Simbawa[1] the Volterra-Fredholm integro-differential equations (VFIDEs) were resolved using the modified Adomian decomposition method (MADM). By utilizing different degrees of TPs, Abdullah[2] and Abdullah[3–6] presented TPs for the numerical resolution of the first order and second kind with conditions of linear Fredholm integro-differential equation (FIDE) and linear Volterra integro-differential equation (VIDE), respectively.

Also, MIDEs solved by Hamoud[7] using MADM and the Adomian decomposition method (ADM). Laplace discrete ADM (LDADM) is used with non-homogeneous nonlinear VFIDEs, according to Dawood.[8] In the Caputo sense, Ahmed and Faeq[9] used the Bessel collocation method to solve the Fredholm-Volterra integral-fractional differential equations. Nazir[10] used TPs evaluated on systems of linear and nonlinear integral equations. Dan[11] used a monic Chebyshev polynomial to solve linear MIDEs and IDEs of the second kind. Based on the duality of LSM with

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* Corresponding author.
E-mail addresses: hameeda.mezban@uobasrah.edu.iq (H. O. Al-Humedi), zahraa.adnan@uobasrah.edu.iq (Z. A. Jameel).

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the Laguerre polynomial, Al-Humedi and Shoushan [12] provides a new method for solving IDEs subject to mixed conditions.

To solve VFIDEs of linear 1st and 2nd order, Zanib and Ahmad [13] approximated Lagrange multipliers. Yassein [14] uses a trustworthy iterative method to resolve a variety of VFIDEs of the second class with beginning conditions, Chebyshev polynomial basis functions used by Deepmala [15] to solve the VFIDEs. Shoushan and Al-Humedi [16] employed Legendre polynomials under mixed conditions to solve high-order linear FIDEs. Higher-order linear VFIDEs utilizing the Chebyshev polynomial were used by Yüksel. [17]

Using the TP and Bernstein polynomial, Mustapha [18] determines the approximations to the second-order integral equations of Fredholm and Volterra. Dhari [19] used a linear programming problem to solve three different kinds of linear integral equations of the second kind, including (VFIEs). To solve linear MIEs and systems of linear MIEs, used the Taylor expansion method by Didgar and Vahidi. [20] Hamoud [21] solved MIDEs using HAM.

Al-Humedi and Jameel [22,23] combined between cubic B-spline least-square method and a quadratic B-spline as a weight function to find a solution of (IDEs) with the weakly singular kernel.

Many issues in mathematical applications, such as modeling and bioinformatics, can be solved using MIDEs, which have been researched in numerous science domains, including biomedicine and biophysics. [24] So, inspired by the above-mentioned publications, we will solve the following mixed integrodifferential equations (MIDEs) and Volterra–Fredholm integro-differential equations (VFIDEs), with the following forms:

$$u^{(\ell)}(x) = f(x) + \lambda_1 \int_0^x K_1(x,t)\,u(t)\,dt + \lambda_2 \int_a^b K_2(x,t)\,u(t)\,dt,\ \ a \le x, t \le b, \tag{1}$$

the mixed condition is

$$\sum_{j=0}^{N-1} \left(a_{j\tau} + b_{j\tau}\right) u^{(\tau)}(x) = \beta_j,\ \ \tau = 0\,(1)\,(N-1), \tag{2}$$

where $u^{(\ell)}(x) = \frac{du^{(\ell)}(x)}{dx^{(\ell)}}$, $\ell = 1, 2, \ldots, K_1(x,t)$ and $K_2(x,t)$ are kernel functions, λ_1 and λ_2 are constants. Also, MIDE defined as

$$u^{(\ell)}(x) = f(x) + \lambda \int_0^x \int_a^b K(r,t)\ u(t)\,dt\,dr,$$

by combining LSM with TPs.

The paper is organized as follows. In second Sections, we recall the definitions of TPs which are used for our main results. In Section Three, deriving the proposed method based on combining LSM with TPs by applying it to VFIDEs is presented. Section four discusses the convergence analysis. In Section five, four examples of different kinds of MIDEs and VFIDEs are given to verify the proposed formulation. Finally, a brief conclusion is given in the sixth Section.

Touchard polynomials

The Tps studied in 1939 by Jacques Touchard, consist of polynomial sequences of binomial. Tps given as [10,25–27]

$$T_n(x) = \sum_{\mathcal{L}=0}^{n} T(n,\mathcal{L})\,x^{\mathcal{L}} = \sum_{\mathcal{L}=0}^{n} \binom{n}{\mathcal{L}} x^{\mathcal{L}}, \tag{3}$$

where $\binom{n}{\mathcal{L}} = \frac{n!}{\mathcal{L}!\,(n-\mathcal{L})!}$, n is the degree of polynomial, $\mathcal{L}$ is the index of polynomials, x is the variable.

First four TPs are

$$T_0(x) = 1,$$

$$T_1(x) = 1 + x,$$

$$T_2(x) = 1 + 2x + x^2,$$

$$T_3(x) = 1 + 3x + 6x^2 + 3x^3,$$

$$\vdots$$

The derivative of the TPs is

$$\frac{dT_n(x)}{dx} = \frac{d}{dx}\sum_{\mathcal{L}=0}^{n}\binom{n}{\mathcal{L}}x^{\mathcal{L}} = \sum_{\mathcal{L}=1}^{n}\binom{n}{\mathcal{L}-1}x^{\mathcal{L}-1}. \tag{4}$$

Solving VFIDE by combining LSM with TPs

This part will solve VFIDEs by collecting LSM [28] with TPs for different degree. Presume that the approximation solution as

$$u_n(x) = \sum_{i=0}^{n} c_i\,\mathcal{T}_i(x),\ \ x \in [a,b]\ , \tag{5}$$

where, $\mathcal{T}_i(x)$ are the TPs of degrees i and c_i are unknown constants. By inserting Eq. (5) into Eq. (1).

yield

$$\sum_{i=0}^{n} c_i \mathcal{T}_i^{(\ell)}(x) = f(x) + \lambda_1 \int_0^x K_1(x,t) \sum_{i=0}^{n} c_i \mathcal{T}_i(t)\, dt + \lambda_2 \int_a^b K_2(x,t) \sum_{i=0}^{n} c_i \mathcal{T}_i(t)\, dt\,. \tag{6}$$

The residual equation is given by the formula

$$D_n(x, c_i) = \sum_{i=0}^{n} c_i \mathcal{T}_i^{(\ell)}(x) - f(x) - \left[\lambda_1 \int_0^x K_1(x,t) \sum_{i=0}^{n} c_i \mathcal{T}_i(t)\, dt + \lambda_2 \int_a^b K_2(x,t) \sum_{i=0}^{n} c_i \mathcal{T}_i(t)\, dt\right]. \tag{7}$$

LSM in Eq. (7) yields

$$r(c_0, c_1, \ldots, c_n) = \int_a^b [D_n(x, c_i)]^2\, w(x)\, dx, \tag{8}$$

set $w(x) = 1$ [29] (for simplicity) to obtain

$$r(c_0, c_1, \ldots, c_n) = \int_a^b \left[\sum_{i=0}^{n} c_i \mathcal{T}_i^{(\ell)}(x) - \left\{f(x) + \left[\lambda_1 \int_0^x K_1(x,t) + \lambda_2 \int_a^b K_2(x,t)\right] \times \sum_{i=0}^{n} c_i \mathcal{T}_i(t)\, dt\right\}\right]^2 dx, \tag{9}$$

where $w(x)$ is the positive weight function defined in the interval $[a, b]$.

It is necessary to find the values of unknown constant c_i, $i = 0(1)n$ in order to reduce r. For c_i $\forall i$ to reduce r, it is a requirement that

$$\frac{\partial r}{\partial c_i} = 0, \quad i = 0(1)n, \tag{10}$$

then by applying Eq. (10) on Eq. (9) to yield:

$$\int_a^b \left[\sum_{i=0}^{n} c_i \mathcal{T}_i^{(\ell)}(x) - \left\{f(x) + \left[\lambda_1 \int_0^x K_1(x,t) + \lambda_2 \int_a^b K_2(x,t)\right] \sum_{i=0}^{n} c_i \mathcal{T}_i(t)\, dt\right\}\right] dx \times \left[\mathcal{T}_j^{(\ell)}(x) - \left\{\lambda_1 \int_0^x K_1(x,t) + \lambda_2 \int_a^b K_2(x,t)\right\} \times \mathcal{T}_j(t)\, dt\right] dx = 0. \tag{11}$$

The matrix form of the Eq. (11), which is an algebraic system of $(n+1)$ equations in $(n+1)$ unknowns c_i, $\forall i$, is as follows:

$$\psi\, C = \mathcal{F}\,, \tag{12}$$

where,

$$\psi = \begin{pmatrix} \int_a^b E_n(x, c_0)\, v_0 dx & \int_a^b E_n(x, c_1)\, v_0 dx & \ldots & \int_a^b E_n(x, c_n)\, v_0 dx \\ \int_a^b E_n(x, c_0)\, v_1 dx & \int_a^b E_n(x, c_1)\, v_1 dx & \ldots & \int_a^b E_n(x, c_n)\, v_1 dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b E_n(x, c_0)\, v_n dx & \int_a^b E_n(x, c_1)\, v_n dx & \ldots & \int_a^b E_n(x, c_n)\, v_n dx \end{pmatrix}, \tag{13}$$

$$\mathcal{F} = \begin{pmatrix} \int_a^b v_0 f(x)\, dx \\ \int_a^b v_1 f(x)\, dx \\ \vdots \\ \int_a^b v_n f(x)\, dx \end{pmatrix}, \tag{14}$$

$$E_n(x, c_i) = \sum_{i=0}^{n} c_i \mathcal{T}_i^{(\ell)}(x) - \left\{\lambda_1 \int_0^x K_1(x,t) + \lambda_2 \int_a^b K_2(x,t)\right\} \sum_{i=0}^{n} c_i \mathcal{T}_i(t)\, dt, \tag{15}$$

and

$$v_i = \mathcal{T}_i^{(\ell)}(x) - \left\{\lambda_1 \int_0^x K_1(x,t) + \lambda_2 \int_a^b K_2(x,t)\right\} \mathcal{T}_i(t)\,dt. \quad (16)$$

To ensure that the LSM is accurately defined, for getting a unique solution, the minimization strategy is required $\forall x \in \bar{\Omega}$, which is equivalent to the matrix's non-singularity ψ.

Property[30]: $\forall x \in \bar{\Omega}$ The matrix ψ defined in Eq. (12) is non-singular.

Eq. (1) is a linear algebraic system of $(n+1)$ equations with unknown orthogonal coefficients $c_i, \forall i$. Applying the requirements, a different version of Eq. (12) can be described as

$$[U_j : \beta_j], \ j = 0(1)(N-1) \quad (17)$$

where

$$U_j = [u_{j0} \quad u_{j1} \quad u_{j2} \quad \ldots \quad u_{jN}], \ j = 0(1)(N-1). \quad (18)$$

By replacing the row matrices Eq. (18) with the last (n) rows of the matrix form Eq. (12), may obtain the solution to Eq. (1) under conditions Eq. (2) and obtain the new augmented matrix[31–33]

$$[\tilde{\psi} : \tilde{\mathcal{F}}] = \begin{pmatrix} \int_a^b E_n(x,c_0)v_0dx & \int_a^b E_n(x,c_1)v_0dx & \ldots & \int_a^b E_n(x,c_n)v_0dx & : & \mathcal{F}_0 \\ \int_a^b E_n(x,c_0)v_1dx & \int_a^b E_n(x,c_1)v_1dx & \ldots & \int_a^b E_n(x,c_n)v_1dx & : & \mathcal{F}_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \int_a^b E_n(x,c_0)v_{n_{N_0}}dx & \int_a^b E_n(x,c_1)v_{n_{N_1}}dx & \ldots & \int_a^b E_n(x,c_n)v_{n_{N_n}}dx & : & \mathcal{F}_{n-N} \\ u_{00} & u_{01} & \ldots & u_{0N} & : & \beta_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{(n-1)0} & u_{(n-1)1} & \ldots & u_{(n-1)N} & : & \beta_{N-1} \end{pmatrix}, \quad (19)$$

if rank $\tilde{\psi} = rank[\tilde{\psi} : \tilde{\mathcal{F}}] = n+1$. The solution to Eq. (19) can be expressed as follows

$$C = (\tilde{\psi})^{-1} \tilde{\mathcal{F}}, \quad (20)$$

As a result, both Eq. (1) and Eq. (2) have unique solutions, as does the matrix C (the coefficients $(c_0, c_1, c_2, \ldots, c_n)$. This solution is provided by Eq. (5). If $\tilde{\psi} = \mathrm{rank}[\tilde{\psi} : \tilde{\mathcal{F}}] < n+1$, when $|\tilde{\psi}| = 0$, then can get an approximation of the solution[34].

There is not a solution if $\tilde{\psi} \neq \mathrm{rank}[\tilde{\psi} : \tilde{\mathcal{F}}] < n+1$. In the same manner, used in the previously stated orthogonal polynomial this yields a $(n+1)$-equations algebraic linear system with $(n+1)$ unknown polynomial coefficients $c_i, \forall i$.

Convergence analysis

Eq. (20) is one of the methods for calculating the unknowing Touchard coefficients $(c_0, c_1, c_2, \ldots, c_n)$. As a result, the solution to Eq. (1) is unique, and the truncated TPs in Eq. (5) provide it. Now, the approximate solution $u_n(x)$ and its derivatives should satisfy the following equation when they are substituted in Eq. (1):[6]

$$x = x_\lambda \in [0,1], \quad \lambda = 0(1)n$$

$$ER_n(x_\lambda) = \left| \left(\sum_{i=0}^n c_i \mathcal{T}_i(x_\lambda)\right)^{(\ell)} - f(x_\lambda) - \left[\lambda_1 \int_0^x K_1(x_\lambda,t) + \lambda_2 \int_a^b K_2(x_\lambda,t)\right] \times \sum_{i=0}^n c_i \mathcal{T}_i(t)dt \right| \cong 0$$

and $ER_n(x_\lambda) \leq 10^{-x_\lambda}$.

If $\max(10^{-x_\lambda}) = 10^{-x}$ is specified, the truncation constraint n is extended until the difference $ER_n(x_\lambda)$ between all the points $x_\lambda \leq 10^{-x}$. Alternatively put, the association can be used to estimate the error function $ER_n(x_\lambda)$:

$$ER_n(x) = \left(\sum_{i=0}^n c_i \mathcal{T}_i(x)\right)^{(\ell)} - f(x) - \left[\lambda_1 \int_0^x K_1(x,t) + \lambda_2 \int_a^b K_2(x,t)\right] \sum_{i=0}^n c_i \mathcal{T}_i(t)dt,$$

therefore, $ER_n(x) \to 0$ when n is extremely large, the error function reduces (3 and 10).

Results and discussion

The study utilized MATLAB 2023 to solve four numerical examples. The tables of absolute error for these examples demonstrate that the method employing the constructed Touchard polynomial is both accurate and converges with fewer iterations compared to alternative approximation methods. Overall, the results suggest that combining the least-squares method with Touchard polynomials is a reliable and effective approach for solving the given numerical examples. More information on the specific examples and methodology, as well as a discussion of the advantages and limitations of combining the least-squares method with Touchard polynomials in relation to other approaches, would further enhance the analysis of the results.

Illustration examples

Four MIDEs will be resolved to demonstrate the precision and effectiveness of the suggested approach.

The following notations will be defined to show the absolute and relative errors ($Abs.\ Error$ and $Rel.\ Error$) of a modern numerical model:

$$Absolute\ Error = |u_n(x) - u(x)|, \quad a \le x \le b, \quad n = 1(1)\infty$$

$$Relative\ Error = \left|\frac{u_n(x) - u(x)}{u(x)}\right|, \quad a \le x \le b, \quad n = 1(1)\infty$$

where the exact solution is $u(x)$ and the approximate solution is $u_n(x)$.

Example 1: Consider the following MIDE in the form[7,21]

$$u'''(x) + \sin(x^2)u(x) = x^2\sin(x^2) - \frac{x^3}{3} + \int_0^x\int_0^1 xtu'(t)\,dx\,dt$$

where $u''(0) = u'(0) = u(0) = 0$ are the initial conditions and $u(x) = x^2$ provides the exact solution.

The comparison of results is shown in Table 1. Additionally, Fig. 1 displays both the exact and approximative solutions for n = 3.

Through comparison of the results that were obtained from the application of[7] and[21] where the ADM and MADM methods in[7] were applied at n = 10, while the HAM approach described in[21] was used to solve the equation at n = 3 and n = 4. Concluding from that the results of the present method, at n = 3 has better and more accurate results when compared to the exact solution and the results of the previous methods, which are shown in Table 1 and Fig. 1.

Example 2: Consider the following MIDE in the form[35]

$$xu''(x) - xu'(x) + 2u(x) = \frac{x^4+17}{12} - \frac{x^3+13x}{6} - \frac{x^2}{2} + \left[\int_0^x (x-t) + \int_0^1 (x+t)\right]u(t)\,dt$$

where $u(0) = u'(0) - 2u(1) + 2u(0) = 1$ are the initial conditions and $u(x) = 1 + x - x^2$ provides the exact solution.

The comparison of results is shown in Table 2. Additionally, Fig. 2 displays both the exact and approximative solutions for n = 3.

Through comparison of the results that were obtained from the application of[35] where the Taylor polynomial method was at n = 5 in.[35] Concluding from that: the results of the present method at n = 3 has better and more accurate results when compared to the exact solution and the results of the previous methods, which are shown in Table 2 and Fig. 2.

Table 1. Comparison of absolute and relative errors of example 1 for n = 3 and h = 0.1.

h	Exact solution	Approximate Solution	$Abs.\ Error_{n=3}$	$Rel.\ Error_{n=3}$
0.1	0.01	0.01	1.1478×10^{-16}	1.1478×10^{-14}
0.2	0.04	0.04	1.4902×10^{-16}	3.7255×10^{-15}
0.3	0.09	0.09	1.8947×10^{-16}	2.1052×10^{-15}
0.4	0.16	0.16	2.3664×10^{-16}	1.4790×10^{-15}
0.5	0.25	0.25	2.9105×10^{-16}	1.1642×10^{-15}
0.6	0.36	0.36	3.5323×10^{-16}	9.8120×10^{-16}
0.7	0.49	0.49	4.2369×10^{-16}	8.6467×10^{-16}
0.8	0.64	0.64	5.0294×10^{-16}	7.8585×10^{-16}
0.9	0.81	0.81	5.9151×10^{-16}	7.3026×10^{-16}

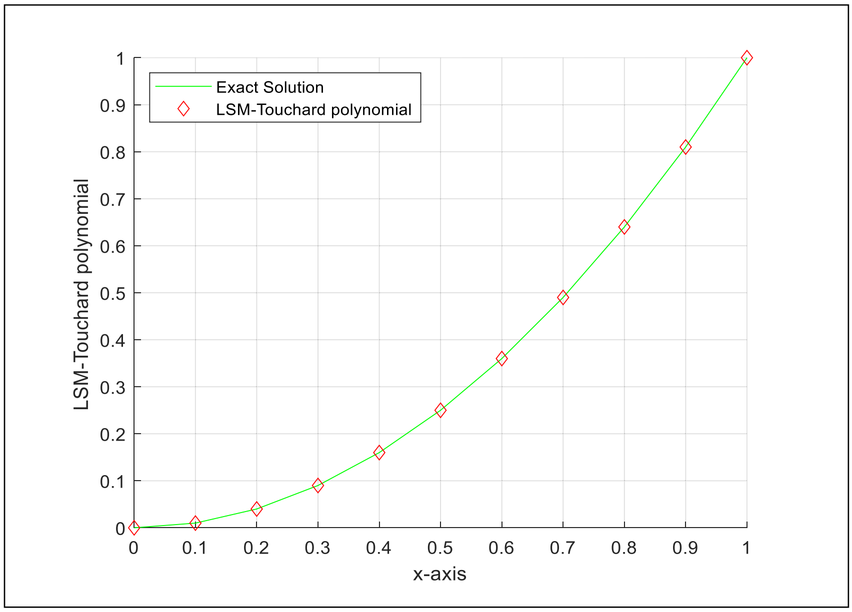

Fig. 1. Exact and approximate solutions of example 1 at n = 3.

Table 2. Comparison of absolute and relative errors of example 2 for n = 3 and h = 0.1.

h	Exact solution	Approximate solution	$Abs.\ Error_{n=3}$	$Rel.\ Error_{n=3}$
0.1	1.0900	1.0900	3.6920×10^{-15}	2.7739×10^{-15}
0.2	1.1600	1.1600	4.7933×10^{-15}	4.1321×10^{-15}
0.3	1.2100	1.2100	6.0942×10^{-15}	5.0365×10^{-15}
0.4	1.2400	1.2400	7.6115×10^{-15}	6.1383×10^{-15}
0.5	1.2500	1.2500	9.3619×10^{-15}	7.4895×10^{-15}
0.6	1.2400	1.2400	1.1362×10^{-14}	9.1628×10^{-15}
0.7	1.2100	1.2100	1.3628×10^{-14}	1.1263×10^{-14}
0.8	1.1600	1.1600	1.6177×10^{-14}	1.3946×10^{-14}
0.9	1.0900	1.0900	1.9026×10^{-14}	1.7455×10^{-14}

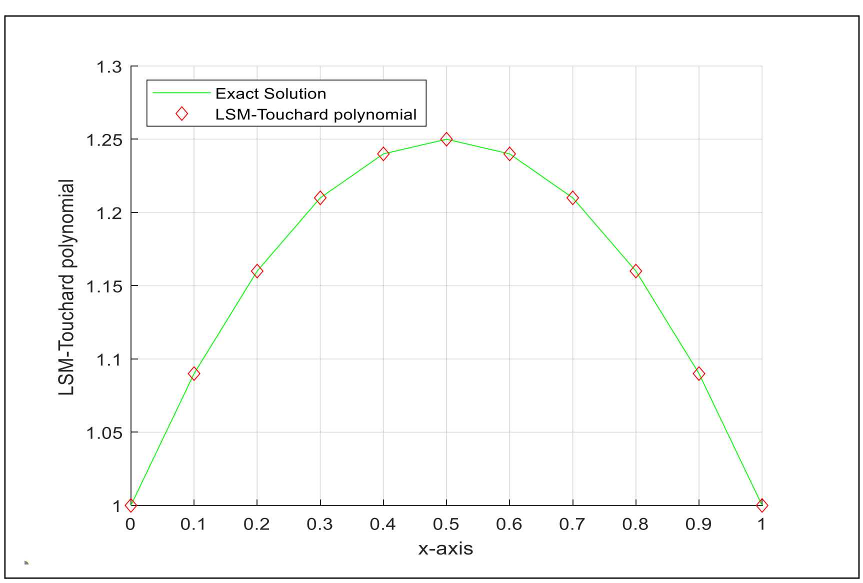

Fig. 2. Exact and approximate solutions of example 2 at n = 3.

Table 3. Comparison of absolute and relative errors of example 3 for n = 7 and h = 0.1.

h	Exact solution	Approximate solution	$Abs.Error_{n=7}$	$Rel.\ Error_{n=7}$
0.1	0.11052	0.11052	3.2116×10^{-9}	2.9060×10^{-8}
0.2	0.24428	0.24428	1.9516×10^{-8}	7.9892×10^{-8}
0.3	0.40496	0.40496	3.3874×10^{-9}	8.3647×10^{-9}
0.4	0.59673	0.59673	2.6585×10^{-8}	4.4551×10^{-8}
0.5	0.82436	0.82436	1.6670×10^{-8}	2.0221×10^{-8}
0.6	1.0933	1.0933	2.3673×10^{-8}	2.1653×10^{-8}
0.7	1.4096	1.4096	2.7897×10^{-8}	1.9790×10^{-8}
0.8	1.7804	1.7804	2.0498×10^{-8}	1.1513×10^{-8}
0.9	2.2136	2.2136	1.7523×10^{-8}	7.9159×10^{-9}

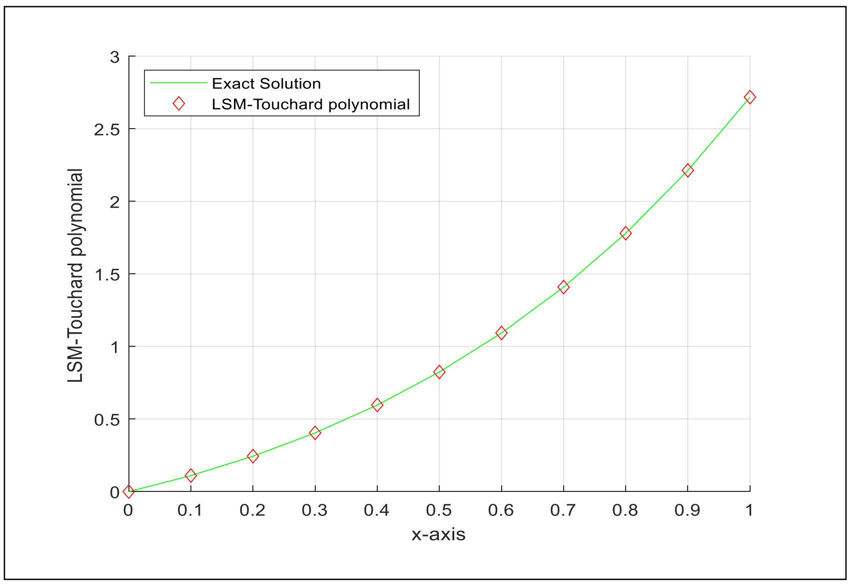

Fig. 3. Exact and approximate solutions of example 3 at n = 7.

Example 3: Consider the following MIDE in the form[36]

$$u'(x) + \left[\int_0^1 \sin x - \frac{1}{2}\int_0^x t\right] u(t)\,dt = 1 + \sin x - \frac{x(x-4)e^x}{2}$$

where $u(0) = 0$ is the initial condition and $u(x) = xe^x$ provides the exact solution.

The comparison of results is shown in Table 3. Additionally, Fig. 3 displays both the exact and approximative solutions for n = 7.

Through comparison of the results obtained from the application of[36] where the power series expansion principle is at n = 10 in.[36] Concluding from that the results of the present method: at n = 3 has better and more accurate results when compared to the exact solution and the results of the previous methods, which are shown in Table 3 and Fig. 3.

Example 4: Consider the following MIDE in the form[37]

$$u'(x) = (x+1)e^{-2} - \frac{2e^{-x}}{3} - \frac{2x}{3} + \left[\int_0^x (x-t) + \int_0^2 (tx+t)\right] u(t)\,dt$$

where $u(0) = \frac{1}{3}$ is the initial condition and $u(x) = \frac{e^{-x}}{3}$ provides the exact solution.

The comparison of results is shown in Table 4. Additionally, Fig. 4 displays both the exact and approximative solutions for n = 9.

Through comparison of the results obtained from the application of[37] where the cubic Legendre spline pooling method in[37] were applied at n = 9. Concluding from that note that the results of the present method at n = 9 have better and more accurate results when compared to the exact solution and the results of the previous methods, which are shown in Table 4 and Fig. 4.

Table 4. Comparison of absolute and relative errors of example 4 for n = 9 and h = 0.1.

h	Exact solution	Approximate solution	$Abs.\ Error_{n=9}$	$Rel.\ Error_{n=9}$
0.1	0.30161	0.30161	1.0916×10^{-10}	3.6192×10^{-10}
0.2	0.27291	0.27291	1.5909×10^{-9}	5.8292×10^{-9}
0.3	0.24694	0.24694	2.4261×10^{-10}	9.8249×10^{-10}
0.4	0.22344	0.22344	1.5788×10^{-9}	7.0659×10^{-9}
0.5	0.20218	0.20218	1.8117×10^{-9}	8.9608×10^{-9}
0.6	0.18294	0.18294	3.9668×10^{-10}	2.1684×10^{-9}
0.7	0.16553	0.16553	1.3814×10^{-9}	8.3455×10^{-9}
0.8	0.14978	0.14978	2.1607×10^{-9}	1.4426×10^{-8}
0.9	0.13552	0.13552	1.4214×10^{-9}	1.0488×10^{-8}

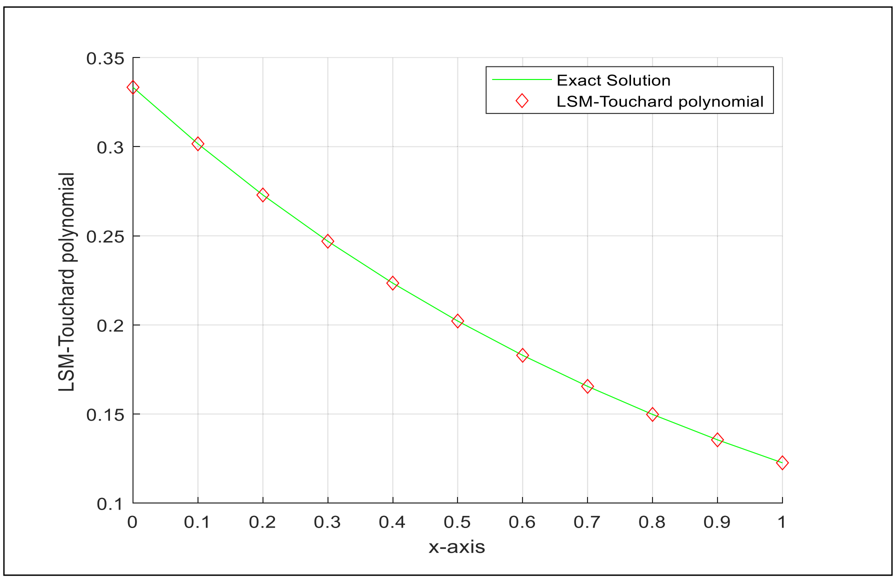

Fig. 4. Exact and approximate solutions of example 4 at n = 9.

Conclusion

Our approach to solving Volterra-Fredholm (mixed) integro-differential equations of the second kind is described in this study. To formulate this method, different degrees of Touchard polynomials were combined with the least-squares weighted residual method. The numerical results are presented and contrasted with various approaches and exact solutions found in the literature. The tables and figures for each example taken into consideration in this paper display these results. As an illustration of the method's computational efficiency, the excellent accuracy and powerful numerical solution it provides for this type of integro-differential equations is noticed. Since it requires a few iterations to achieve high accuracy of results, the proposed method is more flexible than other numerical methods like the Adomian decomposition method and modified Adomian decomposition method, Homotopy analysis method, Taylor polynomials, power series expansion, cubic Legendre spline collocation method, as well as the suggested approach converges quickly, is extremely accurate, efficient, and gives sufficient evidence to provide a good agreement with the exact result.

At last, conclude from that the proposed method described above is a powerful method that is acceptable for solving Volterra-Fredholm (mixed) integro-differential equations of the second kind.

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Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been

included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at The University of Basrah's.

Authors' contribution statement

Z. A. J. performed the computations and verified the analytical method, discussed the results in addition to contributed to the final manuscript. H. O. AL-H. conceived of the presented idea and discussed the results as well as contributed to the final manuscript.

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دمج طريقة المربعات الصغرى مع متعددة حدود توجارد لحل المعادلات التفاضلية-التكاملية المختلطة

حميدة عودة الحميدي ، زهراء عدنان جميل

القسم الرياضيات، كلية التربية للعلوم الصرفة ، جامعة البصرة، البصرة، العراق.

الخلاصة

باستخدام طريقة المربعات الصغرى المعروفة (LSM) مدمجة مع متعددات الحدود من نوع Touchard (TPs) وبدرجات مختلفة لأيجاد الحلول العددية لمعادلات فولتيرا-فريدهولم التفاضلية-التكاملية (VFIDEs) ومعادلات فولتيرا-فريدهولم التكاملية-التفاضلية المختلطة (MIDEs) من النوع الثاني حيث هناك العديد من الطرق التي وجدت الحل التقريبي للمعادلات التفاضلية-التكاملية ((VFIDEs (MIDEs) مثل طريقة تحليل أدوميان وطريقة تحليل أدوميان المعدّلة وطريقة التحليل الهوموتوبي ومتعددات حدود تايلور وتوسيع متسلسلة القوى وطريقة تجميع ليجندر لشريحة تكعيبية. في هذا العمل، قدمنا طريقة تعتمد على دمج كلاً من LSM مع TPs وهذا ما يُمثل عنصر أساسي في هذا البحث ومن خلال تنفيذ مثل هذه الطريقة، يتولد نظام من المعادلات الجبرية التي يمكن حلها بالطرق المعروفة سابقاً في حل الأنظمة الخطية. تم حل عدد من الأمثلة لمعادلات فولتيرا-فريدهولم التفاضلية-التكاملية VFIDEs (MIDEs) لإظهار دقة وفعالية الطريقة المقدمة مع عدد قليل نسبيًا من التكرارات، عند مقارنة نتائج الطريقة الحالية بنتائج الطرق الأخرى المذكورة في أعلاه والمعروفة في الدراسات، فإن النتائج التقريبية التي تمّ الحصول عليها بواسطة الطريقة المقترحة تؤكد دقتها وكفاءتها وكذلك تم إثبات قابلية تطبيق الطريقة المقترحة وتمت مناقشة تحليل التقارب.

الكلمات المفتاحية: الحلول التقريبية، الحلول الدقيقة، طريقة المربعات الصغرى، المعادلة التفاضلية -التكاملية المختلطة، كثيرات الحدود توجارد.