Baghdad Science Journal

Volume 22 | Issue 6 Article 23

6-24-2025

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How to Cite this Article

Abdulkarem, Noor Abdulamer; Abed, Taghreed Hussein; Abdulkareem, Farah Abdulameer; and Mohamad, Hussain Ali (2025) "Mostly oscillation for a system of half linear neutral differential equations of the second order with several arguments," *Baghdad Science Journal*: Vol. 22: Iss. 6, Article 23.
DOI: https://doi.org/10.21123/2411-7986.4973

BAGHDAD SCIENCE JOURNAL 2025;22(6):2002–2017

RESEARCH ARTICLE

Mostly Oscillation for a System of Half Linear Neutral Differential Equations of the Second Order with Several Arguments

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ABSTRACT

This paper studies the oscillation properties and asymptotic behavior of all solutions of the 2×2 system of second-order half-linear neutral differential equations. Four results are obtained in this research. The first and second results are auxiliary results while the third and fourth results are main results. All possible cases of non-oscillating bounded solutions for this system are estimated and analyzed. It is noted that the parameters that affect the volatility of the solutions are Qi,Ri on the one hand and r1 and r2 on the other hand. For this purpose, and through investigation, it is shown that there are only fourteen possible cases of non-oscillating bounded solutions for this system, so all these cases must be treated, in the first result as well as the second, some new necessary and sufficient conditions were obtained to ensure that there are no non-oscillating bounded solutions in these cases, and thus all possible solutions for this system, if they exist, will be only oscillating solutions and there are no non-oscillating solutions for this type of equations. Some examples are included to illustrate all the results obtained.

Keywords: Asymptotic behavior, Half linear system, Neutral differential equations, Oscillation, Second order

Introduction

Consider the half-linear system

$$\left.\begin{aligned}(r_1(t)(\xi_1(t)+\mathcal{P}_1(t)\xi_1(\tau_1(t)))')^{\alpha_1})' &= \sum_{i=1}^{n}\mathcal{Q}_i(t)G_i(\xi_2(\rho_i(t)))\\ (r_2(t)(\xi_2(t)+\mathcal{P}_2(t)\xi_2(\tau_2(t)))')^{\alpha_2})' &= \sum_{i=1}^{n}\mathcal{R}_i(t)\mathcal{H}_i(\xi_1(\sigma_i(t)))\end{aligned}\right\},\ t\ge t_0>0. \tag{1}$$

Under the following assumptions

$$\begin{aligned}&r_j,\mathcal{P}_j,\ \mathcal{F}_j,\ \tau_j,\ \mathcal{Q}_i,\mathcal{R}_i\ \rho_i,\sigma_i\in \mathbb{C}\left[[t_0,\infty);R\right],\ r_j(t)>0,\lim_{t\to\infty}\tau_j(t)=\infty,\\ &\lim_{t\to\infty}\rho_i(t)=\infty,\lim_{t\to\infty}\sigma_i(t)=\infty,\rho_i(t)\le t,\ \sigma_i(t)\le t,\ G_i,\mathcal{H}_i\in\mathbb{C}\left[R;R\right],\\ &zG_i(z)>0,\ z\mathcal{H}_i(z)>0,\ j=1,2,\ i=1,\ 2,\ \ldots,\ n,\end{aligned} \tag{2}$$

Received 17 September 2023; revised 8 September 2024; accepted 10 September 2024.
Available online 24 June 2025

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https://doi.org/10.21123/2411-7986.4973

α_1 and α_2 are the quotient of positive odd integers. Let

$$\begin{aligned} \mathcal{U}_1(t) &= \xi_1(t) + \mathcal{P}_1(t)\,\xi_1(\tau_1(t)) \\ \mathcal{U}_2(t) &= \xi_2(t) + \mathcal{P}_2(t)\,\xi_2(\tau_2(t)), \end{aligned} \tag{3}$$

System 1 can be rewritten as

$$\begin{aligned} \left(r_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1}\right)' &= \sum_{i=1}^{n} \mathcal{Q}_i(t)\, G_i(\xi_2(\rho_i(t))) \\ \left(r_2(t)\left(\mathcal{U}_2'(t)\right)^{\alpha_2}\right)' &= \sum_{i=1}^{n} \mathcal{R}_i(t)\, \mathcal{H}_i(\xi_1(\sigma_i(t))), \end{aligned} \tag{4}$$

By a solution to system 1, means a vector functions $(\xi_1(t), \xi_2(t))$ such that $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1},\ r_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ are continuously differentiable and $(\xi_1(t), \xi_2(t))$ satisfies system 1. The solutions that satisfy the condition $\sup\{|\xi_1(t)| : t \geq T\} > 0, \sup\{|\xi_2(t)| : t \geq T\} > 0,\ T \geq t_0$ are the solutions concerned in this research. As usual, a function is said to be oscillatory if it has arbitrarily large zeros, otherwise, it is nonoscillatory. A solution of system 1 is said to be oscillatory if all of its components are oscillatory. In the articles[1–3] conditions for the oscillation and for the existence of nonoscillatory solutions with polynomial growth at infinity are given for the second order neutral differential equations. It has been proven that every bounded solution of a neutral-type system converges to Equilibrium.[4–6] Some sufficient conditions have been obtained to ensure that all solutions of neutral systems oscillate or converge[7,8]

Some research has been devoted to studying the oscillation of solutions of second-order halflinear neutral differential equations, the classical Riccati transformation technique was used by taking into account that part of the total effect of delay that was neglected in previous results.[9–11] Asymptotic behavior and oscillation of all solutions of impulsive neutral differential equations with positive and negative coefficients and with impulsive integral terms were investigated, the convergence of all nonoscillatory solutions to zero is ensured by some sufficient conditions.[12,13] The existence of nonoscillatory solution of neutral second order differential equations is studied in article.[14] In this article asymptotic behavior and oscillation of second order half linear neutral system with several arguments and several delays are investigated. Before presenting the results, the following lemmas need to be proved:

Lemma 1: *Assume that $(\xi_1(t), \xi_2(t))$ be nonoscillatory bounded solution of sys. 1, $\mathcal{Q}_i(t) \leq 0, \mathcal{R}_i(t) \leq 0,\ i = 1, 2, \ldots n,\ t \geq t_0$, and*

$$\int_T^\infty \left(\frac{1}{r_j(t)}\right)^{\frac{1}{\alpha_j}} dt = \infty, \qquad j = 1, 2, \quad T \geq t_0. \tag{5}$$

Then there are only the following possible cases:

1. $\mathcal{U}_i(t) > 0, \mathcal{U}_i'(t) > 0, (r_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \leq 0,$ and either $\text{Lim}_{t\to\infty}\xi_i(t) = \infty$ or $\xi_i(t)$ is bounded away from zero if $0 \leq \mathcal{P}_i(t) < 1,\ i = 1, 2$.
2. $\mathcal{U}_1(t) > 0, \mathcal{U}_1'(t) > 0, (r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \leq 0,$ and either $\text{Lim}_{t\to\infty}\xi_1(t) = \infty$ or $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, and $\xi_2(t)$ oscillates.
3. $\xi_1(t)$ oscillates and $\mathcal{U}_2(t) > 0, \mathcal{U}_2'(t) > 0, (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \leq 0,$ and either $\text{Lim}_{t\to\infty}\xi_2(t) = \infty,$ or $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$.
4. $\mathcal{U}_i(t) < 0, \mathcal{U}_i'(t) < 0, (r_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \geq 0,$ and either$\text{Lim}_{t\to\infty}\xi_i(t) = -\infty,$ or $\xi_i(t)$ is bounded away from zero if $0 \leq \mathcal{P}_i(t) < 1,\ i = 1, 2$.
5. $\xi_1(t)$ oscillates and $\mathcal{U}_2(t) < 0, \mathcal{U}_2'(t) < 0, (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \geq 0,$ and either $\text{Lim}_{t\to\infty}\xi_2(t) = -\infty,$ or $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$.
6. $\mathcal{U}_1(t) < 0, \mathcal{U}_1'(t) < 0, (r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0,$ and either $\text{Lim}_{t\to\infty}\xi_1(t) = -\infty$ or $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, and $\xi_2(t)$ oscillates.
7. $\mathcal{U}_1(t) > 0, \mathcal{U}_1'(t) > 0, (r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0,\ \text{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty,\ \text{Lim}_{t\to\infty}\xi_1(t) = \infty,$ $\mathcal{U}_2(t) < 0, \mathcal{U}_2'(t) > 0, (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \leq 0,\ \text{Lim}_{t\to\infty}r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0.$

8. $\mathcal{U}_1(t) > 0, \mathcal{U}_1^{'}(t) < 0, (r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1})' \geq 0,$ $\text{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1} = 0,$
   $\mathcal{U}_2(t) < 0,$ $\mathcal{U}_2^{'}(t) < 0,$ $(r_2(t)\mathcal{U}_2^{'}(t))' \leq 0,$ $\text{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty, \text{Lim}_{t\to\infty}\xi_2(t) = -\infty.$
9. $\mathcal{U}_1(t) > 0, \mathcal{U}_1^{'}(t) > 0, (r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1})' \geq 0,$ $\text{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty,$ $\text{Lim}_{t\to\infty}\xi_1(t) = \infty,$
   $\mathcal{U}_2(t) < 0, \mathcal{U}_2^{'}(t) < 0, (r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2})' \leq 0, \text{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty,$ $\text{Lim}_{t\to\infty}\xi_2(t) = -\infty.$
10. $\mathcal{U}_1(t) > 0, \mathcal{U}_1^{'}(t) < 0, (r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1})' \geq 0,$ $\text{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1} = 0,$
   $\mathcal{U}_2(t) < 0, \mathcal{U}_2^{'}(t) > 0,$ $(r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2})' \leq 0,$ $\text{Lim}_{t\to\infty} r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2} = 0.$
11. $\mathcal{U}_1(t) < 0, \mathcal{U}_1^{'}(t) > 0, (r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1})' \leq 0,$ $\text{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1} = 0,$
   $\mathcal{U}_2(t) > 0,$ $\mathcal{U}_2^{'}(t) > 0,$ $(r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2})' \geq 0,$ $\text{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty, \text{Lim}_{t\to\infty}\xi_2(t) = \infty.$
12. $\mathcal{U}_1(t) < 0,$ $\mathcal{U}_1^{'}(t) < 0,$ $(r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1})' \leq 0,$ $\text{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty,$ $\text{Lim}_{t\to\infty}\xi_1(t) = -\infty,$
   $\mathcal{U}_2(t) > 0,$ $\mathcal{U}_2^{'}(t) < 0, (r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2})' \geq 0,$ $\text{Lim}_{t\to\infty} r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2} = 0.$
13. $\mathcal{U}_1(t) < 0, \mathcal{U}_1^{'}(t) > 0, (r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1})' \leq 0,$ $\text{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1} = 0,$
   $\mathcal{U}_2(t) > 0, \mathcal{U}_2^{'}(t) < 0, (r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2})' \geq 0,$ $\text{Lim}_{t\to\infty} r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2} = 0.$
14. $\mathcal{U}_1(t) < 0,$ $\mathcal{U}_1^{'}(t) < 0,$ $(r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1})' \leq 0,$ $\text{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty,$ $\text{Lim}_{t\to\infty}\xi_1(t) = -\infty,$
   $\mathcal{U}_2(t) > 0,$ $\mathcal{U}_2^{'}(t) > 0,$ $(r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2})' \geq 0,$ $\text{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty, \text{Lim}_{t\to\infty}\xi_2(t) = \infty.$

Proof: Assume that sys.1 has nonoscillatory bounded solution $(\xi_1(t), \xi_2(t))$ then there are the following possible cases for $t \geq t_0$:

1. $\xi_1(t) > 0, \xi_2(t) > 0,$
2. $\xi_1(t) < 0, \xi_2(t) < 0,$
3. $\xi_1(t) > 0, \xi_2(t) < 0$
4. $\xi_1(t) < 0, \xi_2(t) > 0.$

Case 1. If $\xi_1(t) > 0, \xi_2(t) > 0,$ $t \geq t_0$ and $\xi_1(\sigma_i(t)) > 0, \xi_2(\rho_i(t)) > 0, i = 1, \ldots, n$ then from sys.1 getting

$$(r_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1})' \leq 0,\ (r_2(t)\left(\mathcal{U}_2'(t)\right)^{\alpha_2})' \leq 0,\ t \geq t_0. \tag{6}$$

That means $r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1}$, $r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2}$ non-increasing hence there exists $t_1 \geq t_0$ such that $r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1}$ and $r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2}$ are eventually positive or eventually negative. So there are the following subcases that can be considered:

$$\left.\begin{array}{ll}\textbf{i.} & r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} > 0, r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} > 0,\\ \textbf{ii.} & r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} < 0, r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} < 0,\\ \textbf{iii.} & r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} > 0, r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} < 0,\\ \textbf{iv.} & r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} < 0, r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} > 0,\end{array}\right. \quad t \geq t_1. \tag{7}$$

i. It follows from this case $\mathcal{U}_1'(t) > 0,$ $\mathcal{U}_2'(t) > 0$
Let $\text{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1} = l_1 \geq 0,$ $\text{Lim}_{t\to\infty} r_2(t)(\mathcal{U}_2^{'}(t))^{\alpha_2} = l_2 \geq 0,$

$$r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \geq l_1,\ r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \geq l_2,\ t \geq t_2 \geq t_1$$

$$\mathcal{U}_1^{'}(t) \geq l_1^{\frac{1}{\alpha_1}}\left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}},\ \mathcal{U}_2^{'}(t) \geq l_2^{\frac{1}{\alpha_2}}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}}.$$

Integrating from t_2 to t, yields:

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_2) \geq l_1^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_1(\theta)}\right)^{\frac{1}{\alpha_1}}d\theta,\ \mathcal{U}_2(t) - \mathcal{U}_2(t_2) \geq l_2^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_2(\theta)}\right)^{\frac{1}{\alpha_2}}d\theta.$$

If $l_1 > 0, l_2 > 0,$ then by letting $t \to \infty$, the last inequality leads to $\text{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty,$ $\text{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty,$ claiming that $\text{Lim}_{t\to\infty}\xi_1(t) = \infty,$ $\text{Lim}_{t\to\infty}\xi_2(t) = \infty,$ otherwise $\text{Lim sup}_{t\to\infty}\xi_1(t) = k_1 < \infty,$ $\text{Lim sup}_{t\to\infty}\xi_2(t) = k_2 < \infty,$ since $\mathcal{P}_1(t), \mathcal{P}_2(t)$ are bounded it follows from 3 that $\mathcal{U}_1(t) \leq k_1 + \mathcal{P}_1(t)k_1 = (1 + \mathcal{P}_1(t))k_1 \leq K_1$ and $\mathcal{U}_2(t) \leq (1 + \mathcal{P}_2(t))k_2 \leq K_2$. As $t \to \infty$, leads to $\text{Lim}_{t\to\infty}\mathcal{U}_1(t) \leq K_1 < \infty,$ $\text{Lim}_{t\to\infty}\mathcal{U}_2(t) \leq K_2 < \infty,$ a contradiction. Or if $l_1 = 0, l_2 = 0,$ then $\text{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1^{'}(t))^{\alpha_1} = 0,$ and $\mathcal{U}_1(t)$ is bounded away from zero, claiming that $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, otherwise there exists t_1 large enough such that $\xi_1(t_1) = 0$, then Eq. (3) implies

$$0 = \xi_1(t_1) = \mathcal{U}_1(t_1) - \mathcal{P}_1(t_1)\xi_1(\tau_1(t_1)) > \mathcal{U}_1(t_1) - \xi_1(\tau_1(t_1))$$

a contradiction in a similar way, it can be shown that $\xi_2(t)$ is bounded away from zero if $0 \le \mathcal{P}_2(t) < 1$.

ii. $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} < 0, r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} < 0, t \ge t_1$, then $\mathcal{U}_1'(t) < 0,\ \mathcal{U}_2'(t) < 0$. There exist $b_1,\ b_2 < 0$ such that $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \le b_1,\ r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \le b_2$

$$\mathcal{U}_1'(t) \le b_1^{\frac{1}{\alpha_1}}\left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}},\ \mathcal{U}_2'(t) \le b_2^{\frac{1}{\alpha_2}}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}}, \quad t \ge t_2 \ge t_1.$$

Integrating from t_2 to t to get:

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_2) \le b_1^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_1(s)}\right)^{\frac{1}{\alpha_1}}ds,\ \mathcal{U}_2(t) - \mathcal{U}_2(t_2) \le b_2^{\frac{1}{\alpha_2}}\int_{t_2}^{t}\left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}}ds.$$

Letting $t \to \infty$, gets $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty$, this is a contradiction. Similar procedures as in Case i or ii can be used to prove Case iii or iv.

Case 2. If $\xi_1(t) < 0, \xi_2(t) < 0,\ t \ge t_0$ and $\xi_1(\sigma_i(t)) < 0, \xi_2(\rho_i(t)) < 0$ then from Eq. (3) and sys 1 getting $\mathcal{U}_1(t) < 0, \mathcal{U}_2(t) < 0$ and

$$\left(r_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1}\right)' \ge 0,\ \left(r_2(t)\left(\mathcal{U}_2'(t)\right)^{\alpha_2}\right)' \ge 0,\ t \ge t_0. \tag{8}$$

That is $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1}$, $r_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ nondecreasing hence there exists $t_1 \ge t_0$ such that $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1}$ and $r_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ are eventually positive or eventually negative. So the subcases in 7 can be considered as follows:

i. It follows $\mathcal{U}_1'(t) > 0,\ \mathcal{U}_2'(t) > 0$ and there exist $b_1,\ b_2 > 0$ such that $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \ge b_1,\ r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \ge b_2$,

$$\mathcal{U}_1'(t) \ge b_1^{\frac{1}{\alpha_1}}\left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}},\ \mathcal{U}_2'(t) \ge b_2^{\frac{1}{\alpha_2}}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}},\ t \ge t_2 \ge t_1\ .$$

Integrating from t_2 to t, gets the following:

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_2) \ge b_1^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_1(s)}\right)^{\frac{1}{\alpha_1}}ds,\ \mathcal{U}_2(t) - \mathcal{U}_2(t_2) \ge b_2^{\frac{1}{\alpha_2}}\int_{t_2}^{t}\left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}}ds.$$

Letting $t \to \infty$, gets $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty$, this is a contradiction.

ii. It follows $\mathcal{U}_1'(t) < 0,\ \mathcal{U}_2'(t) < 0$. There exist $l_1,\ l_2 \le 0$ such that $\mathrm{Lim}_{t\to\infty}r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = l_1 \le 0,\ \mathrm{Lim}_{t\to\infty}r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = l_2 \le 0$,

$$r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \le l_1,\ r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \le l_2, \quad t \ge t_2 \ge t_1$$

$$\mathcal{U}_1'(t) \le l_1^{\frac{1}{\alpha_1}}\left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}},\ \mathcal{U}_2'(t) \le l_2^{\frac{1}{\alpha_2}}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}}.$$

Integrating from t_2 to t, yields:

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_2) \le l_1^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_1(s)}\right)^{\frac{1}{\alpha_1}}ds,\ \mathcal{U}_2(t) - \mathcal{U}_2(t_2) \le l_2^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}}ds\ .$$

Letting $t \to \infty$, if $l_1, l_2 < 0$, the last inequality leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty$. Claiming that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = -\infty,\ \mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$, otherwise $\mathrm{Lim\,inf}_{t\to\infty}\xi_1(t) = k_1 > -\infty,\ \mathrm{Lim\,inf}_{t\to\infty}\xi_2(t) = k_2 > -\infty$, since $\mathcal{P}_1(t),\ \mathcal{P}_2(t)$ are bounded it follows from Eq. (3) that

$$\mathcal{U}_1(t) \ge k_1 + \mathcal{P}_1(t)k_1 = (1 + \mathcal{P}_1(t))k_1 \ge K_1 \text{ and } \mathcal{U}_2(t) \ge (1 + \mathcal{P}_2(t))k_2 \ge K_2.$$

as $t \to \infty$, leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) \geq K_1 > -\infty,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) \geq K_2 > -\infty$, a contradiction. If $l_1 = 0, l_2 = 0$, then $\mathrm{Lim}_{t\to\infty}\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0$, and $\mathcal{U}_1(t)$ is bounded away from zero, claiming that $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, otherwise, there exists t_1 large enough such that $\xi_1(t_1) = 0$, then 3 implies

$$0 = \xi_1(t_1) = \mathcal{U}_1(t_1) - \mathcal{P}_1(t_1)\xi_1(\tau_1(t_1)) < \mathcal{U}_1(t_1) - \xi_1(\tau_1(t_1))$$

a contradiction. in a similar way, it can be shown that $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_2(t) < 1$. Similar procedures as in Case i or ii can be used to prove Case iii or iv.

Case 3. If $\xi_1(t) > 0, \xi_2(t) < 0,\ t \geq t_0$, from sys.1 getting $\mathcal{U}_1(t) > 0,\ \mathcal{U}_2(t) < 0$

$$\left(\mathcal{r}_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1}\right)' \geq 0,\ \left(\mathcal{r}_2(t)\left(\mathcal{U}_2'(t)\right)^{\alpha_2}\right)' \leq 0,\ t \geq t_0. \tag{9}$$

That is $\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1}$ nondecreasing and $\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ non-increasing. So by 7, there are only the following subcases that can be considered for $t \geq t_1 \geq t_0$:

i. This case leads to $\mathcal{U}_1(t) > 0,\ \mathcal{U}_1'(t) > 0$, then similarly as in the proof of case **i** getting $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = \infty$, also $\mathcal{U}_2(t)\langle 0,\ \mathcal{U}_2'(t)\rangle 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0$.

ii. This case leads to $\mathcal{U}_1(t) > 0,\ \mathcal{U}_1'(t) < 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0$ and $\mathcal{U}_2(t) < 0,\ \mathcal{U}_2'(t) < 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$.

iii. This case leads to $\mathcal{U}_1(t) > 0,\ \mathcal{U}_1'(t) > 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty$, which leads to $\mathrm{Lim}_{t\to\infty}\xi_1(t) = \infty$, and $\mathcal{U}_2(t) < 0,\ \mathcal{U}_2'(t) < 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty$, leads to $\mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$.

iv. This case leads to $\mathcal{U}_1(t) > 0,\ \mathcal{U}_1'(t) < 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_2} = 0$ also $\mathcal{U}_2(t)\langle 0,\ \mathcal{U}_2'(t)\rangle 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0$.

Case 4. If $\xi_1(t)\langle 0, \xi_2(t)\rangle 0,\ t \geq t_0$, from sys.1 obtained that $\mathcal{U}_1(t)\langle 0,\ \mathcal{U}_2(t)\rangle 0$

$$\left(\mathcal{r}_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1}\right)' \leq 0,\ \left(\mathcal{r}_2(t)\left(\mathcal{U}_2'(t)\right)^{\alpha_2}\right)' \geq 0,\ t \geq t_0. \tag{10}$$

That is $\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1}$ non-increasing and $\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ non-decreasing. So by (7), there are only the following subcases that can be considered for $t \geq t_1 \geq t_0$:

i. This case leads to $\mathcal{U}_1(t)\langle 0,\ \mathcal{U}_1'(t)\rangle 0$, then similarly as in case 3 getting $\mathrm{Lim}_{t\to\infty}\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0$, also $\mathcal{U}_2(t) > 0,\ \mathcal{U}_2'(t) > 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_2(t) = \infty$.

ii. This case leads to $\mathcal{U}_1(t) < 0,\ \mathcal{U}_1'(t) < 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = -\infty$, and $\mathcal{U}_2(t) > 0,\ \mathcal{U}_2'(t) < 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0$.

iii. This case leads to $\mathcal{U}_1(t)\langle 0,\ \mathcal{U}_1'(t)\rangle 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_2} = 0$ and $\mathcal{U}_2(t) > 0,\ \mathcal{U}_2'(t) < 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0$.

iv. This case leads to $\mathcal{U}_1(t) < 0,\ \mathcal{U}_1'(t) < 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = -\infty$, and $\mathcal{U}_2(t) > 0,\ \mathcal{U}_2'(t) > 0$, implies $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_2(t) = \infty$.

Lemma 2: *Assume that $(\xi_1(t), \xi_2(t))$ be a nonoscillatory bounded solution of sys.1 and let $\mathcal{P}_1(t),\ \mathcal{P}_2(t)$ are bounded, let $\mathcal{Q}_i(t), \mathcal{R}_i(t) \geq 0,\ t \geq t_0$, and 5 holds. Then there are only the following possible cases:*

1. $\mathcal{U}_i(t) > 0, \mathcal{U}_i'(t) > 0, (\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \geq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_i(t) = \infty,\ \mathrm{Lim}_{t\to\infty}\xi_i(t) = \infty$
2. $\mathcal{U}_i(t) > 0, \mathcal{U}_i'(t) < 0, (\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \geq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i} = 0$
3. $\mathcal{U}_1(t) > 0, \mathcal{U}_1'(t) > 0, (\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty,\ \mathrm{Lim}_{t\to\infty}\xi_1(t) = \infty$,
   $\mathcal{U}_2(t) > 0, \mathcal{U}_2'(t) < 0, (\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \geq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0$.
4. $\mathcal{U}_1(t) > 0, \mathcal{U}_1'(t) < 0, (\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0$,
   $\mathcal{U}_2(t) > 0, \mathcal{U}_2'(t) > 0, (\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \geq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty,\ \mathrm{Lim}_{t\to\infty}\xi_2(t) = \infty$.
5. $\mathcal{U}_i(t)\langle 0, \mathcal{U}_i'(t)\rangle 0, (\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \leq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i} = 0$.
6. $\mathcal{U}_i(t) < 0,\ \mathcal{U}_i'(t) < 0,\ (\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \leq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_i(t) = -\infty,\ \mathrm{Lim}_{t\to\infty}\xi_i(t) = -\infty$.
7. $\mathcal{U}_1(t)\langle 0, \mathcal{U}_1'(t)\rangle 0, (\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \leq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0$.
   $\mathcal{U}_2(t) < 0,\ \mathcal{U}_2'(t) < 0,\ (\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \leq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty,\ \mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$.
8. $\mathcal{U}_1(t) < 0,\ \mathcal{U}_1'(t) < 0,\ (\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \leq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty,\ \mathrm{Lim}_{t\to\infty}\xi_1(t) = -\infty$.
   $\mathcal{U}_2(t)\langle 0, \mathcal{U}_2'(t)\rangle 0, (\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \leq 0,\ \mathrm{Lim}_{t\to\infty}\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0$.
9. $\mathcal{U}_1(t) > 0, \mathcal{U}_1'(t) > 0, (\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \leq 0$, and either $\mathrm{Lim}_{t\to\infty}\xi_1(t) = \infty$, *or $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, and $\xi_2(t)$ oscillates.*

10. *$\xi_1(t)$ oscillates and $\mathcal{U}_2(t) < 0, \mathcal{U}_2'(t) < 0, (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \geq 0,$ and either* $\mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$, *or $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_2(t) < 1$*
11. *$\mathcal{U}_1(t) > 0, \mathcal{U}_1'(t) > 0, (r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \leq 0,$* and either $\mathrm{Lim}_{t\to\infty}\xi_1(t) = \infty$, *or $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, $\mathcal{U}_2(t) < 0, \mathcal{U}_2'(t) < 0, (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \geq 0,$ and either* $\mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$, *or $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_2(t) < 1$*
12. *$\xi_1(t)$ oscillates and $\mathcal{U}_2(t) > 0, \mathcal{U}_2'(t) > 0, (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \leq 0,$* and either $\mathrm{Lim}_{t\to\infty}\xi_2(t) = \infty$, *or $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_2(t) < 1$.*
13. *$\mathcal{U}_1(t) < 0, \mathcal{U}_1'(t) < 0, (r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0,$ and either* $\mathrm{Lim}_{t\to\infty}\xi_1(t) = -\infty$, *or $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, and $\xi_2(t)$ oscillates.*
14. *$\mathcal{U}_1(t) < 0, \mathcal{U}_1'(t) < 0, (r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0,$ and either* $\mathrm{Lim}_{t\to\infty}\xi_1(t) = -\infty$, *or $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, and $\mathcal{U}_2(t) > 0, \mathcal{U}_2'(t) > 0, (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \leq 0,$* and either $\mathrm{Lim}_{t\to\infty}\xi_2(t) = \infty$, *or $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_2(t) < 1$.*

Proof: Assume that sys.1 has a nonoscillatory solution $(\xi_1(t), \xi_2(t))$ then there are the following possible cases for $t \geq t_0$:

1. $\xi_1(t) > 0, \xi_2(t) > 0$, 2. $\xi_1(t) < 0, \xi_2(t) < 0$, 3. $\xi_1(t) > 0, \xi_2(t) < 0$, 4. $\xi_1(t)\langle 0, \xi_2(t)\rangle 0$.

Case 1. If $\xi_1(t) > 0, \xi_2(t) > 0,\ t \geq t_0$ and $\xi_1(\sigma_i(t)) > 0, \xi_2(\rho_i(t)) > 0, i = 1, \ldots, n$ then from sys.1 obtaining

$$(r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0,\ (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \geq 0,\ t \geq t_0. \tag{11}$$

That is $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1}$ nondecreasing and $r_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ non-decreasing. So by 7, there are only the following subcases that can be considered for $t \geq t_1 \geq t_0$:

$$\left.\begin{array}{l}\textbf{i.}\ \ r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} > 0, r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} > 0,\\ \textbf{ii.}\ \ r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} < 0, r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} < 0,\\ \textbf{iii.}\ r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} > 0, r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} < 0,\\ \textbf{iv.}\ r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} < 0, r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} > 0,\end{array}\right. \quad t \geq t_1.$$

i. It follows $\mathcal{U}_1(t) > 0,\ \mathcal{U}_1'(t) > 0,$ and $\mathcal{U}_2(t) > 0,\ \mathcal{U}_2'(t) > 0,$ Then, there exist $b_1,\ b_2 > 0$ such that $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \geq b_1,\ r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \geq b_2$

$$\mathcal{U}_1'(t) \geq b_1^{\frac{1}{\alpha_1}}\left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}},\ \mathcal{U}_2'(t) \geq b_2^{\frac{1}{\alpha_2}}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}},\ t \geq t_2 \geq t_1\ .$$

Integrating from t_2 to t to get:

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_2) \geq b_1^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_1(s)}\right)^{\frac{1}{\alpha_1}}ds,\ \mathcal{U}_2(t) - \mathcal{U}_2(t_2) \geq b_2^{\frac{1}{\alpha_2}}\int_{t_2}^{t}\left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}}ds.$$

Letting $t \to \infty$, getting $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty$, $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty$. which implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = \infty$, $\mathrm{Lim}_{t\to\infty}\xi_2(t) = \infty$.

ii. It follows $\mathcal{U}_1'(t) < 0,\ \mathcal{U}_2'(t) < 0$. There exist $l_1,\ l_2 \leq 0$ such that $\mathrm{Lim}_{t\to\infty}r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = l_1 \leq 0$, $\mathrm{Lim}_{t\to\infty}r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = l_2 \leq 0,$

$$r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \leq l_1,\ r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \leq l_2, \quad t \geq t_2 \geq t_1$$

$$\mathcal{U}_1'(t) \leq l_1^{\frac{1}{\alpha_1}}\left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}},\ \mathcal{U}_2'(t) \leq l_2^{\frac{1}{\alpha_2}}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}}.$$

Integrating from t_2 to t, yields:

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_2) \leq l_1^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_1(s)}\right)^{\frac{1}{\alpha_1}}ds,\ \mathcal{U}_2(t) - \mathcal{U}_2(t_2) \leq l_2^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}}ds.$$

Letting $t \to \infty$, if $l_1, l_2 < 0$, the last inequality leads to $\text{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty$, $\text{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty$. a contradiction.

If $l_1 = 0, l_2 = 0$, then $\text{Lim}_{t\to\infty}r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0$, and $\mathcal{U}_1(t)$ is bounded away from zero, claiming that $\xi_1(t)$ is bounded away from zero if $0 \le \mathcal{P}_1(t) < 1$, otherwise there exists t_1 large enough such that $\xi_1(t_1) = 0$, then Eq. (3) implies

$$0 = \xi_1(t_1) = \mathcal{U}_1(t_1) - \mathcal{P}_1(t_1)\xi_1(\tau_1(t_1)) > \mathcal{U}_1(t_1) - \xi_1(\tau_1(t_1))$$

a contradiction. in a similar way, it can be shown that $\xi_2(t)$ is bounded away from zero if $0 \le \mathcal{P}_2(t) < 1$. Procedures similar to **cases i-ii**, can be used to show **cases iii-iv**.

Case 2. If $\xi_1(t) < 0, \xi_2(t) < 0,\ t \ge t_0$ and $\xi_1(\sigma_i(t)) < 0, \xi_2(\rho_i(t)) < 0$ then from Eq. (3) and sys 1, getting $\mathcal{U}_1(t) < 0, \mathcal{U}_2(t) < 0$ and

$$\left(r_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1}\right)' \le 0,\ \left(r_2(t)\left(\mathcal{U}_2'(t)\right)^{\alpha_2}\right)' \le 0,\ t \ge t_0. \tag{12}$$

That is $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1}$ and $r_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ non-increasing hence there exists $t_1 \ge t_0$ such that $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1}$ and $r_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ are eventually positive or eventually negative. So the subcases in 7 can be considered as follows:

i. It follows $\mathcal{U}_1'(t) > 0,\ \mathcal{U}_2'(t) > 0$. There exist $l_1,\ l_2 \ge 0$ such that $\text{Lim}_{t\to\infty}r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = l_1 \ge 0$, $\text{Lim}_{t\to\infty}r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = l_2 \ge 0$,

$$r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \ge l_1,\ r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \ge l_2, \quad t \ge t_2 \ge t_1$$

$$\mathcal{U}_1'(t) \ge l_1^{\frac{1}{\alpha_1}}\left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}},\ \mathcal{U}_2'(t) \ge l_2^{\frac{1}{\alpha_2}}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}}.$$

Integrating from t_2 to t, yields:

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_2) \ge l_1^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_1(s)}\right)^{\frac{1}{\alpha_1}}ds,\ \mathcal{U}_2(t) - \mathcal{U}_2(t_2) \ge l_2^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}}ds\,.$$

Letting $t \to \infty$, if $l_1, l_2 > 0$, the last inequality leads to $\text{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty$, $\text{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty$. a contradiction.

If $l_1 = 0, l_2 = 0$, then $\text{Lim}_{t\to\infty}r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0$, and $\mathcal{U}_1(t)$ is bounded away from zero, claiming that $\xi_1(t)$ is bounded away from zero if $0 \le \mathcal{P}_1(t) < 1$, otherwise there exists t_1 large enough such that $\xi_1(t_1) = 0$, then Eq. (3) implies

$$0 = \xi_1(t_1) = \mathcal{U}_1(t_1) - \mathcal{P}_1(t_1)\xi_1(\tau_1(t_1)) < \mathcal{U}_1(t_1) - \xi_1(\tau_1(t_1))$$

a contradiction. in a similar way, it can be shown that $\xi_2(t)$ is bounded away from zero if $0 \le \mathcal{P}_2(t) < 1$.

ii. It follows $\mathcal{U}_1(t) < 0,\ \mathcal{U}_1'(t) < 0$, and $\mathcal{U}_2(t) < 0,\ \mathcal{U}_2'(t) < 0$, then, there exist $b_1,\ b_2 < 0$ such that $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \le b_1,\ r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \le b_2$

$$\mathcal{U}_1'(t) \le b_1^{\frac{1}{\alpha_1}}\left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}},\ \mathcal{U}_2'(t) \le b_2^{\frac{1}{\alpha_2}}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}},\ t \ge t_2 \ge t_1\,.$$

Integrating from t_2 to t to get:

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_2) \le b_1^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{r_1(s)}\right)^{\frac{1}{\alpha_1}}ds,\ \mathcal{U}_2(t) - \mathcal{U}_2(t_2) \le b_2^{\frac{1}{\alpha_2}}\int_{t_2}^{t}\left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}}ds.$$

Letting $t \to \infty$, getting $\text{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty$, $\text{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty$. which implies that $\text{Lim}_{t\to\infty}\xi_1(t) = -\infty$, $\text{Lim}_{t\to\infty}\xi_2(t) = -\infty$.

Similar procedures as in Case i or ii can be used to prove Case iii or iv.

Case 3. If $\xi_1(t) > 0, \xi_2(t) < 0,\ t \geq t_0$, from sys.1, obtaining $\mathcal{U}_1(t) > 0,\ \mathcal{U}_2(t) < 0$

$$(r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \leq 0,\ (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \geq 0,\ t \geq t_0. \tag{13}$$

That is $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1}$ non-increasing and $r_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ non-decreasing. So by (7), there are only the following subcases that can be considered for $t \geq t_1 \geq t_0$:

i. It follows $\mathcal{U}_1'(t) > 0,\ \mathcal{U}_2'(t) > 0$. Let $\mathrm{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = l_1 \geq 0$,

$$r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \geq l_1,\ ,\quad t \geq t_2 \geq t_1$$

$$\mathcal{U}_1'(t) \geq l_1^{\frac{1}{\alpha_1}}\left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}}.$$

Integrating from t_2 to t, yields:

$\mathcal{U}_1(t) - \mathcal{U}_1(t_2) \geq l_1^{\frac{1}{\alpha_1}} \int_{t_2}^{t} (\frac{1}{r_1(\theta)})^{\frac{1}{\alpha_1}} d\theta$. If $l_1 > 0$, then by letting $t \to \infty$, the last inequality leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty$, claiming that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = \infty$, otherwise $\mathrm{Lim\ sup}_{t\to\infty}\xi_1(t) = k_1 < \infty$, since $\mathcal{P}_1(t),\ \mathcal{P}_2(t)$ are bounded it follows from Eq. (3) that

$$\mathcal{U}_1(t) \leq k_1 + \mathcal{P}_1(t)k_1 = (1 + \mathcal{P}_1(t))k_1 \leq K_1.$$

As $t \to \infty$, leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) \leq K_1 < \infty$, a contradiction. Or if $l_1 = 0$, then $\mathrm{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0$, and $\mathcal{U}_1(t)$ is bounded away from zero, claiming that $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, otherwise there exists t_1 large enough such that $\xi_1(t_1) = 0$, then Eq. (3) implies

$$0 = \xi_1(t_1) = \mathcal{U}_1(t_1) - \mathcal{P}_1(t_1)\xi_1(\tau_1(t_1)) > \mathcal{U}_1(t_1) - \xi_1(\tau_1(t_1))$$

a contradiction.

It follows $\mathcal{U}_2'(t) > 0$ and there exist $b_2 > 0$ such that $r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \geq b_2$,

$$\mathcal{U}_2'(t) \geq b_2^{\frac{1}{\alpha_2}}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}},\ t \geq t_2 \geq t_1.$$

Integrating from t_2 to t, gets the following:

$$\mathcal{U}_2(t) - \mathcal{U}_2(t_2) \geq b_2^{\frac{1}{\alpha_2}} \int_{t_2}^{t}\left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}} ds.$$

Letting $t \to \infty$, getting $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty$, this is a contradiction.

ii. This case leads to $\mathcal{U}_1(t) > 0,\ \mathcal{U}_1'(t) < 0$, implies $\xi_1(t)$ oscillates and $\mathcal{U}_2(t) < 0,\ \mathcal{U}_2'(t) < 0$, either $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$.

Or $\mathrm{Lim}_{t\to\infty} r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0$, and $\mathcal{U}_2(t)$ is bounded away from zero, leads to $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_2(t) < 1$.

iii. This case leads to $\mathcal{U}_1(t) > 0,\ \mathcal{U}_1'(t) > 0$, implies either $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty$, which leads to $\mathrm{Lim}_{t\to\infty}\xi_1(t) = \infty$, or $\mathrm{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1'(t))^{\alpha_2} = 0$ and $\mathcal{U}_2(t) < 0,\ \mathcal{U}_2'(t) < 0$, either $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty$, leads to $\mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$, or $\mathrm{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1'(t))^{\alpha_2} = 0$

iv. This case leads to $\mathcal{U}_1(t) > 0,\ \mathcal{U}_1'(t) < 0$, leads to $\xi_1(t)$ oscillates also $\mathcal{U}_2(t)\langle 0,\ \mathcal{U}_2'(t)\rangle 0$, leads to $\xi_2(t)$ oscillates

Case 4. If $\xi_1(t)\langle 0, \xi_2(t)\rangle 0,\ t \geq t_0$, from sys 1 obtaining $\mathcal{U}_1(t)\langle 0,\ \mathcal{U}_2(t)\rangle 0$

$$(r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0,\ (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \leq 0,\ t \geq t_0. \tag{14}$$

That is $r_1(t)(\mathcal{U}_1'(t))^{\alpha_1}$ nondecreasing and $r_2(t)(\mathcal{U}_2'(t))^{\alpha_2}$ non-increasing. This case can be proven in a similar way case 3.

In the next results, this condition is needed

$$\begin{array}{l} \dfrac{G_i(z)}{z} \geq \lambda_i,\ \lambda = \min\{\lambda_i:\ \lambda_i \geq 0,\ i = 1, 2, ..n\} \\ \dfrac{\mathcal{H}_i(z)}{z} \leq \mu_i,\ \mu = \min\{\mu_i:\ \mu_i \geq 0,\ i = 1, 2, ..n\} \end{array}, \quad z \neq 0. \tag{15}$$

Main results

In this section, two results are given are given, the following results is the first.

Theorem 1: *Assume that* $0 \leq \mathcal{P}_j(t) < 1,\ j = 1, 2,\ \tau_1(t) \leq t,\ \tau_2(t) \leq t,\ \mathcal{Q}_i(t) \leq 0,\ \mathcal{R}_i(t) \leq 0,\ t \geq t_0,\ i = 1, 2, \ldots, n$, *and (5), (15) hold, in addition to the following conditions*

$$\operatorname*{Lim\,sup}_{t\to\infty} \int_T^t \left(\frac{1}{r_1(s)} \int_s^{\delta(s)} \sum_{i=1}^n |\mathcal{Q}_i(v)| \left(1 - \mathcal{P}_2(\rho_i(v))\right) dv \right)^{\frac{1}{\alpha_1}} ds = \infty, \tag{16-a}$$

$$\operatorname*{Lim\,sup}_{t\to\infty} \int_{t_1}^t \left(\frac{1}{r_2(s)} \int_s^{\delta(s)} \sum_{i=1}^n |\mathcal{R}_i(v)| \left(1 - \mathcal{P}_1(\sigma_i(v))\right) d\xi \right)^{\frac{1}{\alpha_2}} ds = \infty. \tag{16-b}$$

$\delta(t) \geq t$. *Then every bounded solution of sys. 1 oscillates.*

Proof: Assume that sys. 1 has nonoscillatory bounded solution $(\xi_1(t), \xi_2(t))$ then by Lemma 1, there are only the following cases:

1. If $\xi_1(t) > 0, \xi_2(t) > 0,$ then $\mathcal{U}_i(t) > 0, \mathcal{U}_i'(t) > 0, (r_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \leq 0,$ and $\xi_i(t)$ is bounded away from zero, $i = 1, 2$.
2. If $\xi_1(t) < 0, \xi_2(t) < 0,$ then $\mathcal{U}_i(t) < 0, \mathcal{U}_i'(t) < 0, (r_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \geq 0,$ and $\xi_i(t)$ is bounded away from zero, $i = 1, 2$.
3. If $\xi_1(t) > 0, \xi_2(t) < 0,$ then $\mathcal{U}_1(t) > 0, \mathcal{U}_1'(t) < 0, (r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0,\ \mathrm{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0,.$ $\mathcal{U}_2(t)\langle 0,\ \mathcal{U}_2'(t)\rangle 0,\ (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \leq 0,\ \mathrm{Lim}_{t\to\infty} r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0.$
4. If $\xi_1(t)\langle 0, \xi_2(t)\rangle 0,$ then $\mathcal{U}_1(t)\langle 0, \mathcal{U}_1'(t)\rangle 0, (r_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \leq 0\ \mathrm{Lim}_{t\to\infty} r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = 0,\ \ \mathcal{U}_2(t) > 0,\ \mathcal{U}_2'(t) < 0,\ (r_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \geq 0,\ \mathrm{Lim}_{t\to\infty} r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = 0$

Case 1. From Eq. (3) it follows

$$\mathcal{U}_i(t) \leq \xi_i(t) + \mathcal{P}_i(t)\,\mathcal{U}_i(\tau_i(t)),\ i = 1, 2.$$

$$\xi_i(t) \geq \mathcal{U}_i(t) - \mathcal{P}_i(t)\,\mathcal{U}_i(\tau_i(t)) \geq (1 - \mathcal{P}_i(t))\,\mathcal{U}_i(t). \tag{17}$$

Integrating the first equation from t to $\delta(t)$ to get

$$r_1(\delta(t))\,(\mathcal{U}_1'(\delta(t)))^{\alpha_1} - r_1(t)\,(\mathcal{U}_1'(t))^{\alpha_1} = \int_t^{\delta(t)} \sum_{i=1}^n \mathcal{Q}_i(s)\,\mathcal{G}_i(\xi_2(\rho_i(s)))\,ds$$

$$-r_1(t)\,(\mathcal{U}_1'(t))^{\alpha_1} \leq \int_t^{\delta(t)} \sum_{i=1}^n \mathcal{Q}_i(s)\,\lambda_i \xi_2(\rho_i(s))\,ds$$

$$r_1(t)\,(\mathcal{U}_1'(t))^{\alpha_1} \geq \lambda \int_t^{\delta(t)} \sum_{i=1}^n |\mathcal{Q}_i(s)|\,(1 - \mathcal{P}_2(\rho_i(s)))\,\mathcal{U}_2(\rho_i(s))\,ds. \tag{18}$$

$$\geq \lambda\,\mathcal{U}_2(\rho(t)) \int_t^{\delta(t)} \sum_{i=1}^n |\mathcal{Q}_i(s)|\,(1 - \mathcal{P}_2(\rho_i(s)))\,ds,\ \ \rho(t) = \min\{\rho_i(t) : i = 1, 2, \ldots n\}$$

$$\mathcal{U}_1'(t) \geq \lambda^{\frac{1}{\alpha_1}} \left(\frac{\mathcal{U}_2(\rho(t))}{r_1(t)} \int_t^{\delta(t)} \sum_{i=1}^n |\mathcal{Q}_i(s)|\,(1 - \mathcal{P}_2(\rho_i(s)))\,ds \right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to get

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_1) \geq \lambda^{\frac{1}{\alpha_1}} \int_{t_1}^{t} \left(\frac{\mathcal{U}_2(\rho(s))}{r_1(s)} \int_s^{\delta(s)} \sum_{i=1}^{n} |\mathcal{Q}_i(v)| (1 - \mathcal{P}_2(\rho_i(v))) \, dv \right)^{\frac{1}{\alpha_1}} ds$$

$$\geq \lambda^{\frac{1}{\alpha_1}} \mathcal{U}_2(\rho(t_1)) \int_{t_1}^{t} \left(\frac{1}{r_1(s)} \right)^{\frac{1}{\alpha_1}} \left(\int_s^{\delta(s)} \sum_{i=1}^{n} |\mathcal{Q}_i(v)| (1 - \mathcal{P}_2(\rho_i(v))) \, dv \right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ and by using the condition (16) leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty$, implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = \infty$, a contradiction, similarly obtaining $\mathrm{Lim}_{t\to\infty}\xi_2(t) = \infty$.

Case 2. Integrating the first equation from t to $\delta(t)$ to get

$$r_1(\delta(t)) (\mathcal{U}_1'(\delta(t)))^{\alpha_1} - r_1(t) (\mathcal{U}_1'(t))^{\alpha_1} = \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s) \mathcal{G}_i(\xi_2(\rho_i(s))) \, ds$$

$$-r_1(t) (\mathcal{U}_1'(t))^{\alpha_1} \geq \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s) \lambda_i \xi_2(\rho_i(s)) \, ds$$

$$-r_1(t) (\mathcal{U}_1'(t))^{\alpha_1} \geq \lambda \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s) (1 - \mathcal{P}_2(\rho_i(s))) \mathcal{U}_2(\rho_i(s)) \, ds$$

$$r_1(t) (\mathcal{U}_1'(t))^{\alpha_1} \leq \lambda \int_t^{\delta(t)} \sum_{i=1}^{n} |\mathcal{Q}_i(s)| (1 - \mathcal{P}_2(\rho_i(s))) \mathcal{U}_2(\rho_i(s)) \, ds$$

$$\leq \lambda \mathcal{U}_2(\rho(t)) \int_t^{\delta(t)} \sum_{i=1}^{n} |\mathcal{Q}_i(s)| (1 - \mathcal{P}_2(\rho_i(s))) \, ds,$$

$$\mathcal{U}_1'(t) \leq \lambda^{\frac{1}{\alpha_1}} \left(\frac{\mathcal{U}_2(\rho(t))}{r_1(t)} \int_t^{\delta(t)} \sum_{i=1}^{n} |\mathcal{Q}_i(s)| (1 - \mathcal{P}_2(\rho_i(s))) \, ds \right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to get

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_1) \leq \lambda^{\frac{1}{\alpha_1}} \int_{t_1}^{t} \left(\frac{\mathcal{U}_2(\rho(s))}{r_1(s)} \int_s^{\delta(s)} \sum_{i=1}^{n} |\mathcal{Q}_i(v)| (1 - \mathcal{P}_2(\rho_i(v))) \, dv \right)^{\frac{1}{\alpha_1}} ds$$

$$\leq \lambda^{\frac{1}{\alpha_1}} \mathcal{U}_2(\rho(t_1)) \int_{t_1}^{t} \left(\frac{1}{r_1(s)} \int_s^{\delta(s)} \sum_{i=1}^{n} |\mathcal{Q}_i(v)| (1 - \mathcal{P}_2(\rho_i(v))) \, dv \right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ and taking into count the condition 16 the last inequality leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty$, implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = -\infty$, a contradiction, similarly obtaining $\mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$.

Case 3. $\xi_1(t) > 0,\ \xi_2(t) < 0$. Integrating the second equation from t to $\delta(t)$ to get

$$r_2(\delta(t)) (\mathcal{U}_2'(\delta(t)))^{\alpha_2} - r_2(t) (\mathcal{U}_2'(t))^{\alpha_2} = \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s) \mathcal{H}_i(\xi_1(\sigma_i(s))) \, ds$$

$$-r_2(t) (\mathcal{U}_2'(t))^{\alpha_2} \leq \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s) \mu_i \xi_1(\sigma_i(s)) \, ds$$

$$\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \geq \int_t^{\delta(t)} \sum_{i=1}^{n} |\mathcal{R}_i(s)| \mu_i (1 - \mathcal{P}_1(\sigma_i(s))) \mathcal{U}_1(\sigma_i(s)) ds$$

$$\mathcal{U}_2'(t) \geq \mu^{\frac{1}{\alpha_2}} \left(\frac{\mathcal{U}_1(\sigma(\delta(t)))}{\mathcal{r}_2(t)} \int_t^{\delta(t)} \sum_{i=1}^{n} |\mathcal{R}_i(s)| (1 - \mathcal{P}_1(\sigma_i(s))) ds \right)^{\frac{1}{\alpha_2}}$$

Integrating from t_1 to t to get

$$\mathcal{U}_2(t) - \mathcal{U}_2(t_1) \geq \mu^{\frac{1}{\alpha_2}} \mathcal{U}_1^{\frac{1}{\alpha_2}}(\sigma(\delta(t))) \int_{t_1}^{t} \left(\frac{1}{\mathcal{r}_2(s)} \int_s^{\delta(s)} \sum_{i=1}^{n} |\mathcal{R}_i(v)| (1 - \mathcal{P}_1(\rho_i(s))) dv \right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ and taking into count the condition 16 the last inequality leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty$, a contradiction unless $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = 0$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = 0$, Integrating the first equation from t to $\delta(t)$ to get

$$\mathcal{r}_1(\delta(t))(\mathcal{U}_1'(\delta(t)))^{\alpha_1} - \mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s) \mathcal{G}_i(\xi_2(\rho_i(s))) ds$$

$$-\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \geq \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s) \lambda_i \xi_2(\rho_i(s)) ds$$

$$\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \leq \int_t^{\delta(t)} \sum_{i=1}^{n} |\mathcal{Q}_i(s)| \lambda_i (1 - \mathcal{P}_2(\rho_i(s))) \mathcal{U}_2(\rho_i(s)) ds$$

$$\mathcal{U}_1'(t) \leq \lambda^{\frac{1}{\alpha_1}} \left(\frac{\mathcal{U}_2(\rho(\delta(t)))}{\mathcal{r}_1(t)} \int_t^{\delta(t)} \sum_{i=1}^{n} |\mathcal{Q}_i(s)| (1 - \mathcal{P}_2(\rho_i(s))) ds \right)^{\frac{1}{\alpha_1}}$$

Integrating from t_1 to t to get

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_1) \leq \lambda^{\frac{1}{\alpha_1}} \mathcal{U}_2^{\frac{1}{\alpha_1}}(\rho(\delta(t))) \int_{t_1}^{t} \left(\frac{1}{\mathcal{r}_1(s)} \int_s^{\delta(s)} \sum_{i=1}^{n} |\mathcal{Q}_i(v)| (1 - \mathcal{P}_2(\rho_i(s))) dv \right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ and taking into count the condition (16) the last inequality leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty$, a contradiction unless $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = 0$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_2(t) = 0$,

Case 4. The proof can be treated in a similar way as in case 3. □

Theorem 2: *Assume that* $0 \leq \mathcal{P}_j(t) < 1$, $j = 1, 2$, $\tau_1(t) \leq t$, $\tau_2(t) \leq t$, $\mathcal{Q}_i(t) \geq 0$, $\mathcal{R}_i(t) \geq 0$, $t \geq t_0$, $i = 1, 2, \ldots, n$, *and 5, 15 hold, in addition to conditions* $16 - a$ *and* $16 - b$ *hold. Then every bounded solution of sys. 1 oscillates.*

Proof: Assume that sys 1 has nonoscillatory bounded solution $(\xi_1(t), \xi_2(t))$ then by Lemma 2, having only the following cases:

1. If $\xi_1(t) > 0, \xi_2(t) > 0$, then $\mathcal{U}_i(t) > 0, \mathcal{U}_i'(t) < 0, (\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \geq 0$, $\mathrm{Lim}_{t\to\infty}\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i} = 0$
2. If $\xi_1(t) < 0, \xi_2(t) < 0$, then $\mathcal{U}_i(t)\langle 0, \mathcal{U}_i'(t)\rangle 0, (\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i})' \leq 0$, $\mathrm{Lim}_{t\to\infty}\mathcal{r}_i(t)(\mathcal{U}_i'(t))^{\alpha_i} = 0$.
3. If $\xi_1(t) > 0, \xi_2(t) < 0$, then $\mathcal{U}_1(t) > 0, \mathcal{U}_1'(t) > 0, (\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \leq 0$, and $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, $\mathcal{U}_2(t) < 0, \mathcal{U}_2'(t) < 0$, and $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_2(t) < 1$
4. If $\xi_1(t)\langle 0, \xi_2(t)\rangle 0$, then $\mathcal{U}_1(t) < 0, \mathcal{U}_1'(t) < 0, (\mathcal{r}_1(t)(\mathcal{U}_1'(t))^{\alpha_1})' \geq 0$. and $\xi_1(t)$ is bounded away from zero if $0 \leq \mathcal{P}_1(t) < 1$, and $\mathcal{U}_2(t) > 0, \mathcal{U}_2'(t) > 0, (\mathcal{r}_2(t)(\mathcal{U}_2'(t))^{\alpha_2})' \leq 0$, and $\xi_2(t)$ is bounded away from zero if $0 \leq \mathcal{P}_2(t) < 1$.

Case 1. From Eq. (3) it follows

$$\mathcal{U}_i(t) \le \xi_i(t) + \mathcal{P}_i(t)\,\mathcal{U}_i(\tau_i(t)), \ i = 1, 2.$$
$$\xi_i(t) \ge \mathcal{U}_i(t) - \mathcal{P}_i(t)\,\mathcal{U}_i(\tau_i(t)) \ge (1 - \mathcal{P}_i(t))\,\mathcal{U}_i(t). \tag{19}$$

Integrating the first equation from t to $\delta(t)$ to get

$$r_1(\delta(t))\left(\mathcal{U}_1'(\delta(t))\right)^{\alpha_1} - r_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1} = \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)\,\mathcal{G}_i(\xi_2(\rho_i(s)))\,ds$$

$$-r_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1} \ge \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)\,\mathcal{G}_i(\xi_2(\rho_i(s)))\,ds$$

$$-r_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1} \ge \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)\,\lambda_i \xi_2(\rho_i(s))\,ds$$

$$r_1(t)\left(\mathcal{U}_1'(t)\right)^{\alpha_1} \le -\lambda \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)\,(1 - \mathcal{P}_2(\rho_i(s)))\,\mathcal{U}_2(\rho_i(s))\,ds, \tag{20}$$

$$\le -\lambda\,\mathcal{U}_2(\rho(\delta(t))) \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)\,(1 - \mathcal{P}_2(\rho_i(s)))\,ds, \ \ \rho(t) = \min\{\rho_i(t) : i = 1, 2, \ldots n\}$$

$$\mathcal{U}_1'(t) \le -\lambda^{\frac{1}{\alpha_1}} \left(\frac{\mathcal{U}_2(\rho(\delta(t)))}{r_1(t)} \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)\,(1 - \mathcal{P}_2(\rho_i(s)))\,ds \right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to get

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_1) \le -\lambda^{\frac{1}{\alpha_1}} \int_{t_1}^{t} \left(\frac{\mathcal{U}_2(\rho(\delta(s)))}{r_1(s)} \int_s^{\delta(s)} \sum_{i=1}^{n} \mathcal{Q}_i(v)\,(1 - \mathcal{P}_2(\rho_i(v)))\,dv \right)^{\frac{1}{\alpha_1}} ds$$

$$\le -\lambda^{\frac{1}{\alpha_1}}\,\mathcal{U}_2(\rho(\delta(t))) \int_{t_1}^{t} \left(\frac{1}{r_1(s)} \right)^{\frac{1}{\alpha_1}} \left(\int_s^{\delta(s)} \sum_{i=1}^{n} \mathcal{Q}_i(v)\,(1 - \mathcal{P}_2(\rho_i(v)))\,dv \right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ and by using the condition 16 leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = -\infty$, implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = -\infty$, a contradiction, similarly obtaining $\mathrm{Lim}_{t\to\infty}\xi_2(t) = -\infty$.

Case 2. $\xi_1(t) < 0, \ \xi_2(t) < 0$. Integrating the second equation from t to $\delta(t)$ to get

$$r_2(\delta(t))\left(\mathcal{U}_2'(\delta(t))\right)^{\alpha_2} - r_2(t)\left(\mathcal{U}_2'(t)\right)^{\alpha_2} = \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s)\,\mathcal{H}_i(\xi_1(\sigma_i(s)))\,ds$$

$$-r_2(t)\left(\mathcal{U}_2'(t)\right)^{\alpha_2} \le \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s)\,\mu_i \xi_1(\sigma_i(s))\,ds$$

$$r_2(t)\left(\mathcal{U}_2'(t)\right)^{\alpha_2} \ge -\int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s)\,\mu_i\,(1 - \mathcal{P}_1(\sigma_i(s)))\,\mathcal{U}_1(\sigma_i(s))\,ds$$

$$\mathcal{U}_2'(t) \ge -\mu^{\frac{1}{\alpha_2}} \left(\frac{\mathcal{U}_1(\sigma(\delta(t)))}{r_2(t)} \int_t^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s)\,(1 - \mathcal{P}_1(\sigma_i(s)))\,ds \right)^{\frac{1}{\alpha_2}}$$

Integrating from t_1 to t to get

$$\mathcal{U}_2(t) - \mathcal{U}_2(t_1) \geq -\mu^{\frac{1}{\alpha_2}} (\mathcal{U}_1(\sigma(\delta(t))))^{\frac{1}{\alpha_2}} \int_{t_1}^{t} \left(\frac{1}{r_2(s)} \int_{s}^{\delta(s)} \sum_{i=1}^{n} \mathcal{R}_i(v)(1 - \mathcal{P}_1(\rho_i(s)))\, dv \right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ and taking into count the condition 16 the last inequality leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = \infty$, a contradiction unless $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = 0$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = 0$, similarly obtaining $\mathrm{Lim}_{t\to\infty}\xi_2(t) = \infty$.

Case 3. $\xi_1(t) > 0,\ \xi_2(t) < 0$. Integrating the second equation from t to $\delta(t)$ to get

$$r_2(\delta(t))(\mathcal{U}_2'(\delta(t)))^{\alpha_2} - r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} = \int_{t}^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s)\mathcal{H}_i(\xi_1(\sigma_i(s)))\, ds$$

$$-r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \geq \int_{t}^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s)\mu_i\xi_1(\sigma_i(s))\, ds$$

$$r_2(t)(\mathcal{U}_2'(t))^{\alpha_2} \leq -\int_{t}^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s)\mu_i(1 - \mathcal{P}_1(\sigma_i(s)))\mathcal{U}_1(\sigma_i(s))\, ds$$

$$\mathcal{U}_2'(t) \leq -\mu^{\frac{1}{\alpha_2}} \left(\frac{\mathcal{U}_1(\sigma(t))}{r_2(t)} \int_{t}^{\delta(t)} \sum_{i=1}^{n} \mathcal{R}_i(s)(1 - \mathcal{P}_1(\sigma_i(s)))\, ds \right)^{\frac{1}{\alpha_2}}$$

Integrating from t_1 to t to get

$$\mathcal{U}_2(t) - \mathcal{U}_2(t_1) \leq -\mu^{\frac{1}{\alpha_2}} \mathcal{U}_1^{\frac{1}{\alpha_2}}(\sigma(t_1)) \int_{t_1}^{t} \left(\frac{1}{r_2(s)} \int_{s}^{\delta(s)} \sum_{i=1}^{n} \mathcal{R}_i(v)(1 - \mathcal{P}_1(\rho_i(s)))\, dv \right)^{\frac{1}{\alpha_2}} ds$$

Letting $t \to \infty$ and taking into count the condition 16 the last inequality leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = -\infty$, a contradiction unless $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = 0$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_1(t) = 0$, Integrating the first equation from t to $\delta(t)$ to get

$$r_1(\delta(t))(\mathcal{U}_1'(\delta(t)))^{\alpha_1} - r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} = \int_{t}^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)\mathcal{G}_i(\xi_2(\rho_i(s)))\, ds$$

$$-r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \leq \int_{t}^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)\lambda_i\xi_2(\rho_i(s))\, ds$$

$$r_1(t)(\mathcal{U}_1'(t))^{\alpha_1} \geq -\int_{t}^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)\lambda_i(1 - \mathcal{P}_2(\rho_i(s)))\mathcal{U}_2(\rho_i(s))\, ds$$

$$\mathcal{U}_1'(t) \geq -\lambda^{\frac{1}{\alpha_1}} \left(\frac{\mathcal{U}_2(\rho(t))}{r_1(t)} \int_{t}^{\delta(t)} \sum_{i=1}^{n} \mathcal{Q}_i(s)(1 - \mathcal{P}_2(\rho_i(s)))\, ds \right)^{\frac{1}{\alpha_1}}$$

Integrating from t_1 to t to get

$$\mathcal{U}_1(t) - \mathcal{U}_1(t_1) \geq -\lambda^{\frac{1}{\alpha_1}} (\mathcal{U}_2(\rho(t_1)))^{\frac{1}{\alpha_1}} \int_{t_1}^{t} \left(\frac{1}{r_1(s)} \int_{s}^{\delta(s)} \sum_{i=1}^{n} \mathcal{Q}_i(v)(1 - \mathcal{P}_2(\rho_i(s)))\, dv \right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ and taking into count the condition (16) the last inequality leads to $\mathrm{Lim}_{t\to\infty}\mathcal{U}_1(t) = \infty$, a contradiction unless $\mathrm{Lim}_{t\to\infty}\mathcal{U}_2(t) = 0$, which implies that $\mathrm{Lim}_{t\to\infty}\xi_2(t) = 0$, Similarly, obtaining $\mathrm{Lim}_{t\to\infty}\xi_1(t) = 0$.

Case 4. The proof can be treated in a similar way as in case 3. □

Applications: 4. Illustrative problems

Two examples are presented to illustrate the main results

Example 1: *Consider the neutral differential system*

$$\begin{array}{l} \left(a\left[\left(\xi_1(t) + \frac{1}{2}\xi_1(t-3\pi)\right)'\right]^1\right)' = -\frac{1}{4}a\xi_2^1\left(t-\frac{\pi}{2}\right) - \frac{1}{4}a\xi_2^1\left(t-\frac{\pi}{2}\right), \\ \left(b\left[\left(\xi_2(t) + \frac{1}{6}\xi_2(t-\pi)\right)'\right]^1\right)' = -b\xi_1^1\left(t-\frac{3\pi}{2}\right) - \frac{1}{6}b\xi_1^1\left(t-\frac{5\pi}{2}\right) \end{array}, \tag{21}$$

In this example $\alpha_1 = \alpha_2 = 1, r_1(t) = a > 0, r_2(t) = b > 0$, $\mathcal{P}_1(t) = \frac{1}{2}, \mathcal{P}_2(t) = \frac{1}{6}$, $\tau_1(t) = t - 3\pi$, $\tau_2(t) = t - \pi, Q_1(t) = Q_2(t) = -\frac{1}{4}a$, $\rho_1(t) = \rho_2(t) = t - \frac{\pi}{2}$, $\sigma_1(t) = t - \frac{3\pi}{2}$, $\sigma_2(t) = t - \frac{5\pi}{2}$, $G_1(\xi_2) = G_2(\xi_2) = \xi_2^1$, $\mathcal{H}_1(\xi_1) = \mathcal{H}_2(\xi_1) = \xi_1^1$, $\mathcal{R}_1(t) = -b$, $\mathcal{R}_2(t) = -\frac{1}{6}b$. *Let* $\delta(t) = 2t$. *To check the condition (16), note that*

$$\operatorname*{Lim\,sup}_{t\to\infty} \int_T^t \left(\frac{1}{r_1(s)} \int_s^{\delta(s)} \sum_{i=1}^n |Q_i(v)| \left(1 - \mathcal{P}_2(\rho_i(v))\right) dv\right)^{\frac{1}{\alpha_1}} ds = \lim_{t\to\infty} \int_T^t \left(\frac{1}{a}\int_s^{2s} \frac{a}{2}\left(1 - \frac{1}{6}\right) dv\right) ds = \infty,$$

$$\operatorname*{Lim\,sup}_{t\to\infty} \int_{t_1}^t \left(\frac{1}{r_2(s)} \int_s^{\delta(s)} \sum_{i=1}^n |\mathcal{R}_i(v)| \left(1 - \mathcal{P}_1(\sigma_i(v))\right) d\xi\right)^{\frac{1}{\alpha_2}} ds = \lim_{t\to\infty} \int_T^t \left(\frac{1}{b}\int_s^{2s} \frac{a}{2}\left(1 - \frac{1}{2}\right) dv\right) ds = \infty$$

So all the conditions of Theorem 1 are satisfied, and hence, according to Theorem 1, every solution of the system 21 either oscillates or tends to zero as $t \to \infty$. *For instance the solution* $(\xi_1(t), \xi_2(t)) = (\sin t,\ \cos t)$, *is such an oscillatory solution.* □

Example 2: *Consider the neutral differential system*

$$\begin{array}{l} \left(t\left[\left(\xi_1(t) + e^{-2}\xi_1(t-1)\right)'\right]^3\right)' = 64e^{-6}(4t-1)\xi_2^6(t-1) + 128e^{-12}t\xi_2^6(t-2), \\ \left(\sqrt{t}\left[\left(\xi_2(t) + e^{-3}\xi_2(t-3)\right)'\right]^3\right)' = 8\frac{e^{-\frac{3}{2}}}{\sqrt{t}}\left(2t-\frac{1}{2}\right)\xi_1^{\frac{3}{2}}\left(t-\frac{1}{2}\right) + 8e^{-\frac{9}{2}}\sqrt{t}\xi_1^{\frac{3}{2}}\left(t-\frac{3}{2}\right) \end{array}, \quad t \geq \frac{1}{4}, \tag{22}$$

In this example $\alpha_1 = \alpha_2 = 3, r_1(t) = t, r_2(t) = \sqrt{t}$, $\mathcal{P}_1(t) = e^{-2}, \mathcal{P}_2(t) = e^{-3}$, $\tau_1(t) = t - 1$, $\tau_2(t) = t - 3$, $q_1(t) = 64e^{-6}(4t-1)$, $q_2(t) = 128e^{-12}t$, $\rho_1(t) = t - 1$, $\rho_2(t) = t - 2$, $\sigma_1(t) = t - \frac{1}{2}$, $\sigma_2(t) = t - \frac{3}{2}, \mathcal{G}_1(\xi_2) = \mathcal{G}_2(\xi_2) = \xi_2^6$, $\mathcal{H}_1(\xi_1) = \mathcal{H}_2(\xi_1) = \xi_1^{\frac{3}{2}}$, $\mathcal{R}_1(t) = 8\frac{e^{-\frac{3}{2}}}{\sqrt{t}}(2t - \frac{1}{2})$, $\mathcal{R}_2(t) = 8e^{-\frac{9}{2}}\sqrt{t}$. *Let To check condition 5, note that*

$$\int_T^\infty \left(\frac{1}{r_1(t)}\right)^{\frac{1}{\alpha_1}} dt = \int_T^\infty \left(\frac{1}{t}\right)^{\frac{1}{3}} dt = \infty, \quad \int_T^\infty \left(\frac{1}{r_2(t)}\right)^{\frac{1}{\alpha_2}} dt = \int_T^\infty \left(\frac{1}{\sqrt{t}}\right)^{\frac{1}{3}} dt = \infty,$$

One can see that all the conditions of Theorem 2 are satisfied, so according to Theorem 2, each solution of the system (22) either oscillates or tends to zero as $t \to \infty$. *For instance the solution* $(\xi_1(t), \xi_2(t)) = (e^{-2t}\ ,\ e^{-t})$, *is such a nonoscillatory convergent solution.* □

Results and discussion

The space of the functions $Q_i(t)$ and $\mathcal{R}_i(t)$ were classified into two classes either $Q_i(t)$ and $\mathcal{R}_i(t) \geq 0$ or $Q_i(t)$ and $\mathcal{R}_i(t) \leq 0$ and obtain some conditions for oscillation of each solution of system 1. Through condition 5, 14 possible cases were identified for the solutions of system 1 to oscillate or for these solutions to converge to zero. As for condition 16, the 14 cases were reduced to only four cases in which the solutions were oscillatory or convergent.

Conclusion

Several new sufficient conditions are introduced to ensure that every bounded solution of the system 1 either oscillates or converges to zero as t tends to infinity. To this end, fourteen cases of a non-oscillating bounded solution are discussed, and thus, under necessary and sufficient conditions, every possible bounded solution of this system is guaranteed to oscillate. Some illustrative examples of the results obtained are given.

Author's declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

Author's contribution

N.A., T.H., F.A., and H.A. contributed to the design and implementation of the research, to the analysis of the results, and to the writing of the manuscript.

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نظام المعادلات التفاضلية المحايدة نصف الخطية متذبذبة على الاغلب من الرتبة الثانية متعدد المعاملات

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الخلاصة

تم تقديم عدة شروط جديدة كافية لضمان تذبذب أو تقارب كل حل محدود للنظام (1) إلى الصفر مع ميل t إلى ما لا نهاية. ولتحقيق هذه الغاية، تمت مناقشة أربع عشرة حالة لحل محدود غير متذبذب، وبالتالي، في ظل الظروف الضرورية والكافية، يتم ضمان تذبذب كل حل محدود ممكن لهذا النظام. تم تقديم بعض الأمثلة التوضيحية للنتائج التي تم الحصول عليها.

الكلمات المفتاحية: السلوك المحاذي، نظام نصف خطي، معادلات تفاضلية محايدة، تذبذب، الرتبة الثانية.