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RESEARCH ARTICLE

Common Fixed Points Results for Three Mappings under Generalized Contraction of Suzuki-Type in ƀ-Metric Spaces with Application

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ABSTRACT

Suzuki used a brilliant strategy in his seminal publications to expand the Banach contraction theorem (BCT). This has undergone numerous intriguing generalizations and extensions during the past few decades and led to a new trend of coincidence and fixed points for countless Suzuki-type contractive and non-expansive maps in the various spaces. In this article, in the setup of ƀ-metric spaces, the aim is to produce a common fixed point of three mappings subjected to a generalized contraction of the Suzuki type. The present work generalizes well- known results of Suzuki, Chandra et al., Roshan et al. and several other results available in the literature. An applied illustration in which graphical and computational analysis has been performed accords the exploratory verification of the produced work making the results more adaptable by a wider class of researchers. The iterative analysis based on iterative methods in the illustration is also supported by an algorithm. Furthermore, an application of the present work to the system of functional equations in dynamic programming shows how the present results are usable. Finally, an example is given to justify the application of the present work.

Keywords: Algorithm, ƀ-Metric space, Common fixed point, Dynamic programming, Generalized Suzuki type contraction, Iterative methods

Introduction

Fixed point theory itself is a very dynamic field of study having multiple dimensions of research in non-linear analysis. It has various applications in mathematical sciences.[1,2] A robust line of research in fixed point theory was initiated as a consequence of the famous BCT.[3] By using different kinds of contractive conditions in various domains, the BCT has been widely generalized and unified in different metric spaces by various authors.[4–6] Researchers have been enjoying fixed point theorems for the last quarter of the Twentieth century under non-identical contractive and contraction conditions. Recently, Induwa et al.[7] presented a fixed point (FP) result in geodisec space and Salisu et al.[8] in Hadamard space. Numerous applications can be made of these generalizations in different diverse fields, viz., artificial intelligence, neural networking, dynamic programming in computer science, robotics, fuzzy networking, physical and chemical sciences etc.[9–11] Suzuki[12] used a clever

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strategy to generalize BCT in his illustrious publications which has led to a new trend of coincidence and FP for countless contractive and non-expansive maps in the different spaces.[13–15]

Suzuki's subsequent result generalizes BCT:

Theorem 1:[12] *Let (Ҳ, đ) be a complete metric space (CMS) and* $Ǵ : Ҳ \to Ҳ$. *Consider a non-increasing function* $\eta : [0,1) \to (\frac{1}{2}, 1]$ *defined by*

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \le s \le \frac{\sqrt{5}-1}{2}, \\ \frac{1-s}{s^2} & \text{if } \frac{\sqrt{5}-1}{2} \le s \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+s} & \text{if } \frac{1}{\sqrt{2}} \le s < 1. \end{cases}$$

Assume $\exists\, s \in [0,\ 1)$ so that

$$\eta(s)\, đ\,(\alpha, Ǵ(\alpha)) \le đ\,(\alpha, \beta)$$

implies

$$đ\,(Ǵ\,(\alpha),\ Ǵ\,(\beta)) \le s\ đ\,(\alpha, \beta),$$

for all $\alpha,\ \beta \in Ҳ$. Then Ǵ possesses a unique FPɎ. In addition, $\lim_{n\to\infty} Ǵ^n(\alpha) = Ɏ$ for all $\alpha \in Ҳ$.

Theorem 1 has now undergone numerous intriguing generalizations and extensions[16–18] during the past few decades. One of them, which is due to Chandra et al.,[19] is as follows:

Theorem 2:[19] *Define a function* $\eta(s)$ *as in Theorem 1. Let* $(Ҳ,\ \varrho)$ *be a CMS and* Ǵ, 'H: $Ҳ \to Ҳ$ *with the condition that for all* $\alpha,\ \beta \in Ҳ$,

$$\eta(s)\ \min\left\{\varrho\left(\alpha,\ Ǵ(\alpha)\right),\ \varrho\left(\alpha,\ 'H(\alpha)\right)\right\} \le \varrho(\alpha,\ \beta)$$

implies

$$\max\left\{\varrho\left(Ǵ\ (\alpha), Ǵ\ (\beta)\right), \varrho\left('H\ (\alpha),\ 'H\ (\beta)\right), \frac{\varrho\left(Ǵ\ (\alpha), 'H\ (\beta)\right) + \varrho\left(Ǵ\ (\beta), 'H\ (\alpha)\right)}{2}\right\} \le s\varrho(\alpha, \beta).$$

Then, Ǵ and 'H have a unique common fixed point (CFP).

In 1989, a famous mathematician Bakhtin innovated the following Definitions 1 and 2 on ƀ-metric space (ƀMS), demonstrated BCT in these spaces, and subsequently various researchers gave their valuable results in these spaces.[20–22]

Definition 1: Let Ҳ represent a nonempty set and $ƀ \ge 1$ is a real number. Consequently, $\varrho : Ҳ \times Ҳ \to R^+$ is a ƀ-metric on Ҳ, if all $\alpha,\ \beta,\ Ɏ \in Ҳ$,

(i) $\varrho(\alpha, \beta) = 0$ iff $\alpha = \beta$.
(ii) $\varrho(\alpha, \beta) = \varrho(\beta, \alpha)$.
(iii) $\varrho(\alpha, Ɏ) \le ƀ[\varrho(\alpha, \beta) + \varrho(\beta, Ɏ)]$.

The pair $(Ҳ,\ \varrho)$ is referred to as a ƀMS".

Since a ƀMS is only a metric space (MS) if $ƀ = 1$. It should be emphasized that in contrast to MS, the family of ƀMS is bigger.

Prior giving our findings, list some fundamental information that will utilize later on. The concepts of convergence and completion in a ƀMS are briefly summarized subsequently:

Definition 2: Assume that $\{\alpha_n\}$ is a sequence in Ҳ and that $(Ҳ,\ \varrho)$ is a ƀMS, then

(i) $\{\alpha_n\}$ is ƀ-convergent to $\alpha \in Ҳ$ if $\lim_{n\to\infty} \varrho(\alpha_n, \alpha) = 0$ holds.
(ii) $\{\alpha_n\}$ is ƀ-Cauchy in Ҳ iff $\varrho(\alpha_n, \alpha_m) \to 0$ when $n, m \to \infty$;
(iii) Ҳ is considered to be ƀ-complete if, in Ҳ, every ƀ-Cauchy sequence is ƀ-convergent.

The sufficient condition for the Cauchyness of a sequence is given by Suzuki[22] in the subsequent lemma:

Lemma 1:[22] *Let* $(\mathcal{X}, \varrho)$ *represent a* ƀMS *with* $ƀ \geq 1$. *Then* $\{\alpha_n\}$ *is a* ƀ*-Cauchy if*

$$\varrho(\alpha_n, \alpha_{n+1}) \leq \lambda\, \varrho(\alpha_{n-1}, \alpha_n), \text{ for all } n \in \mathbb{N}, \lambda \in [0, 1).$$

Remarkably, Lemma 1 can be highly beneficial in demonstrating the existence theorems in complete ƀ-metric space (C ƀMS).

Roshan et al.[23] provided another significant Suzuki-type CFP result in the subsequent manner:

Theorem 3:[23] *Define a function* $\eta(s)$ *as in* Theorem 1. *Let* $(\mathcal{X}, \varrho)$ *be a* CƀMS *and* $\acute{G}, \acute{H}: \mathcal{X} \to \mathcal{X}$ *with the condition that for all* $\alpha, \beta \in \mathcal{X}$,

$$\frac{1}{ƀ}\eta(s)\min\{\varrho(\alpha, \acute{G}(\alpha)), \varrho(\alpha, \acute{H}(\alpha))\} \leq \varrho(\alpha, \beta)$$

implies

$$\max\{\varrho(\acute{G}(\alpha), \acute{G}(\beta)), \varrho(\acute{G}(\alpha), \acute{H}(\beta)), \varrho(\acute{H}(\alpha), \acute{H}(\beta)), \varrho(\acute{G}(\beta), \acute{H}(\alpha))\} \leq \frac{s}{ƀ^2}\varrho(\alpha, \beta).$$

Then, $\acute{G}$ and $\acute{H}$ have a unique CFP.

Nowadays, researchers are enjoying Suzuki-type multivalued contractions in various metric spaces to obtain different applications in various diverse spaces.[24–26]

This research work aims to produce CFP results for three mappings satisfying a generalized Suzuki-type contraction in ƀMS generalizing Theorems 1 to 3, and several others. A suitable example is given to verify the veracity of the work accompanied by the proper graphical and computational analysis along with the algorithmic scheme of the iterative process. An implementation of this work to determine whether a common solution exists for a particular class of functional equations arising in dynamic programming shows the usability of our work. An example is given to justify the application.

Results and discussion

The formulation of the main findings is covered in this section. The subsequent is the primary outcome:

Theorem 4: *Define a function* $\eta(s)$ *as in* Theorem 1. *Let* $f, \acute{G}, \acute{H}: \mathcal{X} \to \mathcal{X}$ *be so that* $\acute{G}(\mathcal{X}) \subseteq f(\mathcal{X})$ *and* $\acute{H}(\mathcal{X}) \subseteq f(\mathcal{X})$, *with the contractive condition*

$$\frac{1}{ƀ}\eta(s)\min\{\varrho(f(\alpha), \acute{G}(\alpha)), \varrho(f(\alpha), \acute{H}(\alpha))\} \leq \varrho(f(\alpha), f(\beta)) \quad (1)$$

implies

$$\max\{\varrho(\acute{G}(\alpha), \acute{G}(\beta)), \varrho(\acute{G}(\alpha), \acute{H}(\beta)), \varrho(\acute{H}(\alpha), \acute{H}(\beta)), \varrho(\acute{G}(\beta), \acute{H}(\alpha))\} \leq \frac{s}{ƀ^2}\varrho(f(\alpha), f(\beta)), \quad (2)$$

for all $\alpha, \beta \in \mathcal{X}$. If any of $f(\mathcal{X})$, $\acute{G}(\mathcal{X})$, or $\acute{H}(\mathcal{X})$ is a ƀ-complete subspace of $\mathcal{X}$ and f commute with both $\acute{G}$ and $\acute{H}$. Then f, $\acute{G}$ and $\acute{H}$ have a unique CFP.

Proof: Construct two sequences $\{\alpha_n\} \subseteq \mathcal{X}$ and $\{\beta_n = f(\alpha_n)\} \subseteq f(\mathcal{X})$ as follows:

$$f(\alpha_{2n+2}) = \acute{G}(\alpha_{2n+1}) \text{ and } f(\alpha_{2n+1}) = \acute{H}(\alpha_{2n}).$$

Now, demonstrate that for each $n \in \mathbb{N}$

$$\varrho(f(\alpha_{n+1}), f(\alpha_n)) \leq \lambda\varrho(f(\alpha_n), f(\alpha_{n-1})), \lambda \in [0, 1).$$

Since, $\frac{1}{ƀ}\ \eta(s)\min\{\varrho(f(\alpha_{2n-1}),Ǵ(\alpha_{2n-1})),\varrho(f(\alpha_{2n}),\text{'H}(\alpha_{2n}))\}\le\varrho(f(\alpha_{2n-1}),f(\alpha_{2n}))$, so Eq. (2) implies that

$$\begin{aligned}\varrho\left(Ǵ\left(\alpha_{2n-1}\right),\text{'H}\left(\alpha_{2n}\right)\right) &\le \max\left\{\begin{array}{l}\varrho\left(Ǵ\ \left(\alpha_{2n-1}\right),\ Ǵ\ \left(\alpha_{2n}\right)\right),\ \varrho(Ǵ\left(\alpha_{2n-1}\right),\ \text{'H}\left(\alpha_{2n}\right),\\ \varrho\left(\text{'H}\left(\alpha_{2n-1}\right),\ \text{'H}\left(\alpha_{2n}\right)\right),\ \varrho\left(Ǵ\left(\alpha_{2n}\right),\ \text{'H}\left(\alpha_{2n-1}\right)\right)\end{array}\right\}\\ &\le \frac{s}{ƀ^2}\varrho\left(f\left(\alpha_{2n-1}\right),f\left(\alpha_{2n}\right)\right).\end{aligned}$$

Therefore,

$$\varrho\left(f\left(\alpha_{2n}\right),\ f\left(\alpha_{2n+1}\right)\right)\ \le\ \frac{s}{ƀ^2}\ \varrho\left(f\left(\alpha_{2n-1}\right),\ f\left(\alpha_{2n}\right)\right). \tag{3}$$

Similarly, this demonstrate that

$$\varrho\left(f\left(\alpha_{2n+1}\right),\ f\left(\alpha_{2n+2}\right)\right)\ \le\ \frac{s}{ƀ^2}\ \varrho\left(f\left(\alpha_{2n}\right),\ f\left(\alpha_{2n+1}\right)\right). \tag{4}$$

Eqs. (3) and (4), lead us to the deduction that

$$\varrho\left(f\left(\alpha_{n+1}\right),f\left(\alpha_{n}\right)\right)\le\frac{s}{ƀ^2}\varrho\left(f\left(\alpha_{n}\right),f\left(\alpha_{n-1}\right)\right).$$

Since $\frac{s}{ƀ^2}\in[0,\ 1)$, therefore $\{\beta_n = f(\alpha_n)\}$ is a ƀ-Cauchy sequence, by Lemma 1. So it has a limit, let it be δ, so that $\beta_n = f(\alpha_n)\to\delta$, so $\exists$ some $Ɣ\in Ӿ$ so that $f(Ɣ)=\delta$.
Since $f(\alpha_n)\to f(Ɣ),\ \exists\ n_0\in Ɲ$ such that

$$\varrho\left(f\left(Ɣ\right),f\left(\alpha_{n}\right)\right)\le\frac{1}{3ƀ}\varrho\left(f\left(Ɣ\right),f\left(\beta\right)\right)\text{ for }\ (\beta)\ne f\left(Ɣ\right),n\ge n_0.$$

Then, as in Singh et al.[27]

$$\begin{aligned}\eta\left(s\right)\ \varrho\left(f\left(\alpha_{2n-1}\right),\ Ǵ\left(\alpha_{2n-1}\right)\right) &\le\varrho\left(f\left(\alpha_{2n-1}\right),\ Ǵ\left(\alpha_{2n-1}\right)\right)\ \le\ \varrho\left(f\left(\alpha_{2n-1}\right),\ f\left(\alpha_{2n}\right)\right)\\ &\le\ ƀ\left[\varrho\left(f\left(\alpha_{2n-1}\right),\ f\ \left(Ɣ\right)\right)\ +\ \varrho\left(f\ \left(Ɣ\right),\ f\left(\alpha_{2n}\right)\right)\right]\\ &\le\ ƀ\frac{2}{3ƀ}\ \varrho\left(f\left(\beta\right),\ f\left(Ɣ\right)\right)=\ \frac{2}{3}\ \varrho\left(f\left(\beta\right),\ f\left(Ɣ\right)\right)\\ &=\ \left[\varrho\left(f\left(\beta\right),\ f\left(Ɣ\right)\right)-\ \frac{ƀ}{3ƀ}\ \varrho\left(f\left(\beta\right),\ f\left(Ɣ\right)\right)\right]\\ &\le\ \varrho\left(f\left(\beta\right),\ f\left(Ɣ\right)\right)-\ ƀ\ \varrho\left(f\left(Ɣ\right),\ f\left(\alpha_{2n-1}\right)\right)\\ &\le\ ƀ\ \varrho\left(f\ \left(Ɣ\right),\ f\left(\alpha_{2n-1}\right)\right)+\ ƀ\ \varrho\left(f\left(\alpha_{2n-1}\right),\ f\left(\beta\right)\right)-\ ƀ\ \varrho\left(f\left(\alpha_{2n-1}\right),\ f\left(Ɣ\right)\right)\\ &=\ ƀ\ \varrho(f(\alpha_{2n-1}),\ f(\beta)).\end{aligned}$$

Therefore,

$$\frac{1}{ƀ}\eta\left(s\right)\ \varrho\ \left(f\left(\alpha_{2n-1}\right),\ Ǵ\left(\alpha_{2n-1}\right)\right)\ \le\varrho\left(f\left(\alpha_{2n-1}\right),\ f\left(\beta\right)\right). \tag{5}$$

Now, either $\varrho(f(\alpha_{2n-1}),Ǵ(\alpha_{2n-1}))\le\varrho(f(\beta),\text{'H}(\beta))$ or $\varrho(f(\beta),\text{'H}(\beta))\le\varrho(f(\alpha_{2n-1}),Ǵ(\alpha_{2n-1}))$.
In either case, by Eq. (5) and the given hypotheses, yield

$$\varrho\left(f\left(\alpha_{2n}\right),\text{'H}\left(\beta\right)\right)\le\max\left\{\varrho\left(Ǵ\left(\alpha_{2n-1}\right),Ǵ\left(\beta\right)\right),\varrho\left(Ǵ\left(\alpha_{2n-1}\right),\text{'H}\left(\beta\right)\right),\varrho\left(\text{'H}\left(\alpha_{2n-1}\right),\text{'H}\left(\beta\right)\right),\varrho\left(Ǵ\left(\beta\right),\text{'H}\left(\alpha_{2n-1}\right)\right)\right\}$$

$$\le\frac{s}{ƀ^2}\varrho\left(f\left(\alpha_{2n-1}\right),f\left(\beta\right)\right)=\frac{s}{ƀ^2}\varrho\left(f\left(Ɣ\right),f\left(\beta\right)\right).$$

Thus,

$$\varrho\left(f\left(Ɣ\right),\ \text{'H}\left(\beta\right)\right)\le\frac{s}{ƀ^2}\varrho\left(f\left(Ɣ\right),\ f\left(\beta\right)\right)\text{ for all }f\left(\beta\right)\ne f\left(Ɣ\right). \tag{6}$$

Similarly $\varrho\left(f\left(\text{Ɏ}\right),\ \acute{G}\ (\beta)\right)\ \leq\ \frac{s}{\text{ƀ}^2}\varrho\left(f\left(\text{Ɏ}\right),\ f\left(\beta\right)\right)$ for all $f\left(\beta\right)\neq f\left(\text{Ɏ}\right).$ (7)

Let us demonstrate that $f(\text{Ɏ})\ =\ \delta$ is a FP of f.

In the case when $\#\{\text{n}: \varrho(f(\alpha_{2n}),\ '\text{H}(\alpha_{2n}))\ >\ \varrho(ff(\alpha_{2n}),\ f(\alpha_{2n}))\}=\infty$, $\exists$ a subsequence $\{\alpha_{2n_j}\}$ of $\{\ \alpha_{2n}\}$ such that

$$\varrho\left(f\left(\alpha_{2n_j}\right),'\text{H}\left(\alpha_{2n_j}\right)\right)>\varrho\left(ff\left(\alpha_{2n_j}\right),f\left(\alpha_{2n_j}\right)\right).$$

$$\begin{aligned}\text{Thus } \varrho(f(\delta),\delta)=lim\varrho(ff(\alpha_{2n_j}),\delta)\ &\leq \text{ƀ} \lim\{\varrho(ff(\alpha_{2n_j}),f(\alpha_{2n_j}))+\varrho(f(\alpha_{2n_j}),\delta)\}\\ &<\ \text{ƀ}\lim\left\{\varrho\left(f\left(\alpha_{2n_j}\right),'\text{H}\left(\alpha_{2n_j}\right)\right)+\varrho\left(f\left(\alpha_{2n_j}\right),\ \delta\right)\right\}\\ &=\ \text{ƀ}\lim\left\{\varrho\left(f\left(\alpha_{2n_j}\right),f\left(\alpha_{2n_{j+1}}\right)\right)+\varrho\left(f\left(\alpha_{2n_j}\right),\delta\right)\right\}]\\ &=\ 0.\end{aligned}$$

This gives $\delta=\ f(\delta)$.

On the other side, if $\#\{\text{n}: \varrho(f(\alpha_{2n}),\ '\text{H}(\alpha_{2n}))\ >\ \varrho(ff(\alpha_{2n}),\ f(\alpha_{2n}))\}\ <\ \infty, \exists \text{v}_2\in \text{N}$ so that

$$\varrho\left(f\left(\alpha_{2n}\right),'\text{H}\left(\alpha_{2n}\right)\right)\leq\left(ff\left(\alpha_{2n}\right),f\left(\alpha_{2n}\right)\right)\forall n\geq v_2.$$

$$\begin{aligned}\text{That is, } \tfrac{\eta(s)}{\text{ƀ}}min\{\varrho(f(\alpha_{2n}),'\text{H}(\alpha_{2n})),\varrho(f(\alpha_{2n}),\acute{G}\ (\alpha_{2n}))\}\ &\leq\varrho(f(\alpha_{2n}),'\text{H}(\alpha_{2n}))\\ &\leq\varrho(ff(\alpha_{2n}),f(\alpha_{2n})).\end{aligned}$$

So,

$$\varrho\left('\text{H}\left(\alpha_{2n}\right),'\text{H}f\left(\alpha_{2n}\right)\right)\leq\max\begin{Bmatrix}\varrho\left('\text{H}\left(\alpha_{2n}\right),'\text{H}f\left(\alpha_{2n}\right)\right),\ \varrho\left(\acute{G}\left(\alpha_{2n}\right),\acute{G}f\left(\alpha_{2n}\right)\right),\\ \varrho\left('\text{H}\left(\alpha_{2n}\right),\acute{G}f\left(\alpha_{2n}\right)\right),\ \varrho\left(\acute{G}\left(\alpha_{2n}\right),\ '\text{H}f\left(\alpha_{2n}\right)\right)\end{Bmatrix}\leq\frac{s}{\text{ƀ}^2}\varrho\left(ff\left(\alpha_{2n}\right),f\left(\alpha_{2n}\right)\right).$$

Thus,

$$\begin{aligned}\varrho(f(\alpha_{2n+1}),ff(\alpha_{2n+1}))\ &\leq\varrho('\text{H}(\alpha_{2n}),f'\text{H}(\alpha_{2n}))=\varrho('\text{H}(\alpha_{2n}),'\text{H}f(\alpha_{2n}))\\ &\leq\ \frac{s}{\text{ƀ}^2}\ \varrho\left(ff\left(\alpha_{2n}\right),f\left(\alpha_{2n}\right)\right)\leq\ \left(\frac{s}{\text{ƀ}^2}\right)^{n-v_2}\varrho\left(ff\left(\alpha_{v_2}\right),f\left(\alpha_{v_2}\right)\right),\end{aligned}$$

and hence $\lim_{n\to\infty}\varrho(f(\alpha_n),\ ff(\alpha_n))=0$ implies $\delta=\ f(\delta)$, that is δ is the fixed point of f in both cases.

Since

$$\begin{aligned}&\tfrac{\eta(s)}{\text{ƀ}}\min\{\varrho(f'\text{H}^{n-1}(\delta),'\text{H}^n(\delta)),\varrho(f'\text{H}^{n-1}(\delta),\acute{G}'\text{H}^{n-1}(\delta))\}\leq\varrho(f'\text{H}^{n-1}(\delta),'\text{H}^n(\delta))\\ &\quad=\ \varrho\left(f'\text{H}^{n-1}\left(\delta\right),'\text{H}^n f\left(\delta\right)\right)=\varrho\left(f'\text{H}^{n-1}\left(\delta\right),f'\text{H}^n\left(\delta\right)\right).\end{aligned}$$

From the given assumption,

$$\begin{aligned}\varrho\left('\text{H}^n\left(\delta\right),\ '\text{H}^{n+1}\left(\delta\right)\right)\ &=\varrho\left('\text{H}'\text{H}^{n-1}\left(\delta\right),\ '\text{H}'\text{H}^n\left(\delta\right)\right)\\ &\leq\ \max\begin{Bmatrix}\varrho\left(\acute{G}'\text{H}^{n-1}\left(\delta\right),\ \acute{G}'\text{H}^n\left(\delta\right)\right),\ \varrho\left('\text{H}'\text{H}^{n-1}\left(\delta\right),\ '\text{H}'\text{H}^n\left(\delta\right)\right),\\ \varrho\left(\acute{G}'\text{H}^{n-1}\left(\delta\right),\ '\text{H}'\text{H}^n\left(\delta\right)\right),\ \varrho\left('\text{H}'\text{H}^{n-1}\left(\delta\right),\ \acute{G}'\text{H}^n\left(\delta\right)\right)\end{Bmatrix}\\ &\leq\ \frac{s}{\text{ƀ}^2}\varrho\left(f'\text{H}^{n-1}\left(\delta\right),f'\text{H}^n\left(\delta\right)\right)=\frac{s}{\text{ƀ}^2}\varrho\left('\text{H}^{n-1}f\left(\delta\right),'\text{H}^n f\left(\delta\right)\right)=\frac{s}{\text{ƀ}^2}\varrho\left('\text{H}^{n-1}\left(\delta\right),'\text{H}^n\left(\delta\right)\right).\end{aligned}$$

That is

$$\varrho\left('\text{H}^n\left(\delta\right),\ '\text{H}^{n+1}\left(\delta\right)\right)\leq\frac{s}{\text{ƀ}^2}\ \varrho\left('\text{H}^{n-1}\left(\delta\right),'\text{H}^n\left(\delta\right)\right). \quad (8)$$

So, by using Eq. (8),

$$\varrho\left('\mathrm{H}^{n}(\delta),\ '\mathrm{H}^{n+1}(\delta)\right) \leq \left(\frac{s}{\text{ƀ}^{2}}\right)^{n} \varrho\left(\delta, '\mathrm{H}(\delta)\right). \tag{9}$$

Next, to demonstrate δ is FP of $'\mathrm{H}$ and $\acute{\mathrm{G}}$, consider the following cases:

Case-I. When $0 \leq s \leq \frac{\sqrt{5}-1}{2}$, Note that $s^{2} + s - 1 \leq 0$ and $2s^{2} < 1$.
Our aim is to prove

$$\eta(s)\,\varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ '\mathrm{H}'\mathrm{H}'\mathrm{H}(\delta)\right) \leq \varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ \delta\right). \tag{10}$$

If not, then since $\varrho('\mathrm{H}'\mathrm{H}(\delta),\ \delta) < \varrho('\mathrm{H}'\mathrm{H}(\delta),\ '\mathrm{H}'\mathrm{H}'\mathrm{H}(\delta))$, by Eq. (9)

$$\begin{aligned}
\varrho\left('\mathrm{H}(\delta),\ \delta\right) &\leq \text{ƀ}\left[\varrho\left('\mathrm{H}(\delta),\ '\mathrm{H}'\mathrm{H}(\delta)\right) + \varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ \delta\right)\right] \\
&< \text{ƀ}\,\varrho\left('\mathrm{H}(\delta),\ '\mathrm{H}'\mathrm{H}(\delta)\right) + \text{ƀ}\,\varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ '\mathrm{H}'\mathrm{H}'\mathrm{H}(\delta)\right) \\
&\leq \text{ƀ}\frac{s}{\text{ƀ}^{2}}\,\varrho\left(\delta, '\mathrm{H}(\delta)\right) + \text{ƀ}\left(\frac{s}{\text{ƀ}^{2}}\right)^{2} \varrho\left(\delta, '\mathrm{H}(\delta)\right) \\
&\leq \left(s^{2} + s\right)\ \varrho\left(\delta,\ '\mathrm{H}(\delta)\right) \leq\ \varrho\left(\delta, '\mathrm{H}(\delta)\right),
\end{aligned}$$

a contradiction. Thus Eq. (10) holds.
Hence

$$\begin{aligned}
\eta(s)\varrho(f'\mathrm{H}'\mathrm{H}(\delta),\ '\mathrm{H}'\mathrm{H}(\delta)) &= \eta(s)\varrho('\mathrm{H}'\mathrm{H}f(\delta),\ '\mathrm{H}'\mathrm{H}(\delta)) \\
&= \eta(s)\,\varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ '\mathrm{H}'\mathrm{H}(\delta)\right) \\
&\leq \varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ \delta\right) \\
&= \varrho\left('\mathrm{H}'\mathrm{H}f(\delta), f(\delta)\right) = \varrho\left(f'\mathrm{H}'\mathrm{H}(\delta), f(\delta)\right).
\end{aligned}$$

So by the given assumption

$$\varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ '\mathrm{H}(\delta)\right)$$

$$\leq \max\left\{\varrho\left(\acute{\mathrm{G}}'\mathrm{H}'\mathrm{H}(\delta),\ \acute{\mathrm{G}}(\delta)\right),\ \varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ '\mathrm{H}(\delta)\right),\ \varrho\left(\acute{\mathrm{G}}'\mathrm{H}'\mathrm{H}(\delta),\ '\mathrm{H}(\delta)\right),\ \varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ \acute{\mathrm{G}}(\delta)\right)\right\}$$

$$\leq\ \frac{s}{\text{ƀ}^{2}}\varrho\left(f'\mathrm{H}'\mathrm{H}(\delta),\ f(\delta)\right) =\ \frac{s}{\text{ƀ}^{2}}\varrho\left('\mathrm{H}'\mathrm{H}(\delta),\ \delta\right). \tag{11}$$

Using contradiction as our argument, Suppose $'\mathrm{H}'\mathrm{H}(\delta) \neq \delta$.
Then note that $f'\mathrm{H}'\mathrm{H}(\delta) \neq \delta$ and $f'\mathrm{H}(\delta) = '\mathrm{H}(\delta)\ \neq \delta$.
Using Eqs. (6) and (7),

$$\varrho\left(\delta, '\mathrm{H}'\mathrm{H}(\delta)\right) \leq \frac{s}{\text{ƀ}^{2}}\varrho\left(\delta, f'\mathrm{H}'\mathrm{H}(\delta)\right) = \frac{s}{\text{ƀ}^{2}}\varrho\left(\delta, '\mathrm{H}'\mathrm{H}(\delta)\right) \leq \frac{s}{\text{ƀ}^{2}}\cdot\frac{s}{\text{ƀ}^{2}}\varrho\left(\delta, f'\mathrm{H}(\delta)\right) = \left(\frac{s}{\text{ƀ}^{2}}\right)^{2}\varrho\left(\delta, '\mathrm{H}(\delta)\right). \tag{12}$$

Here,

$$\begin{aligned}\varrho(\delta,\ Ɦ(\delta)) &\le ƀ[\varrho(\delta,\ ƦƦ(\delta)) + \varrho(ƦƦ(\delta),\ Ɦ(\delta))]\\ &\le ƀ.\left(\frac{s}{ƀ^2}\right)^2 \varrho\left(\delta,\ Ɦ(\delta)\right) + ƀ.\ \frac{s}{ƀ^2}\,\varrho\left(ƦƦ(\delta),\ \delta\right)\\ &= 2\frac{s^2}{ƀ^3}\varrho\left(\delta, Ɦ(\delta)\right) \le 2s^2\varrho\left(\delta, Ɦ(\delta)\right) < \varrho\left(\delta, Ɦ(\delta)\right).\end{aligned}$$

This is a contradiction. Thus $ƦƦ(\delta) = \delta$.
By Eq. (9), $\varrho(Ɦ(\delta), \delta) = \varrho(Ɦ(\delta), ƦƦ(\delta)) \le \frac{s}{ƀ^2}\varrho(\delta, Ɦ(\delta))$, which implies $Ɦ(\delta) = \delta$.
Analogously $Ɠ(\delta) = \delta$.

Case-II. When $\frac{\sqrt{5}-1}{2} \le s \le \frac{1}{\sqrt{2}}$, Note that $2s^2 < 1$.
If Eq. (10) does not hold, then from Eq. (9)

$$\begin{aligned}\varrho\left(\delta,\ Ɦ(\delta)\right) &\le ƀ\left[\varrho\left(\delta,\ ƦƦ(\delta)\right) + \varrho\left(ƦƦ\delta,\ Ɦ(\delta)\right)\right]\\ &< ƀ.\ \eta(s)\ \varrho\left(ƦƦ(\delta),\ ƦƦƦ(\delta)\right) + ƀ.\ \varrho\left(ƦƦ(\delta),\ Ɦ(\delta)\right)\\ &= \frac{1-s}{ƀ^3}\varrho\left(\delta, Ɦ(\delta)\right) + \frac{s}{ƀ}\varrho\left(\delta, Ɦ(\delta)\right) \le \varrho\left(\delta, Ɦ(\delta)\right).\end{aligned}$$

Thus $\varrho(\delta,\ Ɦ(\delta)) <\ \varrho(\delta,\ Ɦ(\delta))$.
This is in conflict with itself. Thus Eq. (10) holds, as in the case-I, $Ɦ(\delta) = \delta$. Analogously $Ɠ(\delta) = \delta$.

Case-III. When $\frac{1}{\sqrt{2}} \le s < 1$
Now, our aim is to show $\varrho(Ɠ(Ɣ), Ɦ(\beta)) \le \frac{s}{ƀ^2}\varrho(f(Ɣ), f(\beta))$.
Assume that $f(\beta) \ne f(Ɣ)$. Then for each $n \in ℕ$, $\exists\ Ɣ_n \in Ɦ(\beta)$ in such a way that

$$\varrho\left(f(Ɣ), Ɣ_n\right) \le \varrho\left(f(Ɣ), Ɦ(\beta)\right) + \frac{1}{n}\varrho\left(f(\beta), f(Ɣ)\right).$$

Therefore,

$$\begin{aligned}\varrho\left(f(\beta),\ Ɦ(\beta)\right) &\le \varrho\left(f(\beta),\ Ɣ_n\right) \le ƀ\left[\varrho\left(f(\beta),\ f(Ɣ)\right) + \varrho\left(f(Ɣ),\ Ɣ_n\right)\right]\\ &\le ƀ\left[\varrho\left(f(\beta),\ f(Ɣ)\right) + \varrho\left(f(Ɣ),\ Ɦ(\beta)\right) + \tfrac{1}{n}\varrho\left(f(\beta),\ f(Ɣ)\right)\right]\\ &\le ƀ\left(1 + \tfrac{1}{n}\right)\varrho\left(f(\beta), f(Ɣ)\right) + ƀ\frac{s}{ƀ^2}\varrho\left(f(\beta), f(Ɣ)\right)\\ &= ƀ\left(1 + s + \tfrac{1}{n}\right)\varrho\left(f(\beta), f(Ɣ)\right).\end{aligned}$$

Hence, $\varrho(f(\beta), Ɦ(\beta)) \le ƀ(1+s)\varrho(f(\beta), \delta) = ƀ(1+s)\varrho(f(\beta), f(\delta))$.
Now, either $\varrho(f(\delta),\ Ɠ(\delta)) \le \varrho(f(\beta),\ Ɦ(\beta))$ or $\varrho(f(\beta),\ Ɦ(\beta)) \le \varrho(f(\delta),\ Ɠ(\delta))$.
This gives, $\min\{\varrho(f(\delta), Ɠ(\delta)), \varrho(f(\beta), Ɦ(\beta))\} \le \varrho(f(\beta), Ɦ(\beta)) \le ƀ(1+s)\varrho(f(\beta), f(\delta))$.
In other words, $\frac{1}{ƀ}\eta(s)\min\{\varrho(f(\delta),\ Ɠ(\delta)),\ \varrho(f(\beta),\ Ɦ(\beta))\} \le \varrho(f(\beta),\ f(\delta))$.
So Eq. (2) implies

$$\max\left\{\varrho\left(Ɠ(\beta), Ɠ(\delta)\right), \varrho\left(Ɠ(\beta), Ɦ(\delta)\right), \varrho\left(Ɦ(\beta), Ɦ(\delta)\right), \varrho\left(Ɠ(\delta), Ɦ(\beta)\right)\right\} \le \frac{s}{ƀ^2}\varrho\left(f(\beta), f(\delta)\right).$$

Now taking $\beta = \alpha_{2n}$,

$$\begin{aligned}\varrho\left(Ɠ(\delta),\ Ɦ(\alpha_{2n})\right) &\le \max\left\{\varrho\left(Ɠ(\alpha_{2n}),\ Ɠ(\delta)\right),\ \varrho\left(Ɠ(\alpha_{2n}),\ Ɦ(\delta)\right),\ \varrho\left(Ɦ(\alpha_{2n}),\ Ɦ(\delta)\right),\ \varrho\left(Ɠ(\delta),\ Ɦ(\alpha_{2n})\right)\right\}\\ &\le \frac{s}{ƀ^2}\ \varrho\left(f(\alpha_{2n}),\ f(\delta)\right),\end{aligned}$$

letting $n \to \infty$,

$$\varrho\left(Ɠ(\delta), f(\delta)\right) \le \frac{s}{ƀ^2}\varrho\left(\delta, f(\delta)\right) = 0.$$

This gives, $\varrho(Ɠ(\delta),\ f(\delta)) = 0$.

Thus, $f(\delta) = \acute{G}(\delta)$.
Analogously, $f(\delta) = \acute{H}(\delta)$.
Thus $\delta = f(\delta) = \acute{G}(\delta) = \acute{H}(\delta)$, in each of the three cases.
To demonstrate the uniqueness of the CFP δ to finish the proof, assume θ to be another CFP of f, $\acute{G}$ and $\acute{H}$.
Since $\eta(s)\varrho(f(\delta), \acute{H}(\delta)) = 0 \le \varrho(f(\delta), f(\theta))$.
So, by the assumption,

$$\begin{aligned}\varrho(\delta, \theta) &= \varrho(\acute{H}(\delta), \acute{H}(\theta))\\ &\le \max\{\varrho(\acute{H}(\delta), \acute{G}(\theta)), \varrho(\acute{H}(\delta), \acute{H}(\theta)), \varrho(\acute{G}(\delta), \acute{G}(\theta)), \varrho(\acute{G}(\delta), \acute{H}(\theta))\}\\ &\le \frac{s}{b^2}\varrho(f(\delta), f(\theta))\\ &= \frac{s}{b^2}\varrho(\delta, \theta) < \varrho(\delta, \theta) \text{ and hence } \delta = \theta.\end{aligned}$$

Corollary 1: *Theorem 3*.

Proof: Theorem 4 directly yields the outcome when $f = I$.

Corollary 2: *Theorem 2*.

Proof: Theorem 4 directly yields the outcome when $f = I$ and $b = 1$.

Corollary 3: *Theorem 1*.

Proof: Theorem 4 directly yields the outcome when $\acute{H} = \acute{G}$, $f = I$ and $b = 1$.

Corollary 4: *Let (X, ϱ) be a CbMS and $g, G, H : X \to X$ satisfying*

$$\frac{1}{b}\eta(s)\min\{\varrho(\alpha, g\acute{H}(\alpha)), \varrho(\alpha, g\acute{G}(\alpha))\} \le \varrho(\alpha, \beta)$$

implies

$$\max\{\varrho(g\acute{G}(\alpha), g\acute{G}(\beta)), \varrho(g\acute{G}(\alpha), g\acute{H}(\beta)), \varrho(g\acute{H}(\alpha), g\acute{H}(\beta)), \varrho(g\acute{G}(\beta), g\acute{H}(\alpha))\} \le \frac{s}{b^2}\varrho(\alpha, \beta),$$

for all $\alpha, \beta \in X$. Also, if g is one to one, $g\acute{G} = \acute{G}g$ and $g\acute{H} = \acute{H}g$, then the mappings $g, \acute{G}, \acute{H}$ have a unique CFP in X.

Proof: By Corollary 1, $g\acute{H}, gG$ have a single CFP $\delta \in X$, then, $g\acute{G}(\delta) = g\acute{H}(\delta) = \delta$, since g is one to one it follows that $\acute{G}(\delta) = \acute{H}(\delta)$ and

$$0 = \frac{1}{b}\eta(s)min\{\varrho(\delta, g\acute{H}(\delta)), \varrho(\delta, g\acute{G}(\delta))\} \le \varrho(\delta, \acute{H}(\delta)).$$

Consequently,

$$\begin{aligned}\varrho(\delta, \acute{H}(\delta)) &\le \max\{\varrho(g\acute{G}(\delta), g\acute{G}\acute{H}(\delta)), \varrho(g\acute{G}(\delta), g\acute{H}^2(\delta)), \varrho(g\acute{H}(\delta), g\acute{H}^2(\delta)), \varrho(g\acute{G}\acute{H}(\delta), g\acute{H}(\delta))\}\\ &= \max\{\varrho(g\acute{G}(\delta), \acute{G}g\acute{H}(\delta)), \varrho(g\acute{H}(\delta), \acute{H}g\acute{H}(\delta)), \varrho(g\acute{G}(\delta), \acute{H}g\acute{H}(\delta)), \varrho(\acute{G}g\acute{H}(\delta), g\acute{H}(\delta))\}\\ &= \max\{\varrho(\delta, \acute{G}(\delta)), \varrho(\delta, \acute{H}(\delta)), \varrho(\delta, \acute{H}(\delta)), \varrho(\acute{G}(\delta), \delta)\}\\ &\le \tfrac{s}{b^2}\varrho(\delta, \acute{H}(\delta)),\end{aligned}$$

it follows that $\acute{H}(\delta) = \acute{G}(\delta) = \delta$, hence, $g(\delta) = g\acute{H}(\delta) = \delta$.

Corollary 5: *Let (X, ϱ) be a CMS and $g, \acute{G}, \acute{H} : X \to X$ satisfying the following contractive condition*

$$\eta(s)\min\{\varrho(\alpha, g\acute{G}(\alpha)), \varrho(\alpha, g\acute{H}(\alpha))\} \le \varrho(\alpha, \beta)$$

implies

$$\max\left\{\varrho\left(g\acute{G}(\alpha),\ g\acute{G}(\beta)\right),\ \varrho\left(g\acute{G}(\alpha),\ g'H(\beta)\right),\ \varrho\left(g'H(\alpha),\ g'H(\beta)\right),\ \varrho\left(g\acute{G}(\beta),\ g'H(\alpha)\right)\right\} \leq s\,\varrho(\alpha,\ \beta),$$

for all $\alpha,\ \beta\ \in\ X$. Also, if g is one to one, $g\acute{G} = \acute{G}g$ and $g'H = 'Hg$, then $g,\ \acute{G},\ 'H$ have a unique CFP in X.

Proof: Corollary 4 directly yields the outcome when $ƀ = 1$.

Theorem 4 is demonstrated by the illustration given below:

Example 1: *Take* $X = [0,\ \infty)$. *Define* $\varrho : X \times X \rightarrow R^+$ *by*

$$\varrho(\alpha,\beta) = \begin{cases} 0, & \alpha = \beta, \\ (\alpha+\beta)^2 & \alpha \neq \beta, \end{cases}$$

for all $\alpha, \beta \in X$. Then $(X,\ \varrho)$ is a CƀMS for $ƀ = 2$.

Consider $f,\ \acute{G},\ 'H : X \rightarrow X$ by

$$f(\alpha) = \frac{\alpha}{2},\ \acute{G}(\alpha) = ln\left(1 + \frac{1}{4\sqrt{2}}\alpha\right) \text{ and } 'H(\alpha) = ln\left(1 + \frac{1}{8\sqrt{2}}\alpha\right)$$

for all $\alpha \in X$. Then for all $\alpha,\ \beta\ \in X$,

$$\begin{aligned} \tfrac{1}{2}\eta(s)\ \min\left\{\varrho\left(f(\alpha),\ \acute{G}(\alpha)\right),\ \varrho\left(f(\alpha), 'H(\alpha)\right)\right\} &= \tfrac{1}{2}\cdot\tfrac{1}{3}\min\left\{\begin{array}{l}\left(\frac{\alpha}{2} + ln\left(1 + \frac{1}{4\sqrt{2}}\alpha\right)\right)^2, \\ \left(\frac{\alpha}{2} + ln\left(1 + \frac{1}{8\sqrt{2}}\alpha\right)\right)^2\end{array}\right\} \\ &= \tfrac{1}{6}\left(\tfrac{\alpha}{2} + ln\left(1 + \tfrac{1}{8\sqrt{2}}\alpha\right)\right)^2 \\ &\leq \tfrac{1}{6}\left(\tfrac{\alpha}{2} + \tfrac{1}{8\sqrt{2}}\alpha\right)^2 = \tfrac{1}{6}\left(\tfrac{1}{2} + \tfrac{1}{8\sqrt{2}}\right)^2\alpha^2 \\ &\leq \tfrac{1}{4}\alpha^2 \leq \tfrac{1}{4}(\alpha+\beta)^2 = d\left(f(\alpha), f(\beta)\right), \end{aligned}$$

and

$$\max\ \left\{\varrho\left(\acute{G}(\alpha), \acute{G}(\beta)\right),\ \varrho\left('H(\alpha), 'H(\beta)\right), \varrho\left(\acute{G}(\alpha), 'H(\beta)\right), \varrho\left(\acute{G}(\beta), 'H(\alpha)\right)\right\}$$

$$\leq \max\left\{\begin{array}{l}\left(\frac{1}{8\sqrt{2}}\alpha + \frac{1}{8\sqrt{2}}\beta\right)^2 \\ \left(\frac{1}{4\sqrt{2}}\alpha + \frac{1}{4\sqrt{2}}\beta\right)^2 \\ \left(\frac{1}{8\sqrt{2}}\alpha + \frac{1}{4\sqrt{2}}\beta\right)^2 \\ \left(\frac{1}{4\sqrt{2}}\alpha + \frac{1}{8\sqrt{2}}\beta\right)^2\end{array}\right\} \leq \left(\frac{1}{4\sqrt{2}}\alpha + \frac{1}{4\sqrt{2}}\beta\right)^2$$

$$= \left[\frac{1}{4\sqrt{2}}(\alpha+\beta)\right]^2$$

$$= \frac{1}{32}(\alpha+\beta)^2 \leq \frac{1}{24}(\alpha+\beta)^2 = \frac{2}{3}\cdot\frac{1}{4}\cdot\frac{1}{4}(\alpha+\beta)^2 = \frac{s}{ƀ^2}\varrho\left(f(\alpha), f(\beta)\right).$$

Since all the hypotheses of Theorem 4 are met; therefore the mappings $f,\ \acute{G}$ and $'H$ have a CFP in X.

The index clearly shows that the blue line represents the line $y = x$, whereas, black, green, and red lines represent the functions $f,\ \acute{G}$ and $'H$, respectively. Clearly, Fig. 1 suggests that the mappings $f,\ \acute{G},$ and $'H$ have a unique CFP 0, which is intersected at the origin by the curves of these three mappings and the line $y = x$.

Table 1 gives us the complete computational analysis of the iterations generated in Example 1. As a matter of fact, the proof of Theorem 4 demonstrates that $f(\alpha_{2n+2}) = \acute{G}(\alpha_{2n+1})$ and $f(\alpha_{2n+1}) = 'H(\alpha_{2n})$, which simply

Table 1. Computation analysis of iterations using software MATLAB.

Steps	α_i	$f(\alpha_i)$	Ή(α_i)	Ǵ(α_i)
1	α_0 = 10.000000	5.000000	0.442130	
2	α_1 = 0.884259	0.442130		0.118143
3	α_2 = 0.236286	0.118143	0.017772	
4	α_3 = 0.035544	0.017772		0.005424
5	α_4 = 0.010847	0.005424	0.000832	
6	α_5 =0.001664	0.000832		0.000255
7	α_6 = 0.000511	0.000255	0.000039	
8	α_7 = 0.000078	0.000039		0.000012
9	α_8 = 0.000024	0.000012	0.000002	
10	α_9 = 0.000004	0.000002		0.000001
11	α_{10} = 0.000001	0.000001	0.000000	
12	α_{11} = 0.000000	0.000000		0.000000
13	α_{12} = 0.000000	0.000000	0.000000	
14	α_{13} = 0.000000	0.000000		0.000000

means that

$f(\alpha_2)$ = Ǵ(α_1) and $f(\alpha_1)$ = Ή(α_0),
$f(\alpha_4)$ = Ǵ(α_3) and $f(\alpha_3)$ = Ή(α_2)
$f(\alpha_6)$ = Ǵ(α_5) and $f(\alpha_5)$ = Ή(α_4),
$f(\alpha_8)$ = Ǵ(α_7) and $f(\alpha_7)$ = Ή(α_6), and so on.

For the purpose of experimental verification of the above suggested iterative scheme, MATLAB software is used. On taking the initiator x (that is, an initial value of x or, can say α) to be 10, the numerical values of the above suggested iterative scheme are obtained in Table 1.

Now elaborate on Table 1 as per the below-mentioned steps:

Step 1 suggests that for the initial value of x that is α_0 (as chosen in the proof of Theorem 4), that is, the initiator is 10.000000 at which function f takes the value 5.000000 and Ή takes 0.442130.
Step 2 suggests that the value of the next iteration, that is, α_1 is 0.884259, for which f assumes the value 0.442130 so that $f(\alpha_1)$ = Ή(α_0), is verified. Again, at α_1, Ǵ assumes the value 0.118143.
Step 3 suggests the value of α_2 which is 0.236286, for which f assumes the value 0.118143 so that $f(\alpha_2)$ = Ǵ(α_1) is verified. Moreover, at α_2, the function Ή takes the value 0.017772.

This process continues in the successive steps and using Table 1, In general, it may be confirmed that $f(\alpha_{2n+2})$ = Ǵ(α_{2n+1}) and $f(\alpha_{2n+1}\alpha)$ = Ή(α_{2n}) for $n \geq 0$.

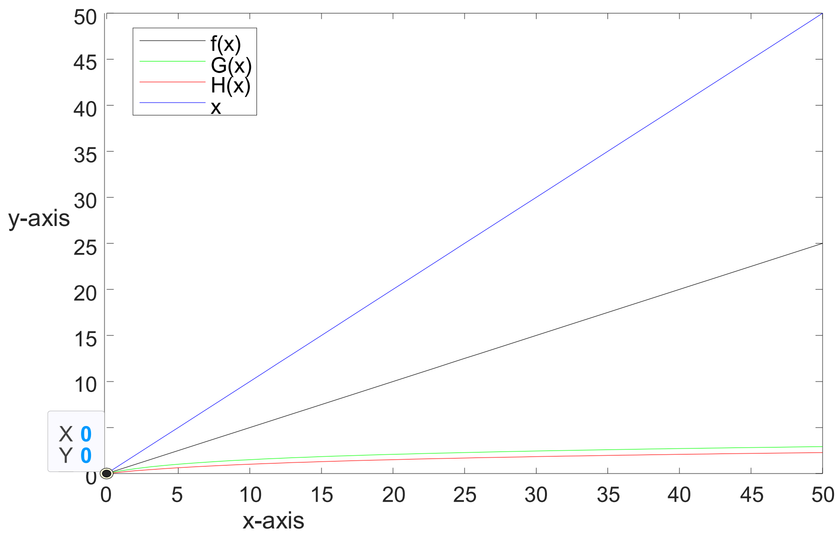

Fig. 1. Illustrates the graphical representation of Example 1.

Steps 12, 13, and 14 suggest that

$$f(\alpha_{12}) = \acute{G}(\alpha_{11}) = 0.000000 \quad \text{and} \quad f(\alpha_{13}) = {}'H(\alpha_{12}) = 0.000000.$$

Interestingly, $\alpha_{11} = \alpha_{12} = \alpha_{13} = 0.000000$, for which $f(\alpha_{12}) = \acute{G}(\alpha_{11}) = 0.000000$ and $f(\alpha_{13}) = {}'H(\alpha_{12}) = 0.000000$. Hence, it can be concluded that 0 is the CFP of the mappings f, $\acute{G}$ and $'H$, which indeed is unique.

Algorithm 1 is used for the generation of iterations in Table 1 as under:

Algorithm 1: for generating the iterations.

Input: $\boldsymbol{f(\alpha)} = \frac{\alpha}{2}$; $'H(\boldsymbol{\alpha}) = \boldsymbol{ln}(1 + \frac{1}{8\sqrt{2}}\boldsymbol{\alpha})$; $\acute{G}(\boldsymbol{\alpha}) = \boldsymbol{ln}(1 + \frac{1}{4\sqrt{2}}\boldsymbol{\alpha})$; $\alpha 1$—initial guess; itr—maximum number of iterations; Assign tol $= 1e-7$ (allowed error for convergence of sequence); Assign $\boldsymbol{\alpha}(1) = \boldsymbol{\alpha}1$.
Output: CFP of mappings $\boldsymbol{f}$, $\acute{G}$ **and** $'H$.

1. for $\boldsymbol{i} = 2$: itr
2. if mod($\boldsymbol{i}$, 2) == 0 then
3. $\boldsymbol{\alpha(i)} = 2 * \acute{G}(\boldsymbol{\alpha(i-1)})$
4. if **abs**($\boldsymbol{\alpha(i) - \alpha(i-1)}$) < **tol** then
5. break
6. end if
7. **print values of** $\boldsymbol{i}$, $\boldsymbol{\alpha(i)}$, $\boldsymbol{f(\alpha(i))}$, $'H(\boldsymbol{\alpha(i)})$
8. end if
9. else
10. $\boldsymbol{\alpha(i) = 2 * {}'H(\alpha(i-1))}$
11. if abs($\boldsymbol{\alpha(i)} - \boldsymbol{\alpha(i} - 1)$) < tol then
12. break
13. end if
14. print values of $\boldsymbol{i}$, $\boldsymbol{\alpha(i)}$, $\boldsymbol{f(\alpha(i))}$, $\acute{G}(\boldsymbol{\alpha(i)})$
15. end else
16. end for

Remark 1: Roshan et al.[23] provided an example where they used $\mu(s) = 1/2$. But $1/2$ does not fall under the co-domain of the definition of μ.

Application to dynamic programming problem

Here, the focus is to yield the application of the work done by us in the previous section to the dynamic programming problem. Take $\mathfrak{X}$ as Banach spaces such that $A,\ B \subseteq \mathfrak{X}$. Let $\mathbb{R}$ represent the field of real numbers and consider the mappings $\zeta : A \times B \to A$ and $p,\ q : A \times B \to \mathbb{R}$ and $S, T_1, T_2 : A \times B \times \mathbb{R} \to \mathbb{R}$. Consider A as the state space and B as the decision space, the dynamic programming problem reduces to the solution of functional equations;

$$u_i = \sup_{\beta\in B} \{p(\alpha, \beta) + T_i(\alpha, \beta, u_i(\zeta(\alpha, \beta)))\},\ \alpha \in A,\ i = 1, 2. \tag{13}$$

$$v = \sup_{\beta\in B} \{q(\alpha, \beta) + S(\alpha, \beta, v(\zeta(\alpha, \beta)))\},\ \alpha \in A. \tag{14}$$

There are some functional equations that naturally arise in multistage processes.[9,10] In this section, our aim is to investigate the common solution of Eqs. (13) and (14).

Consider $\mathcal{B}(A) = \{m :\ m$ is bounded real-valued function on $A\}$. For any $m \in \mathcal{B}(A)$, define $\|m\| = \sup_{\alpha\in A} |m(\alpha)|^2$. Then $(\mathcal{B}(A),\ \|.\|)$ is a Banach space.[28] Considering the aforementioned conditions to be true:

(ÐϷ - 1) $S,\ T_1, T_2,\ p$ and q are bounded functions.

(ÐϷ - 2) Take $\eta(s)$ as defined in Theorem 1. Suppose $\exists\, s \in [0,\ 1)$ so that for all $(\alpha,\ \beta) \in A \times B,\ m,\ n \in \mathcal{B}(A)$ and $t \in A$,

$$\frac{1}{ƀ}\eta(s)\ \min\left\{|Jm(t) - J_1m(t)|^2,\ |Jn(t) - J_2n(t)|^2\right\} \le |Jm(t) - Jn(t)|^2$$

implies

$$\max\left\{\begin{array}{l}|T_1(\alpha, \beta, m(t)) - T_2(\alpha, \beta, n(t))|^2,\ |T_1(\alpha, \beta, m(t)) - T_1(\alpha, \beta, n(t))|^2,\\ |T_2(\alpha, \beta, m(t)) - T_2(\alpha, \beta, n(t))|^2,\ |T_2(\alpha, \beta, m(t)) - T_1(\alpha, \beta, n(t))|^2\end{array}\right\} \le \frac{s}{ƀ^2}|Jm(t) - Jn(t)|^2,$$

where J_1, J_2 and $J : \mathcal{B}(A) \to \mathcal{B}(A)$ have the following definitions:

$$J_im(\alpha) = \sup_{\beta\in B} \{p(\alpha, \beta) + T_i(\alpha, \beta, m(\zeta(\alpha, \beta)))\}, \alpha \in A, m \in \mathcal{B}(A), i = 1, 2.$$

$$Jm(\alpha) = \sup_{\beta\in B} \{q(\alpha, \beta) + S(\alpha, \beta, m(\zeta(\alpha, \beta)))\}, \alpha \in A, m \in \mathcal{B}(A).$$

(ÐϷ - 3) For all $m,\ n \in \mathcal{B}(A)$, $\exists\, x,\ y \in \mathcal{B}(A)$ so that

$$J_1m(\alpha) = Jx(\alpha) \quad \text{and} \quad J_2n(\alpha) = Jy(\alpha), \alpha \in A.$$

(ÐϷ - 4) $\exists\, m,\ n \in \mathcal{B}(A)$ so that

$$Jm(\alpha) = J_1m(\alpha) \quad \text{implies} \quad JJ_1m(\alpha) = J_1Jm(\alpha)$$

and $Jn(\alpha) = J_2n(\alpha)$ implies $JJ_2n(\alpha) = J_2Jn(\alpha)$.

Theorem 5: *Let us assume that (ÐϷ - 1) to (ÐϷ - 4) are true and $J(\mathcal{B}(A))$ be closed and subspace of $\mathcal{B}(A)$. Then Eq. (13), $i = 1, 2$, and Eq. (14) possess a unique common solution in $\mathcal{B}(A)$.*

Proof: For any $m,\ n \in \mathcal{B}(A)$, let $\varrho(m, n) = \sup\{|m(\alpha) - n(\alpha)|^2 : \alpha \in A\}$. Then $(\mathcal{B}(A),\ \varrho)$ is a CƀMS. Let $\varepsilon > 0$ be a given and $m_1, m_2 \in \mathcal{B}(A)$. Take $\alpha \in A$, and choose $\beta_1, \beta_2 \in B$ such that

$$J_im_j < p(\alpha,\ \beta_j) + T_i(\alpha, \beta_j, m_j(\alpha_j)) + \varepsilon,\ i = 1,\ 2, \tag{15}$$

where $\alpha_j = \zeta(\alpha,\ \beta_j)$.

Further,

$$J_1 m_1 \geq p(\alpha,\ \beta_1) + T_1(\alpha, \beta_1, m_1(\alpha_1)), \tag{16}$$

$$J_2 m_2 \geq p(\alpha,\ \beta_1) + T_2(\alpha, \beta_1, m_2(\alpha_1)), \tag{17}$$

$$J_1 m_2 \geq p(\alpha,\ \beta_1) + T_1(\alpha, \beta_1, m_2(\alpha_2)), \tag{18}$$

$$J_2 m_1 \geq p(\alpha,\ \beta_2) + T_2(\alpha, \beta_2, m_1(\alpha_1)). \tag{19}$$

Therefore, the first inequality in (ÐϷ - 2) becomes

$$\frac{1}{ƀ}\eta(s)\min\left\{|Jm_1 - J_1 m_1|^2,\ |Jm_2 - J_2 m_2|^2\right\} \leq |Jm_1 - Jm_2|^2, \tag{20}$$

and this together with Eqs. (15) to (20) implies

$$|J_1 m_1 - J_2 m_2|^2 < |T_1(\alpha, \beta_1, m_1(\alpha_1)) - T_2(\alpha, \beta_1, m_2(\alpha_1)) + \varepsilon|^2.$$

Since $\varepsilon > 0$ is an arbitrary positive real number, so

$$\begin{aligned} |J_1 m_1 - J_2 m_2|^2 &< |T_1(\alpha, \beta_1, m_1(\alpha_1)) - T_2(\alpha, \beta_1, m_2(\alpha_1))|^2 \\ &\leq \frac{s}{ƀ^2}|Jm_1 - Jm_2|^2. \end{aligned} \tag{21}$$

And,

$$|J_1 m_1 - J_1 m_2|^2 \leq \frac{s}{ƀ^2}|Jm_1 - Jm_2|^2. \tag{22}$$

$$|J_2 m_1 - J_2 m_2|^2 \leq \frac{s}{ƀ^2}|Jm_1 - Jm_2|^2. \tag{23}$$

$$|J_1 m_2 - J_2 m_1|^2 \leq \frac{s}{ƀ^2}|Jm_1 - Jm_2|^2. \tag{24}$$

As Eqs. (21) to (24), holds for all $\alpha \in A$, taking supremum, and from Eqs. (20) to (24), that

$$\varrho(J_1 m_1, J_2 m_2) \leq \frac{s}{ƀ^2}\varrho(Jm_1, Jm_2),$$

$$\varrho(J_1 m_2, J_2 m_1) \leq \frac{s}{ƀ^2}\varrho(Jm_1, Jm_2),$$

$$\varrho(J_1 m_1, J_1 m_2) \leq \frac{s}{ƀ^2}\varrho(Jm_1, Jm_2),$$

$$\varrho(J_2 m_1, J_2 m_2) \leq \frac{s}{ƀ^2}\varrho(Jm_1, Jm_2).$$

Therefore,

$$\frac{1}{ƀ}\eta(s)\min\{\varrho(Jm_1, J_1 m_1),\ \varrho(Jm_2, J_2 m_2)\} \leq \varrho(Jm_1, Jm_2)$$

gives

$$\max\{\varrho(J_1 m_1, J_2 m_2), \varrho(J_1 m_2, J_2 m_1), \varrho(J_1 m_1, J_1 m_2), \varrho(J_2 m_1, J_2 m_2)\} \leq \frac{s}{ƀ^2}\varrho(Jm_1, Jm_2)$$

As a result, Theorem 4 is applicable, where $J,\ J_1$ and J_2 correspond to the mappings $f,\ Ǵ,$ and Ή respectively. Consequently, $J,\ J_1$ and J_2 have a unique CFP w^*, in other words, $w^*(\alpha)$ is the unique bounded common solution of Eqs. (13) and (14), $i = 1, 2$.

To illustrate Theorem 5, an example is provided below:

Example 2: *Take* $\mathbb{X} = \mathbb{R}$ *be a Banach space with the norm* $\|.\|$ *specified as* $\|\alpha\| = |\alpha|,\ \forall\, \alpha \in \mathbb{X}$.

Take $A = [0, 1] \subset \mathbb{X}$ as state space, and $B = [0, \infty) \subset \mathbb{X}$ as decision space. Consider $\zeta : A \times B \to A$ by $\zeta(\alpha,\ \beta) = \frac{\alpha}{\beta^2+1},\ \alpha \in A,\ \beta \in B.$

For any m, n in $\mathcal{B}(A)$ and $i = 1, 2$, define $y_i,\ z : A \to R$ by

$$y_i(\alpha) = z(\alpha) = \alpha^3 + \frac{3}{8}.$$

Define $T_i,\ S : A \times B \times \mathbb{R} \to \mathbb{R}$ by

$$T_i(\alpha,\ \beta,\ t) = \frac{1}{8}\left[\frac{\alpha}{(\alpha+2)(\beta+1)}\sin\left(\frac{\beta}{\beta+1}\right) + 3\right],\ i = 1,\ 2;$$

$$S(\alpha,\ \beta, t) = \frac{3}{8}\sin\ t.$$

Take $p(\alpha,\ \beta) = \frac{\alpha^3\beta^5}{\alpha+\beta^5}$ and $q(\alpha,\ \beta) = \frac{\alpha^3\beta^7}{\alpha+\beta^7}$.

Notice that T_1, T_2, S, p, and q are bounded.

Also, for any $\alpha \in A$ and $m, n \in \mathcal{B}(A)$

$$Jm(\alpha) = \sup_{\beta\in B}\{q(\alpha, \beta) + S(\alpha, \beta, m(\zeta(\alpha, \beta)))\} = \alpha^3 + \frac{3}{8} = y_i(\alpha) = z(\alpha),$$

$$Jn(\alpha) = \sup_{\beta\in B}\{q(\alpha, \beta) + S(\alpha, \beta, n(\zeta(\alpha, \beta)))\} = \alpha^3 + \frac{3}{8} = z(\alpha),$$

$$J_1m(\alpha) = \sup_{\beta\in B}\{p(\alpha, \beta) + T_1(\alpha, \beta, m(\zeta(\alpha, \beta)))\} = \alpha^3 + \frac{3}{8},$$

$$J_2n(\alpha) = \sup_{\beta\in B}\{p(\alpha, \beta) + T_2(\alpha, \beta, n(\zeta(\alpha, \beta)))\} = \alpha^3 + \frac{3}{8},$$

$$J_1n(\alpha) = J_2m(\alpha) = \alpha^3 + \frac{3}{8}.$$

Now,

$$\frac{1}{\flat}\eta(s)\min\left\{|Jm(t) - J_1m(t)|^2,\ |Jn(t) - J_2n(t)|^2\right\} = 0 = |Jm(t) - Jn(t)|^2.$$

Thus,

$$\frac{1}{\flat}\eta(s)\min\left\{|Jm(t) - J_1m(t)|^2,\ |Jn(t) - J_2n(t)|^2\right\} = |Jm(t) - Jn(t)|^2$$

implies

$$\max\left\{\begin{array}{l}|T_1(\alpha, \beta, m(t)) - T_2(\alpha, \beta, n(t))|^2, |T_1(\alpha, \beta, m(t)) - T_1(\alpha, \beta, n(t))|^2,\\ |T_2(\alpha, \beta, m(t)) - T_2(\alpha, \beta, n(t))|^2, |T_2(\alpha, \beta, m(t)) - T_1(\alpha, \beta, n(t))|^2\end{array}\right\} = 0 \leq \frac{s}{\flat^2}|Jm(t) - Jn(t)|^2$$

Eventually, for any $m, n \in B(A)$ with $J_1m = Jm$,
$J_1Jm = y_1(\alpha) = z(\alpha) = JJm = JJ_1m$,
that is, $JJ_1m = J_1Jm$, and with $J_2n = Jn$,

$J_2Jn = y_2(\alpha) = z(\alpha) = JJn = JJ_2n.$

Thus, all of the conditions of Theorem 5 hold. Therefore, the solution of Eqs. (13) and (14) is unique in $\mathcal{B}(A)$.

The following results are the direct consequences of Theorems 4 and 5:

Corollary 6: *Considering the conditions from (ÐϘ - 1) to (ÐϘ - 4) to be true with:*

(i) T_1, T_2 and p *are bounded functions.*
(ii) For $0 \le s < 1$ *and* $\forall\ (\alpha, \beta) \in A \times B, m, n \in \mathcal{B}(A),\ t \in A$,
$\frac{1}{\mathfrak{b}}\ \eta(s)\ min\{|m(t) - J_1m(t)|^2,\ |n(t) - J_2n(t)|^2\} \le |m(t) - n(t)|^2$

implies

$$\max\left\{\begin{matrix}|T_1(\alpha,\beta,m(t)) - T_2(\alpha,\beta,n(t))|^2, |T_1(\alpha,\beta,m(t)) - T_1(\alpha,\beta,n(t))|^2, \\ |T_2(\alpha,\beta,m(t)) - T_2(\alpha,\beta,n(t))|^2, |T_2(\alpha,\beta,m(t)) - T_1(\alpha,\beta,n(t))|^2\end{matrix}\right\} \le \frac{s}{\mathfrak{b}^2}|m(t) - n(t)|^2.$$

where J_1 and J_2 have the following definitions:

$$\text{for } \alpha \in A, m \in \mathcal{B}(A), i = 1, 2; J_im(\alpha) = \sup_{\beta \in B}\{p(\alpha,\beta) + T_i(\alpha,\beta,m(\zeta(\alpha,\beta)))\}.$$

Then the solution of Eq. (13) for $i = 1, 2$ is unique in $\mathcal{B}(A)$.

Proof: Theorem 5 directly yields the outcome when $q = 0$, $\zeta(\alpha,\ \beta) = \alpha$ and $S(\alpha, \beta, t) = t$.

Corollary 7: *Considering the conditions from (ÐϘ - 1) to (ÐϘ - 4) to be true with:*

(i) 'Ħ and p *are bounded functions.*
(ii) For $0 \le s < 1$ *and* $\forall\ (\alpha, \beta) \in A \times B, m, n \in \mathcal{B}(A), t \in A$,
$\frac{1}{\mathfrak{b}}\ \eta(s)\ |m(t) - J_1m(t)|^2 \le |m(t) - n(t)|^2$ *implies*

$$|T(\alpha,\beta,m(t)) - T(\alpha,\beta,n(t))|^2 \le \frac{s}{\mathfrak{b}^2}|m(t) - n(t)|^2.$$

where J_1 has the following definition:

$$\text{for } \alpha \in A, m \in \mathcal{B}(A); J_1m(\alpha) = \sup_{\beta \in B}\left\{p(\alpha,\beta) + T\left(\alpha,\beta,m\left(\zeta\left(x,y\right)\right)\right)\right\}.$$

Then the solution to the system of Eq. (13) with $T_1 = T_2 = T$ is unique in $\mathcal{B}$(A).

Proof: Corollary 6 directly yields the outcome when $T_1 = T_2 = T$.

Conclusion

This study has successfully extended the Suzuki-type contraction framework to a more generalized form, facilitating the identification of common fixed points for three mappings in ƀ-metric spaces. Under Suzuki-type contraction, we demonstrated how our obtained results could be applied to solve specific classes of functional equations in dynamic programming. Our approach included detailed graphical and computational analyses, along with an algorithmic iterative process, which provided a robust verification of our theoretical findings. This work not only generalizes the results of Suzuki, Chandra, et al., and Roshan et al., but also makes these results more accessible and adaptable for a broader audience, enhancing their practical utility in various mathematical and computational fields. The work presented here opened the way for future research and applications in the realm of fixed point theory and dynamic programming.

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Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Qassim University, Saudi Arabia.

Authors' contribution statement

Conceptualization M.D, M J, and S S C ; methodology, M D, M.J , M S, and S S C ; formal analysis, M D, and M.J ; investigation, M J , M.S, and S S C; writing—original draft preparation, M J, and S S C ; writing—review and editing, M D , M J , M S, and S S C ; supervision, S S C , funding, M S. After reading the published version of the manuscript, all writers have given their approval.

Availability of data and materials

MATLAB version 2022b is the software that is used to process and validate the computational data included in the research.

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نتائج النقاط الثابتة المشتركة لثلاث رسومات تخطيطية بموجب الانكماش المعمم من نوع سوزوكي في «ƀ» - المساحات المترية مع التطبيق

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الخلاصة

استخدم سوزوكي استراتيجية رائعة في منشوراته الأساسية لتوسيع مبرهنة تقلص باناخ (BCT). وقد شهد ذلك العديد من التعميمات والتمديدات المثيرة للاهتمام خلال العقود القليلة الماضية وأدى إلى اتجاه جديد من المصادفة والنقطة الثابتة لعدد لا يحصى من الخرائط التعاقدية وغير التوسعية من نوع سوزوكي في مختلف الأماكن. . في هذه المقالة، في إعداد فضاءات قياسية ƀ، الهدف هو إنتاج نقطة ثابتة مشتركة من ثلاثة خرائط معرضة لانكماش عام من نوع سوزوكي. ويعمم العمل الحالي النتائج المعروفة جيدا لسوزوكي وشاندرا وآخرين وروشان وآخرين. وعدة نتائج أخرى متاحة في الأدبيات .يتوافق الرسم التوضيحي التطبيقي الذي تم فيه إجراء التحليل الرسومي والحسابي مع التحقق الاستكشافي للعمل المنتج مما يجعل النتائج أكثر قابلية للتكيف من قبل فئة أوسع من الباحثين. التحليل التكراري المستند إلى الأساليب التكرارية في الرسم التوضيحي مدعوم أيضًا بخوارزمية. علاوة على ذلك، فإن تطبيق العمل الحالي على نظام المعادلات الوظيفية في البرمجة الديناميكية يوضح كيف يمكن استخدام النتائج الحالية. وأخيرا، يقدم مثال لتبرير تطبيق هذا العمل.

الكلمات المفتاحية: خوارزمية، ƀ - فضاء متري، نقطة ثابتة مشتركة، برمجة ديناميكية، تقلص نوع سوزوكي المعمم، طرق تكرارية.