Posterior Estimates for the scale parameter of the Laplace Distribution Using Double Informative Priors Based on GELF with Simulation Study

التقديرات اللاحقة لمعلمة القياس باستعمال معلومات أولية مضاعفة بالاعتماد على GELF مع دراسة محاكاة

Jinan A. Naser Al-obedy ا.م.د. جنان عباس ناصر

Department of Quality Management Techniques, Technical College of Management-Middle Technical university, Baghdad, Iraq.
jinan.abbas@mtu.edu.iq

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Abstract

The Laplace distribution is a continuous probability distribution, also called the double exponential distribution and it is named after the French mathematician Pierre-Simon Laplace. Laplace distribution used in many fields and applications in real life. It is often used to model phenomena with heavy tails or when data has a higher peak than the normal distribution. In present study, we deal with Bayesian analysis of Laplace distribution to derive different estimators for the unknown scale parameter (α) using different double informative priors which are represented by the inverted gamma prior-inverse chi squared prior and the inverted gamma prior-levy prior and inverse chi squared prior-levy prior distributions. We derived the posterior distribution under different double informative priors for the scale parameter (α) of the Laplace distribution. Then Bayes estimators are derived based on the general entropy loss function (GELF) by assuming different values for the shape parameter of GELF, and we compared it with classical Estimators which is obtained by the maximum likelihood estimation (MLE). We used computer programs in Matlab R2018b to obtain the simulation results of this study, we generated different sample sizes of Laplace distribution with several values for the true value of the scale parameter (α) assuming the location parameter (β) be known. Also, we assumed different values for the parameters of different double informative priors. We depend on the mean square error criterion (MSE) to compare the accuracy of the different estimates for α, and the estimates with the smallest MSE's will be the best estimates. We concluded that Bayes estimators for the unknown scale parameter (α) under three different double informative priors based on GELF showed better performance than ML estimator according to the smallest value for MSE criterion, when the location parameter (β) be known and the true value of the scale parameters of the Laplace distribution as $(\alpha = 0.5, \beta_0 = 0)$ and $(\alpha = \beta_0 = 1)$. We recommend to use Bayes estimation under the Inverted gamma -Standard levy distribution as double informative prior to estimate the scale parameter of the Laplace distribution (α) based on different loss functions such as the quadratic loss function (QLF) and Degroot Loss Function (DLF) and the modified loss function (MLF) and the modified linear exponential (MLINEX) loss function to compare the accuracy of the different estimates.

Keywords: The Laplace distribution, Maximum likelihood estimation, Informative Prior, Posterior Distribution, General Entropy Loss Function, Mean Square Error.

المستخلص

يعد توزيع لابلاس من احد التوزيعات المستمرة , سمي بالتوزيع الاسي المضاعف نسبة الى عالم الرياضيات الفرنسي Pierre–Simon Laplace. اذ استعمل في العديد من المجالات والتطبيقات في الواقع العملي. وغالبا ما يستعمل لنمذجة الظواهر التي تمتاز بالذيول الثقيلة (Heavy tails) اوعندما تكون البيانات تمتلك قمة اعلى من القمة في حالة التوزيع الطبيعي. في هذا البحث نتعامل مع أسلوب بيز لتوزيع لابلاس لاشتقاق مقدرات مختلفة لمعلمة القياس المجهولة (α) باستعمال معلومتين مختلفتين كتوزيع اولي التي تمثلت بالتوزيع (معكوس كاما –معكوس مربع كاي) وتوزيع (معكوس كاما – ليفي القياسي) وتوزيع (معكوس مربع كاي – ليفي القياسي). فقد تم اشتقاق التوزيع اللاحق لمعلمة القياس (α) لتوزيع لابلاس باستعمال التوزيعات الأولية المضاعفة. فقد تم اشتقاق التوزيع اللاحق لمعلمة القياس (α)باستعمال التوزيعات الأولية المضاعفة ومن ثم اشتقت مقدرات بيز بالاعتماد على دالة الخسارة للانتروبي العامة ((The general entropy loss function (GELF), بافتراض عدة قيم مختلفة لمعلمة الشكل لدالة GELF , وقورنت مقدرات بيز مع المقدرات الكلاسيكية التي استحصلت بتقدير الإمكان الأعظم (MLE). اذ استخدمت برامج حاسبة بالماتلاب R2018b للحصول على نتائج المحاكاة, ولدنا حجوم من العينات المختلفة من توزيع لابلاس بعدة قيم مختلفة (true values) لمعلمة القياس بافتراض ان معلمة الموقع معلومة. أيضا افترضنا عدة قيم مختلفة لمعلمات التوزيعات الأولية المضاعفة . اعتمدنا على معيار متوسط مربع الخطاء(MSE) لمقارنة دقة التقديرات المختلفة باقل قيمة لمتوسط مربع الخطاء (MSE) سيكون افضل التقديرات. واستنتجنا بان مقدرات بيز لمعلمة القياس المجهولة تحت افتراض التوزيعات الأولية المضاعفة الثلاثة بالاعتماد على دالة الخسارة للانتروبي العامة (GELF) تبين بان ادائهاء افضل من مقدر الإمكان الأعظم (MLE) وفقا لاقل قيمة لمعيار متوسط مربع الخطاء (MSE). عندما تكون معلمة الموقع معلومة والقيمة الحقيقية لمعلمة القياس لتوزيع لابلاس مساوية لـ ($\alpha = 0.5, \beta_0 = 0$)و ($\alpha = \beta_0 = 1$). نوصي باستعمال تقدير بيز بافتراض التوزيعين الأوليين المضاعفين لتوزيع معكوس كاما وتوزيع ليفي القياسي لتقدير معلمة القياس لتوزيع لابلاس (α)باستعمال دوال خسارة أخرى مثل دالة الخسارة التربيعية (QLF) ودالة خسارة Degroot (DLF) و دالة الخسارة المحورة (MLF) ودالة الخسارة الاسية الخطية المحورة (MLINEX) لمقارنة دقة التقديرات المختلفة .

الكلمات المفتاحية: توزيع لابلاس, الامكان الأعظم, المعلومة الأولية, التوزيع اللاحق, دالة الخسارة للانتروبي العامة , متوسط مربعات الخطأ.

1. Introduction

The Laplace distribution is a continuous probability distribution, also called the double exponential distribution and it is named after the French mathematician Pierre-Simon Laplace. Laplace distribution used in many fields and applications in real life such as in engineering, financial to model and analyze data sets and industrial data, environmental and biological data. Many authors derived different modifications of the Laplace distribution in the literature. We review some of these studies, such as Nadarajah in 2010 [1] derived two posterior distributions for the location parameter of the Laplace distribution. He used two random variables distributed independently from Student's t and Laplace which are represented by X and Y. He derived the distributions of the product XY and the ratio X|Y. He used computer programs to generate the distributions and obtain tables for the percentage points for the distributions of the product XY and the ratio X|Y. Also, Ali and Aslam and Abbas and Kazmi in 2012 [2] discussed classical approach and Bayesian analysis of Laplace model assuming the location parameter is zero using complete and censored data. They derived Bayes estimators under different priors which are represented by inverted gamma, and inverted chi-squared, and Levy and Gumbel Type-II. They study the properties of posterior distribution, credible interval, highest posterior density region (HPDR) and Bayes Factor. They derived Bayes estimators under squared error loss function (SELF), precautionary loss function, weighted squared error loss function and modified (quadratic) squared error loss function. They concluded that Gumbel Type II prior results were more precised than other prior, also they concluded that posterior risk has smallest values under the modified squared error loss. In 2015 Liu and Kozubowski [3] used Folded Laplace (FL) distribution and derived its properties, which are represented by probability density function and distribution function and quantile function and hazard rate and moments, Also, they study the properties of the mixture representation and Lorenz curve and mean residual life and entropy. Also they discussed the estimation parameter stochastic model. Also they applied the properties of FL distribution to real data, they used the FL distribution in modeling West Texas Intermediate (WTI) and Brent Oil historical oil prices. They concluded that the underlying phenomena are restrictive to positive values. And Lanping in 2017 [4] used maximum likelihood estimation and Bayes estimation to obtain the estimators of the scale parameter of Laplace distribution. He derived Bayes estimator of the scale parameter of Laplace distribution, under the prior distribution of the parameter as the gamma prior distribution based on a new class of LINEX based symmetric loss function. He used Monte Carlo simulation to investigate the properties of the estimators by the mean square errors, as a measure of good estimation standards. He noted that Bayes estimators of the scale parameter obtained by using LINEX-based loss function is affected by the value of the shape parameter estimation of c, when sample size less than or equals to 50, the value of parameter has a larger influence on the estimation results. Also, Rahman and Roy in 2018 [5] derived different estimators of scale parameter of Laplace double exponential distribution, by using maximum likelihood estimation and Bayes estimation. They estimated Bayes estimators for the scale parameter of Laplace double exponential distribution under the gamma prior distribution for the scale parameter based on the squared error loss function and Quadratic and MLINEX and NLINEX loss functions. They used Monte Carlo simulation method to obtain the estimated value and MSE of the estimator. They concluded that Bayes estimator under quadratic loss is better than other estimators. In 2018 Chaturvedi and Dubey [6] described the Bayesian inference and prediction of the Truncated Skew Laplace distribution. They derived the expressions for posterior distribution and the posterior mean of the unknown parameters. They assumed inverted gamma prior distribution for the scale parameter, and gamma prior distribution for shape parameter of the Truncated Skew Laplace distribution. Also, they obtained the Bayes' estimator of survival function for Truncated Skew Laplace distribution based on the scale and shape parameters under the squared error loss function. Also, Jan and Ahmad in 2018 [7] obtained Bayes estimates of the scale parameter of the classical Laplace distribution, which is also known as first law of Laplace using two approximation techniques, which are represented by normal approximation and Tierney and Kadane (T-K) approximation, under three different informative priors that represented by the inverted gamma prior and Inverse chi square prior and Gumbel type II prior distributions. Also they applied the two approximation techniques on two life datasets and simulated data. They compared the performance of Bayes estimates of the scale parameter of the Laplace using mean and posterior variance and mean square errors (MSE). The simulation results showed that the posterior variance of the scale parameter of the Laplace distribution have smallest values under inverse gamma prior with $\alpha=\beta=2$, based on normal approximation and Tierney and Kadane (T-K) approximation. And Agu and Onwukwe in 2019 [8] used the exponentiation method to obtain the modified Laplace distribution. They proved some of the basic statistical properties of the modified Laplace distribution (MLD). They estimated the parameters of the distributions using of maximum likelihood estimation. And they applied the proposed modified Laplace distribution on two life datasets and simulated data. They compared the modified Laplace distribution with Laplace distribution and Generalized error distribution using Schwartz Criteria (SC) measure of fitness. The results of study show that as the sample size increases, the biasedness and root mean square error (RMSE) of the proposed modified

Laplace distribution reduces. And Hasan in 2019 [9] used classical method by using maximum likelihood estimation and Bayesian method to derive different estimators of scale parameter of Double exponential distribution. He estimated Bayes estimators for the scale parameter of Double exponential distribution under the gamma prior distribution for the scale parameter based on different loss functions, which are represented by squared error and Quadratic and MLINEX and NLINEX loss functions. He used R- Code simulation from the Double exponential distribution to obtain the estimated value, Bias and MSE of different estimators of scale parameter of Double exponential distribution. He concluded that Bayes estimator under Quadratic loss function is better than all other estimators for simulated data. Also, Alwan and Abdullah and Mohammed and Hameed in 2020 [10] used the maximum likelihood method and the moment method and Bayes method to derive an estimator of reliability function for Laplace distribution with two parameters. They derive Bayes estimator of reliability function under Jeffrey's variant prior based on squared error loss function. They used simulation technique to obtain the results of this study under different Laplace distribution parameters and sample sizes. They concluded that Bayes estimator best of the maximum likelihood estimator and moment estimator in all sample sizes according to the smallest value of the MSE. And De Luca and Magnus and Peracchi in 2021 [11] derived the posterior distribution of normal distribution when location parameter is random variable from Laplace distribution and $\sigma^2 = 1$. Also, they obtained the expressions for posterior quantiles and posterior moments of the posterior distribution. They show that the posterior moments of any order is more complex integral, and they explain that his integral can be solved by using a recursive formula which is known by the recursion of Hermite polynomials.

Many authors study the effect of the two different kind of information as double priors in Bayes analysis, such as Radha and Vekatesan in 2013 [12] and Patel and Patel in 2017 [13]. The purpose of this study is to derive closed-form expressions for Bayes estimators for the scale parameter of the Laplace distribution by using Simulation. In addition to the classical estimator which is obtained by the Maximum likelihood estimation. We assume that the scale parameter of the Laplace distribution is a random variable. And we have two different informative priors (double priors) for the scale parameter. We derived the posterior distribution under different double priors which are represented by the Inverted Gamma prior - Inverse Chi squared prior and the Inverted Gamma prior - Levy prior and Inverse Chi squared prior - Levy prior distributions. We derived Bayes estimators based on the general entropy loss function (GELF) by assuming different values for the shape parameter. We depend on the Mean Square Error (MSE) criterion to compare between the estimators for α.

2. The Laplace Distribution

Let us consider $(x_1, x_2, \dots, x_n)$ is a random sample of continuous random variables, we say that $(x_1, x_2, \dots, x_n)$ from the Laplace distribution with two parameters (β, α) having probability density function (pdf) defined by [14, 15];

$$f(x;\beta,\alpha) = \frac{1}{2\alpha}\exp\left(-\frac{|x-\beta|}{\alpha}\right) \quad , \; -\infty < x < \infty \qquad \dots (1)$$

With the location parameter $(-\infty < \beta < \infty)$ and the scale parameter $(\alpha > 0)$ with the cumulative distribution function is given by

$$F(x;\beta,\alpha) = \frac{1}{2}\exp\left(-\frac{x-\beta}{\alpha}\right) \quad , \; \text{for} \quad x < \beta$$

$$F(x;\beta,\alpha) = 1 - \frac{1}{2}\exp\left(-\frac{x-\beta}{\alpha}\right) \quad , \; \text{for} \quad x \ge \beta \qquad \dots (2)$$

And the mean is $\mu = \beta$, and variance is $\text{var}(x) = 2\alpha^2$.

3. Maximum Likelihood Estimation (MLE)

The likelihood function for a random sample $(x_1, x_2, \dots, x_n)$ which is taken from the Laplace distribution with the density $f(x;\beta,\alpha)$ can be defined by (1) [16, 17, 18]:

$$L(\alpha \mid \underline{x}) = \prod_{i=1}^{n} f(x_i;\beta,\alpha) = \prod_{i=1}^{n} \frac{1}{2\alpha} \exp\left(-\frac{|x_i - \beta|}{\alpha}\right) = (2\alpha)^{-n} \exp\left(-\frac{\sum_{i=1}^{n}|x_i - \beta|}{\alpha}\right) \quad \dots (3)$$

By taking the log-likelihood function $\ell = \ln(L)$:

$$\ell(\alpha,\beta \mid \underline{x}) = -n[\log(2) + \log(\alpha)] - \frac{\sum_{i=1}^{n}|x_i - \beta|}{\alpha} \quad \dots (4)$$

Take the derivative of log of the likelihood $\ell(\alpha \mid \underline{x})$ with respect to the scale parameter α and setting to zero

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n}{\alpha} + \frac{\sum_{i=1}^{n}|x_i - \beta|}{\alpha^2} = 0 \;\Rightarrow\; \frac{-n\alpha + \sum_{i=1}^{n}|x_i - \beta|}{\alpha^2} = 0 \quad \dots (5)$$

When the location parameter (β) be known as $(\beta = \beta_0)$,then we obtained the maximum likelihood estimator(MLE) of α

$$\hat{\alpha}_{MLE} = \frac{\sum_{i=1}^{n}|x_i - \beta_0|}{n} \quad \dots (6)$$

4. Bayesian estimation

To obtain Bayesian estimation , we need to define priors distributions, and then we derive posterior distributions under different double informative priors for the scale parameter (α) of the Laplace distribution as follows :

<u>First</u>, we define different double informative priors for the scale parameter (α) of the Laplace distribution which are represented by (the Inverted Gamma prior - Inverse Chi squared prior) and (the Inverted Gamma prior - the Standard Levy prior) and (Inverse Chi squared prior - the Standard Levy prior)distributions by combining two priors as follows :

Let α as informative prior density defined by using <u>Inverted Gamma</u> prior information with the shape parameter $(\lambda > 0)$ and the scale parameter $(\theta > 0)$ which is given by[19, 20]:

$$h_1(\alpha;\lambda,\theta) = \frac{\theta^{\lambda}}{\Gamma(\lambda)} \alpha^{-(\lambda+1)} \exp\left(-\frac{\theta}{\alpha}\right) \quad , \alpha > 0 \quad \dots (7)$$

Where $\Gamma(.)$ denotes the gamma function. Also Suppose that another prior knowledge for α defined by using the <u>Inverse Chi-squared</u> prior with (k) degrees of freedom is positive number and $\Gamma(.)$ denotes the gamma function .which is given by[21, 22] :

$$h_2(\alpha;k) = \frac{\left(\frac{k}{2}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \alpha^{-\left(\frac{k}{2}+1\right)} \exp\left(-\frac{k}{2\alpha}\right) \quad , \alpha > 0 \quad \dots (8)$$

So the double priors of the parameter α will be $g_{12}(\alpha;\lambda,\theta,k) = h_1(\alpha;\lambda,\theta) \times h_2(\alpha;k)$,then we have

$$g_{12}(\alpha;\lambda,\theta,k) = \left[\frac{\theta^{\lambda}}{\Gamma(\lambda)} \frac{\left(\frac{k}{2}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\right] \alpha^{-\left(\lambda+\frac{k}{2}+1+1\right)} \exp\left(-\frac{1}{\alpha}\left(\theta+\frac{k}{2}\right)\right) \quad \alpha,\ \theta,\ \lambda, \text{and } k > 0 \quad \dots (9)$$

Also, let α defined by using Inverted Gamma prior information which is given by equation (7) and another prior knowledge for α defined by using Standard Levy prior with the scale parameter ε, which is given by[23, 24]:

$$h_3(\alpha;\ \varepsilon) = (\frac{\varepsilon}{2\pi})^{\frac{1}{2}}\ \alpha^{-(\frac{3}{2})} \exp(-\frac{\varepsilon}{2\alpha}) \quad , \quad \alpha > 0 \qquad ...\ (10)$$

we define the double priors of the parameter α as $g_{13}(\alpha; \lambda, \theta, \varepsilon) = h_1(\alpha; \lambda, \theta) \times h_3(\alpha; \varepsilon)$, it means we have:

$$g_{13}(\alpha; \lambda, \theta, k) = [\frac{\theta^{\lambda}}{\Gamma(\lambda)} (\frac{\varepsilon}{2\pi})^{\frac{1}{2}}]\ \alpha^{-(\lambda + \frac{3}{2} + 1)} \exp(-\frac{1}{\alpha}(\theta + \frac{\varepsilon}{2})) \quad , \quad \alpha > 0 \qquad ...\ (11)$$

And let α defined by using Inverse Chi squared prior with (k) degrees of freedom is positive number which is given by equation (8) , and another prior knowledge for α defined by using Standard Levy prior with the scale parameter ε which is given by equation (10).Then we define the double priors of the parameter α as $g_{23}(\alpha; k, \varepsilon) = h_2(\alpha; k) \times h_3(\alpha; \varepsilon)$, it means we have:

$$g_{23}(\alpha; k,\ , \varepsilon) = [\frac{(\frac{k}{2})^{\frac{k}{2}}}{\Gamma(\frac{k}{2})} (\frac{\varepsilon}{2\pi})^{\frac{1}{2}}]\ \alpha^{-(\frac{k}{2} + \frac{3}{2} + 1)} \exp(-\frac{1}{\alpha}(\frac{k}{2} + \frac{\varepsilon}{2})) \quad , \quad \alpha > 0 \qquad ...\ (12)$$

Second , we derive the posterior distributions for the unknown scale parameter (α) when the location parameter be known $(\beta = \beta_0)$ for given the data $\underline{x} = (x_1, x_2, ... , x_n)$ using the different three types of double priors will be as follow[17, 18, 19]:

$$w_{ij}(\alpha | \underline{x}) = \frac{L(\alpha \backslash \underline{x})\, g_{ij}(\alpha)}{\int_{\alpha=0}^{\infty} L(\alpha \backslash \underline{x}) g_{ij}(\alpha) d\alpha} \qquad \text{for} \quad i, j = 1,2,3 \qquad ...\ (13)$$

By substituting the equation (3) and for each of the following equations (9,11,12) in the equation (13), we obtain the posterior distributions as shown below:

i –Under the Inverted Gamma - Inverse Chi squared as double priors

By substituting the equation (3) and the equation (9) in the equation (13), we have [16, 17], [18]:

$$w_{12}(\alpha | \underline{x}) = \frac{L(\alpha \backslash \underline{x})\, g_{12}(\alpha)}{\int_{\alpha=0}^{\infty} L(\alpha \backslash \underline{x}) g_{12}(\alpha) d\alpha}$$

$$w_{12}(\alpha | \underline{x}) = \frac{(2\alpha)^{-n} \exp(-\frac{\sum_{i=1}^{n}|x_i - \beta_0|}{\alpha}) [\frac{\theta^{\lambda}}{\Gamma(\lambda)} \frac{(\frac{k}{2})^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}]\ \alpha^{-(\lambda + \frac{k}{2} + 1 + 1)} \exp(-\frac{1}{\alpha}(\theta + \frac{k}{2}))}{\int_{\alpha=0}^{\infty} (2\alpha)^{-n} \exp(-\frac{\sum_{i=1}^{n}|x_i - \beta_0|}{\alpha}) [\frac{\theta^{\lambda}}{\Gamma(\lambda)} \frac{(\frac{k}{2})^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}]\ \alpha^{-(\lambda + \frac{k}{2} + 1 + 1)} \exp(-\frac{1}{\alpha}(\theta + \frac{k}{2}))\, d\alpha}$$

... (14)

$$w_{12}(\alpha|\underline{x}) = \frac{\alpha^{-((n+\lambda+\frac{k}{2}+1)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2}))}{\int_{\alpha=0}^{\infty} \alpha^{-((n+\lambda+\frac{k}{2}+1)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2}))\, d\alpha} \quad \ldots (15)$$

After multiplying the integral in equation (15) by the quantity which equals to $[\frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2})^{(n+\lambda+\frac{k}{2}+1)}}{\Gamma(n+\lambda+\frac{k}{2}+1)} \quad \frac{\Gamma(n+\lambda+\frac{k}{2}+1)}{(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2})^{(n+\lambda+\frac{k}{2}+1)}}]$, Where $\Gamma(.)$ denotes the gamma function. We obtain

$$w_{12}(\alpha|\underline{x}) = \frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2})^{(n+\lambda+\frac{k}{2}+1)}}{\Gamma(n+\lambda+\frac{k}{2}+1)\, A1(\alpha|\underline{x})} \alpha^{-((n+\lambda+\frac{k}{2}+1)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2})) \quad \ldots (16)$$

Where $A1(\alpha|\underline{x}) = \int_{\alpha=0}^{\infty} \frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2})^{(n+\lambda+\frac{k}{2}+1)}}{\Gamma(n+\lambda+\frac{k}{2}+1)} \alpha^{-((n+\lambda+\frac{k}{2}+1)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2}))\, d\alpha = 1$ is the integral of the pdf of the Inverted Gamma distribution. So the posterior distribution for the unknown scale parameter (α) is inverted gamma distribution with the shape parameter $\lambda_{new} = (n+\lambda+\frac{k}{2}+1)$ and the scale parameter $\theta_{new} = (\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2})$ is given by

$$w_{12}(\alpha|\underline{x}) = \frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2})^{(n+\lambda+\frac{k}{2}+1)}}{\Gamma(n+\lambda+\frac{k}{2}+1)} \alpha^{-((n+\lambda+\frac{k}{2}+1)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{k}{2})),\ \alpha > 0 \quad \ldots (17)$$

ii - Under the Inverted Gamma - the Standard Levy as double priors

By substituting the equation (3) and the equation (11) in the equation (13), we have

$$w_{13}(\alpha|\underline{x}) = \frac{(2\alpha)^{-n} \exp(-\frac{\sum_{i=1}^{n}|x_i - \beta_0|}{\alpha})[\frac{\theta^{\lambda}}{\Gamma(\lambda)}(\frac{\varepsilon}{2\pi})^{\frac{1}{2}}]\, \alpha^{-(\lambda+\frac{3}{2}+1)} \exp(-\frac{1}{\alpha}(\theta+\frac{\varepsilon}{2}))}{\int_{\alpha=0}^{\infty}(2\alpha)^{-n} \exp(-\frac{\sum_{i=1}^{n}|x_i - \beta_0|}{\alpha})[\frac{\theta^{\lambda}}{\Gamma(\lambda)}(\frac{\varepsilon}{2\pi})^{\frac{1}{2}}]\, \alpha^{-(\lambda+\frac{3}{2}+1)} \exp(-\frac{1}{\alpha}(\theta+\frac{\varepsilon}{2}))\, d\alpha} \quad \ldots (18)$$

$$w_{13}(\alpha|\underline{x}) = \frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{\varepsilon}{2})^{(n+\lambda+\frac{3}{2})}}{\Gamma(n+\lambda+\frac{3}{2})A2(\alpha|\underline{x})} \alpha^{-((n+\lambda+\frac{3}{2})+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{\varepsilon}{2})) \quad \ldots (19)$$

$$w_{13}(\alpha|\underline{x}) = \frac{\alpha^{-(n+\lambda+\frac{3}{2}+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{\varepsilon}{2}))}{\int_{\alpha=0}^{\infty} \alpha^{-(n+\lambda+\frac{3}{2}+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{\varepsilon}{2}))\, d\alpha} \quad \ldots (20)$$

After multiplying the integral in equation (20) by the quantity which equals to $[\frac{(\sum_{i=1}^{n}|x_i-\beta_0|+\theta+\frac{\varepsilon}{2})^{(n+\lambda+\frac{3}{2})}}{\Gamma(n+\lambda+\frac{3}{2})} \frac{\Gamma(n+\lambda+\frac{3}{2})}{(\sum_{i=1}^{n}|x_i-\beta_0|+\theta+\frac{\varepsilon}{2})^{(n+\lambda+3)}}]$, Where $\Gamma(.)$ denotes the gamma function. We obtain

Where $A2(\alpha\,|\,\underline{x})=\int_{\alpha=0}^{\infty}\frac{(\sum_{i=1}^{n}|x_i-\beta_0|+\theta+\frac{\varepsilon}{2})^{(n+\lambda+\frac{3}{2})}}{\Gamma(n+\lambda+\frac{3}{2})}\alpha^{-((n+\lambda+\frac{3}{2})+1)}\exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i-\beta_0|+\theta+\frac{\varepsilon}{2}))\,d\alpha=1$ is the integral of the pdf of the Inverted Gamma distribution. Then the posterior distribution for the unknown scale parameter (α) is Inverted Gamma distribution with the shape parameter $\lambda_{new}=(n+\lambda+\frac{3}{2})$ and the scale parameter $\theta_{new}=(\sum_{i=1}^{n}|x_i-\beta_0|+\theta+\frac{\varepsilon}{2})$ is given by

$$w_{13}(\alpha\,|\,\underline{x})=\frac{(\sum_{i=1}^{n}|x_i-\beta_0|+\theta+\frac{\varepsilon}{2})^{(n+\lambda+\frac{3}{2})}}{\Gamma(n+\lambda+\frac{3}{2})}\alpha^{-((n+\lambda+\frac{3}{2})+1)}\exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i-\beta_0|+\theta+\frac{\varepsilon}{2})),\ \alpha>0 \quad \dots (21)$$

iii. Using the Inverse chi squared -the standard Levy as double priors

By substituting the equation (3) and the equation (12) in the equation (13), we have

$$w_{23}(\alpha\,|\,\underline{x})=\frac{(2\alpha)^{-n}\exp(-\frac{\sum_{i=1}^{n}|x_i-\beta_0|}{\alpha})[\frac{(\frac{k}{2})^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}(\frac{\varepsilon}{2\pi})^{\frac{1}{2}}]\ \alpha^{-(\frac{k}{2}+\frac{3}{2}+1)}\exp(-\frac{1}{\alpha}(\frac{k}{2}+\frac{\varepsilon}{2}))}{\int_{\alpha=0}^{\infty}(2\alpha)^{-n}\exp(-\frac{\sum_{i=1}^{n}|x_i-\beta_0|}{\alpha})[\frac{(\frac{k}{2})^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}(\frac{\varepsilon}{2\pi})^{\frac{1}{2}}]\ \alpha^{-(\frac{k}{2}+\frac{3}{2}+1)}\exp(-\frac{1}{\alpha}(\frac{k}{2}+\frac{\varepsilon}{2}))\,d\alpha}$$

... (22)

$$w_{23}(\alpha\,|\,\underline{x})=\frac{\alpha^{-((n+\frac{k}{2}+\frac{3}{2})+1)}\exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i-\beta_0|+\frac{k}{2}+\frac{\varepsilon}{2}))}{\int_{\alpha=0}^{\infty}\alpha^{-((n+\frac{k}{2}+\frac{3}{2})+1)}\exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i-\beta_0|+\frac{k}{2}+\frac{\varepsilon}{2}))\,d\alpha} \quad \dots (23)$$

After multiplying the integral in equation (23) by the quantity which equals to $[\frac{(\sum_{i=1}^{n}|x_i-\beta_0|+\frac{k}{2}+\frac{\varepsilon}{2})^{(n+\frac{k}{2}+\frac{3}{2})}}{\Gamma(n+\frac{k}{2}+\frac{3}{2})}\frac{\Gamma(n+\frac{k}{2}+\frac{3}{2})}{(\sum_{i=1}^{n}|x_i-\beta_0|+\frac{k}{2}+\frac{\varepsilon}{2})^{(n+\frac{k}{2}+\frac{3}{2})}}]$,Where $\Gamma(.)$ denotes the gamma function. We obtain

$$w_{23}(\alpha\,|\,\underline{x})=\frac{(\sum_{i=1}^{n}|x_i-\beta_0|+\frac{k}{2}+\frac{\varepsilon}{2})^{(n+\frac{k}{2}+\frac{3}{2})}}{\Gamma(n+\frac{k}{2}+\frac{3}{2})A3(\alpha\,|\,\underline{x})}\alpha^{-((n+\frac{k}{2}+\frac{3}{2})+1)}\exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i-\beta_0|+\frac{k}{2}+\frac{\varepsilon}{2})) \quad \dots (24)$$

Where

$A3(\alpha|\underline{x}) = \int_{\alpha=0}^{\infty} \frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2})^{(n+\frac{k}{2}+\frac{3}{2})}}{\Gamma(n+\frac{k}{2}+\frac{3}{2})} \alpha^{-((n+\frac{k}{2}+\frac{3}{2})+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2}))\, d\alpha = 1$ is the integral of the pdf of the Inverted Gamma distribution. Then the posterior distribution for the unknown scale parameter (α) is Inverted Gamma distribution with the shape parameter $\lambda_{new} = (n + \frac{k}{2} + \frac{3}{2})$ and the scale parameter $\theta_{new} = (\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2})$ is given by

$$w_{23}(\alpha|\underline{x}) = \frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2})^{(n+\frac{k}{2}+\frac{3}{2})}}{\Gamma(n+\frac{k}{2}+\frac{3}{2})} \alpha^{-((n+\frac{k}{2}+\frac{3}{2})+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2})) \quad \ldots (25)$$

4.2 Bayes estimators based on general entropy loss function

The general entropy loss function (GELF) introduced by Calabria and Pulcini [25] (1994), then several authors used the GELF, such as Kumar and Kumar and Singh and Singh [26] by setting the shape parameter of GELF to be (3, -3). Also Kumari and Tripathi and Sinha and Wang [27] used GELF, by setting the shape parameter of GELF to be (1,2). Adegoke and Obisesan and Oladoja and Adegoke [28] used GELF, by setting the shape parameter of general entropy loss function to be (1). Here we used GELF to obtain Bayes estimators, which is defined as follows:

$$R(\hat{\alpha}, \alpha) = (\frac{\hat{\alpha}}{\alpha})^{\gamma} - \gamma \log_e(\frac{\hat{\alpha}}{\alpha}) - 1 \quad , \quad \gamma > 0 \quad \ldots (26)$$

Where γ is the shape parameter of general entropy loss function. Bayes estimator of α based on general entropy loss function is obtained by solving the following equation:

$$\frac{\partial R(\hat{\alpha}, \alpha)}{\partial \hat{\alpha}} = \frac{\partial}{\partial \hat{\alpha}} \int_{\alpha=0}^{\infty} ((\frac{\hat{\alpha}}{\alpha})^{\gamma} - \gamma \log_e(\frac{\hat{\alpha}}{\alpha}) - 1)\, w(\alpha \backslash x)\, d\alpha = 0 \quad \ldots (27)$$

$$\frac{\partial R(\hat{\alpha}, \alpha)}{\partial \hat{\alpha}} = \gamma\, (\hat{\alpha})^{\gamma-1} E(\alpha^{-\gamma} \backslash x) - \frac{\gamma}{\hat{\alpha}} = 0 \quad \ldots (28)$$

we have $\gamma\, (\hat{\alpha})^{\gamma-1} E(\alpha^{-\gamma} \backslash x) = \gamma\, (\hat{\alpha})^{-1}$ ... (29)

Bayes estimator based on general entropy loss function (GELF) of α denoted by $\hat{\alpha}$ under different double priors as follows

$$\hat{\alpha}_{GELF} = [E(\alpha^{-\gamma} | x)]^{-\frac{1}{\gamma}} \Rightarrow \hat{\alpha}_{GELF} = [\int_{\alpha=0}^{\infty} \alpha^{-\gamma} w(\alpha | x)\, d\alpha\ ]^{-\frac{1}{\gamma}} \quad \ldots (30)$$

We obtain Bayes estimator of α based on entropy loss function if $\gamma = 1$. And Bayes estimator of α based on weighted square error loss function [28, 30] if $\gamma = 1$. Also we obtain Bayes estimator of α based on square error loss function [28] if $\gamma = -1$.

Bayes estimator for the unknown scale parameter (α) of the Laplace distribution based on general entropy loss function (GELF) can be derived as follows.

i. Bayes Estimator Using the Inverted Gamma - Inverse Chi squared as double priors

By substituting equation (17) in equation (30), yields:

$$\hat{\alpha}_{GELF(1)} = [\int_{\alpha=0}^{\infty} \alpha^{-\gamma} w_{12}(\ \alpha \mid x)\, d\alpha\ ]^{-\frac{1}{\gamma}} \qquad \text{... (30)}$$

$$\hat{\alpha}_{GELF(1)} = [\int_{\alpha=0}^{\infty} \alpha^{-\gamma} \frac{(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{k}{2})^{(n+\lambda+\frac{k}{2}+1)}}{\Gamma(n+\lambda+\frac{k}{2}+1)} \alpha^{-((n+\lambda+\frac{k}{2}+1)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{k}{2}))\ d\alpha\,]^{-\frac{1}{\gamma}} \quad \text{... (31)}$$

$$\hat{\alpha}_{GELF(1)} = [\int_{\alpha=0}^{\infty} \frac{(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{k}{2})^{(n+\lambda+\frac{k}{2}+1)+\gamma-\gamma}}{\Gamma(n+\lambda+\frac{k}{2}+1)} \alpha^{-((n+\lambda+\frac{k}{2}+\gamma+1)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{k}{2}))\ d\alpha\ ]^{-\frac{1}{\gamma}} \quad \text{... (32)}$$

By multiplying the integral in equation (32) by the quantity which equals to $(\frac{\Gamma(n+\lambda+\frac{k}{2}+\gamma+1)}{\Gamma(n+\lambda+\frac{k}{2}+\gamma+1)})$

,where $\Gamma(.)$ is a Gamma function. After some simplification we have:

$$\hat{\alpha}_{GELF(1)} = [\frac{\Gamma(n+\lambda+\frac{k}{2}+\gamma+1)}{\Gamma(n+\lambda+\frac{k}{2}+1)(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{k}{2})^{\gamma}} B1(x,\alpha)]^{-\frac{1}{\gamma}} \qquad \text{... (33)}$$

Where $B1(x,\alpha) = [\int_{\alpha=0}^{\infty} \frac{(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{k}{2})^{(n+\lambda+\frac{k}{2}+\gamma+1)}}{\Gamma(n+\lambda+\frac{k}{2}+\gamma+1)} \alpha^{-((n+\lambda+\frac{k}{2}+\gamma+1)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{k}{2}))\ d\alpha = 1$ is the integral of the pdf of inverse Gamma distribution, then the Bayes estimator for (α) will be:

$$\hat{\alpha}_{GELF(1)} = [\frac{\Gamma(n+\lambda+\frac{k}{2}+\gamma+1)}{\Gamma(n+\lambda+\frac{k}{2}+1)(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{k}{2})^{\gamma}}]^{-\frac{1}{\gamma}} \qquad \text{... (34)}$$

ii. Bayes Estimator Using the Inverted Gamma - the Standard Levy as double priors

By substituting equation (21) in equation (30), yields:

$$\hat{\alpha}_{GELF(2)} = [\int_{\alpha=0}^{\infty} \alpha^{-\gamma} w_{13}(\ \alpha \mid x)\, d\alpha\ ]^{-\frac{1}{\gamma}} \qquad \text{... (30)}$$

$$\hat{\alpha}_{GELF(2)} = [\int_{\alpha=0}^{\infty} \alpha^{-\gamma} \frac{(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{\varepsilon}{2})^{(n+\lambda+\frac{3}{2})+\gamma-\gamma}}{\Gamma(n+\lambda+\frac{3}{2})} \alpha^{-((n+\lambda+\frac{3}{2})+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{\varepsilon}{2}))\ d\alpha\ ]^{-\frac{1}{\gamma}} \quad \text{... (35)}$$

$$\hat{\alpha}_{GELF(2)} = [\int_{\varphi=0}^{\infty} \frac{(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{\varepsilon}{2})^{(n+\lambda+\frac{3}{2})+\gamma-\gamma}}{\Gamma(n+\lambda+\frac{3}{2})} \alpha^{-((n+\lambda+\frac{3}{2}+\gamma)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}\left|x_i - \beta_0\right| + \theta + \frac{\varepsilon}{2}))\ d\alpha\ ]^{-\frac{1}{\gamma}} \quad \text{... (36)}$$

By multiplying the integral in equation (36) by the quantity which equals to $\left(\frac{\Gamma(n+\lambda+\frac{3}{2}+\gamma)}{\Gamma(n+\lambda+\frac{3}{2}+\gamma)}\right)$,where $\Gamma(.)$ is a Gamma function. After some simplification, it yields:

$$\hat{\alpha}_{GELF(2)} = \left[\frac{\Gamma(n+\lambda+\frac{3}{2}+\gamma)}{\Gamma(n+\lambda+\frac{3}{2})(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{\varepsilon}{2})^{\gamma}} B2(x,\alpha)\right]^{-\frac{1}{\gamma}} \quad ... (37)$$

Where $B2(x,\alpha) = \int_{\alpha=0}^{\infty} \frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{\varepsilon}{2})^{(n+\lambda+\frac{3}{2}+\gamma)}}{\Gamma(n+\lambda+\frac{3}{2}+\gamma)} \alpha^{-((n+\lambda+\frac{3}{2}+\gamma)+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{\varepsilon}{2}))d\alpha = 1$

is the integral of the pdf of inverse Gamma distribution, then the Bayes estimator for (α) will be:

$$\hat{\alpha}_{GELF(2)} = \left[\frac{\Gamma(n+\lambda+\frac{3}{2}+\gamma)}{\Gamma(n+\lambda+\frac{3}{2})(\sum_{i=1}^{n}|x_i - \beta_0| + \theta + \frac{\varepsilon}{2})^{\gamma}}\right]^{-\frac{1}{\gamma}} \quad ... (38)$$

iii. Bayes Estimator using the Inverse chi squared - the standard Levy as double priors

By substituting equation (25) in equation (30), yields:

$$\hat{\alpha}_{GELF(3)} = \left[\int_{\alpha=0}^{\infty} \alpha^{-\gamma} w_{23}(\alpha \mid x)\, d\alpha\right]^{-\frac{1}{\gamma}} \quad ...(30)$$

$$\hat{\alpha}_{GELF(3)} = \left[\int_{\alpha=0}^{\infty} \alpha^{-\gamma} \frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2})^{(n+\frac{k}{2}+\frac{3}{2})}}{\Gamma(n+\frac{k}{2}+\frac{3}{2})} \alpha^{-((n+\frac{k}{2}+\frac{3}{2})+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2}))\, d\alpha\right]^{-\frac{1}{\gamma}}$$

... (39)

$$\hat{\alpha}_{GELF(3)} = \left[\int_{\alpha=0}^{\infty} \frac{(\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2})^{(n+\frac{k}{2}+\frac{3}{2})+\gamma-\gamma}}{\Gamma(n+\frac{k}{2}+\frac{3}{2})} \alpha^{-((n+\frac{k}{2}+\gamma+\frac{3}{2})+1)} \exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2}))\, d\alpha\right]^{-\frac{1}{\gamma}}$$

... (40)

By multiplying the integral in equation (40) by the quantity which equals to $\left(\frac{\Gamma(n+\frac{k}{2}+\gamma+\frac{3}{2})}{\Gamma(n+\frac{k}{2}+\gamma+\frac{3}{2})}\right)$,where $\Gamma(.)$ is a Gamma function. After some simplification, it yields:

$$\hat{\alpha}_{GELF(3)} = \left[\frac{\Gamma(n+\frac{k}{2}+\gamma+\frac{3}{2})}{\Gamma(n+\frac{k}{2}+\frac{3}{2})(\sum_{i=1}^{n}|x_i - \beta_0| + \frac{k}{2} + \frac{\varepsilon}{2})^{\gamma}} B3(x,\alpha)\right]^{-\frac{1}{\gamma}} \quad ... (41)$$

Where

$$B2(x,\alpha)=\int_{\alpha=0}^{\infty}\frac{(\sum_{i=1}^{n}|x_i-\beta_0|+\frac{k}{2}+\frac{\varepsilon}{2})^{(n+\frac{k}{2}+\gamma+\frac{3}{2})}}{\Gamma(n+\frac{k}{2}+\gamma+\frac{3}{2})}\alpha^{-((n+\frac{k}{2}+\gamma+\frac{3}{2})+1)}\exp(-\frac{1}{\alpha}(\sum_{i=1}^{n}|x_i-\beta_0|+\frac{k}{2}+\frac{\varepsilon}{2}))\ d\alpha=1$$ is the integral of the pdf of inverse Gamma distribution ,then the Bayes estimator for (α) will be:

$$\hat{\alpha}_{GELF(3)}=[\frac{\Gamma(n+\frac{k}{2}+\gamma+\frac{3}{2})}{\Gamma(n+\frac{k}{2}+\frac{3}{2})(\sum_{i=1}^{n}|x_i-\beta_0|+\frac{k}{2}+\frac{\varepsilon}{2})^{\gamma}}]^{-\frac{1}{\gamma}} \quad \text{... (42)}$$

5. Simulation and Discussion

In this section , we discussed a detailed simulation study which is used in order to compare between the maximum likelihood estimator (MLE) and the Bayes estimators for unknown scale parameter (α) of the Laplace distribution by using matlabR2018b with replications number of the experiments (r=10000).

The data is generated for different samples sizes n= (25,50,100,150) from the Laplace distribution using the quantile function from equation (2) as follows:

- We generated U_i from the uniform distribution with (0,1) for all sample sizes with r=10000.
- Then we used the quantile function from equation (2) to generate x_i as

$-x_i=\beta-\alpha\ln(2(1-F_i))$ if $F_i\geq 0.5$ where $F_i=U_i$ or

$-x_i=\beta+\alpha\ln(2F_i)$ if $F_i<0.5$ where $F_i=U_i$

for several values of the true value of the scale parameter (α) when the location parameter (β) be known as $(\alpha=0.5,\beta_0=0)$, $(\alpha=\beta_0=1)$ and $(\alpha=\beta_0=2)$ of the Laplace distribution, with the values for the hyper parameters of the double prior distributions can be chosen arbitrarily to compare the accuracy of the different estimates for α as follows :

- The values for the parameters of the inverted gamma -inverse chi -squared priors have been selected arbitrarily to be $(\theta=1.2,\lambda=3,k=1)$ and $(\theta=2,\lambda=4,k=2)$.
- The value for a parameters of the inverted gamma - standard Levy priors have been selected arbitrarily to be $(\theta=1.2,\lambda=3,\varepsilon=0.8)$ and $(\theta=2,\lambda=4,\varepsilon=1)$.
- The value for a parameters of the inverse chi -squared - standard Levy prior have been selected arbitrarily to be $(k=1,\varepsilon=0.8)$ and $(k=2,\varepsilon=1)$.

Based on the general entropy loss function (GELF) with the shape parameter can be chosen arbitrarily as $(\gamma=-1,1,2)$.

In order to compare the accuracy of the different estimates for α , we depend on the mean square error criterion, i.e. the estimates with the smallest MSE's will be the best estimates.

$$MSE(\alpha)=\frac{1}{10000}\sum_{r=1}^{10000}(\hat{\alpha}(r)-\alpha)^2 \quad (42)$$

The results of simulation study listed in Table.1 to Table.4 for each estimator and for all sample sizes.

Table .1 The MSE of the estimated values for α of the Laplace distribution by using MLE when β be known with values $\alpha = 0.5, \beta_0 = 0$, $\alpha = \beta_0 = 1$ and $\alpha = \beta_0 = 2$.

n	Criteria	$\alpha = 0.5, \beta_0 = 0$	$\alpha = \beta_0 = 1$	$\alpha = \beta_0 = 2$
25	$\hat{\alpha}$	0.4982	0.9963	1.9926
	MSE	**0.0097**	**0.039**	**0.156**
50	$\hat{\alpha}$	0.5004	1.0008	2.0017
	MSE	**0.005**	**0.02**	**0.0798**
100	$\hat{\alpha}$	0.5002	1.0004	2.0007
	MSE	**0.0025**	**0.0099**	**0.0397**
150	$\hat{\alpha}$	0.4998	0.9995	1.999
	MSE	**0.0017**	**0.0067**	**0.0268**

* The MLE gives the smallest MSE's for all n with $(\alpha = 0.5, \beta_0 = 0)$.

From the results listed in table.1, we concluded that the MSE is

- Increased with increasing the assumed value of scale parameter (α) .
- Decreasing with increasing the sample size (n).
- The smallest values for all sample sizes (n) when the true value of the scale parameter (α) when the location parameter (β) be known as $(\alpha = 0.5, \beta_0 = 0)$

Table .2 The MSE of the estimated values for α of the Laplace distribution when β be known with $\alpha = 0.5, \beta_0 = 0$ by using Bayes estimation under different double priors based on GELF with $(\gamma = -1,1,2)$.

N	γ		inverted gamma -inverse chi -squared		inverted gamma -standard levy		inverse chi -squared - standard levy	
			$\theta = 1.2, \lambda = 3$, $k = 1$	$\theta = 2, \lambda = 4$, $k = 2$	$\theta = 1.2, \lambda = 3$, $\varepsilon = 0.8$	$\theta = 2, \lambda = 4$, $\varepsilon = 1$	$k = 1$, $\varepsilon = 0.8$	$k = 2$, $\varepsilon = 1$
25	-1	$\hat{\alpha}$	0.4966	0.5151	0.4931	0.5069	0.5136	0.5266
		MSE	**0.0075**	**0.007**	**0.0075**	**0.007**	**0.0092**	**0.0094**
	1	$\hat{\alpha}$	0.4798	0.4985	0.4764	0.4903	0.4946	0.5074
		MSE	**0.0074**	**0.0063**	**0.0076**	**0.0066**	**0.0084**	**0.0081**
	2	$\hat{\alpha}$	0.4719	0.4907	0.4685	0.4824	0.4857	0.4984
		MSE	**0.0076**	**0.0062**	**0.0078**	**0.0066**	**0.0083**	**0.0078**
50	-1	$\hat{\alpha}$	0.4995	0.505	0.4976	0.5141	0.5083	0.52
		MSE	**0.0044**	**0.0042**	**0.0044**	**0.0044**	**0.0049**	**0.0052****
	1	$\hat{\alpha}$	0.4903	0.4959	0.4885	0.5049	0.4985	0.51
		MSE	**0.0043**	**0.0041**	**0.0043**	**0.0041**	**0.0046**	**0.0047**
	2	$\hat{\alpha}$	0.4859	0.4915	0.484	0.5004	0.4938	0.5052
		MSE	**0.0043**	**0.004**	**0.0044**	**0.004**	**0.0046**	**0.0046**
100	-1	$\hat{\alpha}$	0.4997	0.5026	0.4987	0.5074	0.5041	0.5101
		MSE	**0.0023**	**0.0023**	**0.0023**	**0.0023**	**0.0024**	**0.0025**
	1	$\hat{\alpha}$	0.4949	0.4978	0.494	0.5025	0.4992	0.5051
		MSE	**0.0023**	**0.0022**	**0.0023**	**0.0022**	**0.0024**	**0.0024**
	2	$\hat{\alpha}$	0.4926	0.4955	0.4916	0.5002	0.4968	0.5026
		MSE	**0.0023**	**0.0022**	**0.0023**	**0.0022**	**0.0024**	**0.0024**
150	-1	$\hat{\alpha}$	0.4994	0.5014	0.4988	0.5046	0.5024	0.5064
		MSE	**0.0016**	**0.0016**	**0.0016**	**0.0016**	**0.0017**	**0.0017**
	1	$\hat{\alpha}$	0.4962	0.4982	0.4956	0.5014	0.4991	0.5031
		MSE	**0.0016**	**0.0016**	**0.0016**	**0.0016**	**0.0016**	**0.0016**
	2	$\hat{\alpha}$	0.4946	0.4966	0.494	0.4998	0.4975	0.5014
		MSE	**0.0016**	**0.0016**	**0.0016**	**0.0016**	**0.0016**	**0.0016**

Note.1: The shadow cells represent the smallest value of MSE.

Note.2: (**) i.e. the value of MSE is greater than the MSE of MLE.

Table .3 The MSE of the estimated values for α of the Laplace distribution when β be known with $\alpha = \beta_0 = 1$ by using Bayes estimation under different double priors based on GELF with (γ = -1,1,2) .

N	γ		inverted gamma - inverse chi -squared		inverted gamma - standard levy		inverse chi -squared - standard levy	
			$\theta = 1.2, \lambda = 3$, $k = 1$	$\theta = 2, \lambda = 4$, $k = 2$	$\theta = 1.2, \lambda = 3$, $\varepsilon = 0.8$	$\theta = 2, \lambda = 4$, $\varepsilon = 1$	$k = 1$, $\varepsilon = 0.8$	$k = 2$, $\varepsilon = 1$
25	-1	$\hat{\alpha}$	0.9336	0.9291	0.9301	0.946	0.9926	1.0157
		MSE	**0.0344**	**0.033**	**0.0349**	**0.0309**	**0.0361**	**0.0363**
	1	$\hat{\alpha}$	0.902	0.8986	0.8986	0.915	0.9558	0.9781
		MSE	**0.0376**	**0.0365**	**0.0383**	**0.0334**	**0.0354**	**0.0339**
	2	$\hat{\alpha}$	0.887	0.8842	0.8837	0.9004	0.9386	0.9604
		MSE	**0.0398****	**0.0388**	**0.0406****	**0.0353**	**0.036**	**0.0338**
50	-1	$\hat{\alpha}$	0.9671	0.9641	0.9653	0.9733	0.9989	1.0106
		MSE	**0.0185**	**0.0181**	**0.0186**	**0.0175**	**0.0192**	**0.0193**
	1	$\hat{\alpha}$	0.9494	0.9467	0.9476	0.9557	0.9797	0.9912
		MSE	**0.0194**	**0.019**	**0.0195**	**0.0182**	**0.0189**	**0.0185**
	2	$\hat{\alpha}$	0.9408	0.9383	0.939	0.9472	0.9704	0.9818
		MSE	**0.02**	**0.0197**	**0.0202****	**0.0187**	**0.019**	**0.0184**
100	-1	$\hat{\alpha}$	0.983	0.9812	0.982	0.986	0.9994	1.0053
		MSE	**0.0096**	**0.0094**	**0.0096**	**0.0093**	**0.0097**	**0.0098**
	1	$\hat{\alpha}$	0.9736	0.9719	0.9726	0.9767	0.9896	0.9955
		MSE	**0.0098**	**0.0097**	**0.0098**	**0.0095**	**0.0096**	**0.0096**
	2	$\hat{\alpha}$	0.9689	0.9673	0.968	0.9721	0.9848	0.9906
		MSE	**0.01**	**0.0099**	**0.01**	**0.0096**	**0.0097**	**0.0095**
150	-1	$\hat{\alpha}$	0.9878	0.9866	0.9872	0.9898	0.9989	1.0028
		MSE	**0.0066**	**0.0065**	**0.0066**	**0.0064**	**0.0066**	**0.0066**
	1	$\hat{\alpha}$	0.9814	0.9802	0.9808	0.9835	0.9923	0.9962
		MSE	**0.0067**	**0.0066**	**0.0067**	**0.0065**	**0.0066**	**0.0065**
	2	$\hat{\alpha}$	0.9783	0.9771	0.9776	0.9803	0.9890	0.9930
		MSE	**0.0068****	**0.0067**	**0.0068****	**0.0066**	**0.0066**	**0.0065**

Note.1: The shadow cells represent the smallest value of MSE.

Note.2: (**) i.e. the value of MSE is greater than the MSE of MLE.

Table .4 The MSE of the estimated values for α of the Laplace distribution when β be known with $\alpha = \beta_0 = 2$ by using Bayes estimation under different double priors based on GELF with (γ = -1,1,2) .

N	γ		inverted gamma -inverse chi -squared		inverted gamma -standard levy		inverse chi -squared - standard levy	
			$\theta = 1.2, \lambda = 3$, $k = 1$	$\theta = 2, \lambda = 4$, $k = 2$	$\theta = 1.2, \lambda = 3$, $\varepsilon = 0.8$	$\theta = 2, \lambda = 4$, $\varepsilon = 1$	$k = 1$, $\varepsilon = 0.8$	$k = 2$, $\varepsilon = 1$
25	-1	$\hat{\alpha}$	1.8075	1.7734	1.804	1.7903	1.9506	1.9737
		MSE	**0.157**	**0.1633****	**0.1584****	**0.1559****	**0.1466**	**0.1449**
	1	$\hat{\alpha}$	1.7463	1.7152	1.7429	1.7316	1.8783	1.9006
		MSE	**0.1764****	**0.1859****	**0.1781****	**0.1768****	**0.1485**	**0.1436**
	2	$\hat{\alpha}$	1.7174	1.6878	1.7141	1.7039	1.8445	1.8663
		MSE	**0.1882****	**0.1989****	**0.1901****	**0.1891****	**0.1531**	**0.1468**
50	-1	$\hat{\alpha}$	1.9025	1.8823	1.9006	1.8915	1.9801	1.9919
		MSE	**0.0792**	**0.081****	**0.0796**	**0.079**	**0.0771**	**0.0768**
	1	$\hat{\alpha}$	1.8676	1.8484	1.8658	1.8574	1.942	1.9535
		MSE	**0.0847****	**0.0878****	**0.0852****	**0.0851****	**0.0771**	**0.0759**
	2	$\hat{\alpha}$	1.8507	1.8319	1.8489	1.8409	1.9236	1.935
		MSE	**0.0883****	**0.0919****	**0.0888****	**0.0889****	**0.0782**	**0.0766**
100	-1	$\hat{\alpha}$	1.9495	1.9385	1.9485	1.9433	1.9898	1.9958
		MSE	**0.0396**	**0.0401****	**0.0397**	**0.0396**	**0.039**	**0.0389**
	1	$\hat{\alpha}$	1.9309	1.9201	1.9299	1.9249	1.9703	1.9762
		MSE	**0.0411****	**0.042****	**0.0413****	**0.0413****	**0.039**	**0.0387**
	2	$\hat{\alpha}$	1.9217	1.9111	1.9207	1.9158	1.9607	1.9666
		MSE	**0.0421****	**0.0432****	**0.0423****	**0.0424****	**0.0393**	**0.0389**
150	-1	$\hat{\alpha}$	1.9645	1.957	1.9639	1.9602	1.9918	1.9957
		MSE	**0.0269****	**0.0271****	**0.0269****	**0.0269****	**0.0265**	**0.0265**
	1	$\hat{\alpha}$	1.9518	1.9444	1.9512	1.9476	1.9787	1.9826
		MSE	**0.0276****	**0.0281****	**0.0277****	**0.0277****	**0.0266**	**0.0264**
	2	$\hat{\alpha}$	1.9455	1.9382	1.9449	1.9414	1.9722	1.9761
		MSE	**0.0281****	**0.0286****	**0.0282****	**0.0282****	**0.0267**	**0.0265**

Note.1: The shadow cells represent the smallest value of MSE.

Note.2: (**) i.e. the value of MSE is greater than the MSE of MLE.

For the results of the Bayes estimators which are listed in table.2 to table.4, that are included the estimated value of (α) and their MSE of the Laplace distribution when (β) be known under different double priors based on GELF with the shape parameter $(\gamma = -1, 1, 2)$. We obtain the best estimates according to the smallest value of MSE as shown bellows:

<u>For the results listed in table.2</u> , when the location parameter (β) be known and the true value of the scale parameters as $(\alpha = 0.5, \beta_0 = 0)$, we obtain

- The smallest value of MSE the estimated value of (α) under the double priors is inverted gamma - inverse chi – squared with $(\theta = 2, \lambda = 4, k = 2)$, for all sample sizes (n) and for all the assumed values of the shape parameter $(\gamma = -1, 1, 2)$ of the general entropy loss function (GELF).
- The same value of MSE the estimated value of (α) under the double priors are Inverted Gamma - Inverse chi – squared with $(\theta = 2, \lambda = 4, k = 2)$ and inverted gamma - standard levy with $(\theta = 2, \lambda = 4, \varepsilon = 1)$
    - For the sample size n=25 , when the shape parameter is $(\gamma = -1)$ of GELF.
    - For the sample size n=50 , when the shape parameter is $(\gamma = 1,2)$ of GELF.
- The same value of MSE the estimated value of (α) when the double priors are inverted gamma - inverse chi –squared with $(\theta = 1.2, \lambda = 3, k = 1)$ and $(\theta = 2, \lambda = 4, k = 2)$, and inverted gamma - standard levy with $(\theta = 1.2, \lambda = 3, \varepsilon = 0.8)$ and $(\theta = 2, \lambda = 4, \varepsilon = 1)$ for the sample sizes n=100 , when the shape parameter is $(\gamma = -1)$ of GELF.
- The same value of MSE under all different values of the hyper parameters of the double prior distributions for the sample sizes n=150 when the shape parameter is $(\gamma = 1,2)$ of GELF.

For the results listed in table.3, when the location parameter (β) be known and the true value of the scale parameters as $(\alpha = \beta_0 = 1)$,we obtain the smallest value of MSE the estimated value of (α) when the double priors is

- Inverted gamma - standard levy with $(\theta = 2, \lambda = 4, \varepsilon = 1)$,for all sample sizes (n) when the shape parameter is $(\gamma = -1,1)$ of GELF.
- Inverse chi -squared - standard levy prior with $(k = 2, \varepsilon = 1)$, for all sample sizes (n) and the shape parameter is $(\gamma = 2)$ of GELF.
- Inverse chi -squared - standard levy prior with $(k = 2, \varepsilon = 1)$, for the sample size n=150 and the shape parameter is $(\gamma = 1)$ of GELF.

For the results listed in table.4, when the location parameter (β) be known and the true value of the scale parameters as $(\alpha = \beta_0 = 2)$, we obtain the smallest value of MSE the estimated value of (α) when the double priors is inverse chi -squared - standard levy prior with $(k = 2, \varepsilon = 1)$, for all sample sizes(n) when the shape parameter is $(\gamma = -1,1,2)$ of GELF.

6. Conclusion

In this study, we have proposed to use three different double informative priors which are represented by the inverted gamma prior-inverse chi squared prior and the inverted gamma prior-levy prior and inverse chi squared prior-levy prior distributions to derive Bayes estimators of the scale parameter (α) of the Laplace distribution when (β) be known under different double priors based on GELF with the shape parameter $(\gamma = -1, 1, 2)$. In addition to the maximum likelihood estimator (MLE) of the scale parameter of the Laplace distribution.

In general from simulation the results in tables.2 and table.3, we noted that Bayes estimators under three different double informative priors based on general entropy loss function (GELF) showed better performance than the maximum likelihood estimator (MLE) using the mean square error criterion (MSE), when the location parameter (β) be known and the true value of the scale parameters of the Laplace distribution as $(\alpha = 0.5, \beta_0 = 0)$ and $(\alpha = \beta_0 = 1)$.

Also the results in table.4 showed that the posterior distribution obtained under inverse chi -squared - standard levy prior with $(k = 2, \varepsilon = 1)$ and $(k = 1, \varepsilon = 0.8)$, when the shape parameter is $(\gamma = -1, 1, 2)$ of GELF gave more accurate results in terms of minimum MSE for all samples sizes(n). when the location parameter (β) be known and the true value of the scale parameters of the Laplace distribution as $(\alpha = \beta_0 = 2)$.

7. Recommendation

From the simulation results of Bayes estimators for the scale parameters of the Laplace distribution (α), We recommend to use Bayes estimation under the Inverted gamma -Standard levy distribution as double informative prior to estimate the scale parameter of the Laplace distribution (α) based on different loss functions such as quadratic loss function (QLF) and Degroot Loss Function (DLF) and modified loss function (MLF) and the modified linear exponential (MLINEX) loss function to compare the accuracy of the different estimates.

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