A generalization of P-extending modules via Goldie extending property

L. Dr. Enas Mustafa Kamil[1] and A. L. Huda Emad AL-Deen Jameel[1]
inas.m@csw.uobaghdad.edu.iq - huda.e@csw.uobaghdad.edu.iq
Dept. of Math., College of Science for Women, University of Baghdad, city Baghdad, Iraq

Abstract.

We introduce a class of modules in this article that is comparable to that of G^z-extending and G^r-extending modules. We specify what a module M is as G^p-extending if and only if for every cyclic submodule A of M, there exists a direct summand D of M such that $A\cap D$ is essential in A as well as D. We look into G^p-extending modules find this inference in relation to the other extending properties. We provide a few descriptions of G^p-extending condition. We demonstrate the direct sum of G^p-extending should not beG^p-extending and deal with decompositions for G^p-extending concept.

Keywords: Cyclic submosules, Goldie extending modules, P-extending modules.

تعميم لمقاسات التوسع من نمط - P من خلال خاصية التوسع من النمط ـ غولدي

م.د. ايناس مصطفى كامل[1] م.م. هدى عماد الدين جميل[1]
inas.m@csw.uobaghdad.edu.iq - huda.e@csw.uobaghdad.edu.iq
قسم الرياضيات ، كلية العلوم للبنات، جامعة بغداد، العراق

الملخص:

سنقدم صنف من المقاسات في هذا البحث والذي يكون مماثل الى مقاس التوسع من النمط G^z ومقاس التوسع من النمط G^r. نسمي المقاس M مقاس توسع من النمط G^p اذا وفقط اذا كان كل مقاس جزئي دائري A من M يكون $A \cap D$ جوهري في كل من A و D. وايضا سنحدد العلاقة بين مقاس التوسع من النمط G^p وبعض تعميمات مقاسات التوسع. سنشرح بعض من الخواص الاساسية لمقاسات التوسع من النمط- G^p. ان الجداء المباشر لمقاسات التوسع من النمط- G^p ليس من الضروري ان يكون مقاس توسع من النمط- G^p ايضا سنفسر بعض المبرهنات حول الجداء المباشر لمقاسات التوسع من النمط - G^p.

الكلمات المفتاحية: المقاسات الجزئية الدائرية، مقاسات التوسع من النمط – غولدي، مقاسات التوسع من النمط - p.

1. Introduction.

Throughout this paper, all rings are associative with unitary, R denotes such a ring, and all modules are unital right R- modules. In the spirit of [1] , for a module M, think of the following relations on the set of submodules of M:

(i)$A\alpha B$ if and only if there exists a submodule C of M such that $A \leq_e C$ and $B \leq_e C$.

(ii)$A\beta B$ if and only if $A \cap B \leq_e A$ and $A \cap B \leq_e B$. Recall that β is an equivalence relation. It's clear that a module M is extending (CS) iff for all submodule A of M, there is a direct summand D of M such that $A \alpha D$, (see [1,2]). Further a module M is called Goldie extending module (G-extending) iff for all submodule A of M , there is a direct summand D of M such that $A \beta D$ or equivalently, for each closed submodule A in M, there is a direct summand D of M such that $A \beta D$ (see [1]). Obviously, every extending module is G-extending.

A generalization of CS-modules is p-extending (see [3]). Recall that a module M is called p-extending if every cyclic submodule of M is essential in a direct summand of M.

In this paper, we study a module condition including the β relation on the set of all cyclic submodules of a module. We call a module M is G^p-extending if for every cyclic submodule A of M, there is a direct summand D of M such that $A\beta D$. A ring R is G^p-extending if R_R is G^p-extending module. It is clear that the class of G^p-extending modules property contains the class of G-extending modules. The notion of G^p-extending generalizes each of G-extending, extending and p-extending modules.

In part 2, we look at relationships between G^p-extending property, p-extending and G-extending conditions. In addition, we provide adequate circumstances for p-extending and G^p-extending modules are equivalent.

The characterizations of G^p-extending modules are covered in Section 3. Due to the fact that the total of G^p-extending modules should not be G^p-extending, we concentrate when a direct sum of G^p-extending modules is again G^p-extending. Also, sufficient conditions under which the direct summand of G^p-extending is also G^p-extending are introduced. These are introduced in section 4.

Following [4], if each submodules of M have a unique closure in M then M is called UC -module.

2. Preliminary results.

The G^p-extending notion is based on two tools, namely an equivalence relation on cyclic submodules of a module M. Let us begin by mentioning basic facts about them. First recall the following relations on the set of submodules of M (see [1]).

(i)$A\alpha B$ if and only if there exists a submodule C of M such that $A \leq_e C$ and $B \leq_e C$.
(ii)$A\beta B$ if and only if $A \cap B \leq_e A$ and $A \cap B \leq_e B$.
Observe that α is reflexive and symmetric , but it may not be transitive . However, β is an equivalence relation. For submodules of a module M note that if $A\alpha B$, then $A\beta B$.

Proposition 2.1: A module M is p-extending iff for every cyclic submdule A of M, there is a direct summand D of M such that $A\alpha D$.

Proof: The evidence is typical.
Propositional motivation 2.1 and Akalan, Birkenmeier, Tercan's use of the β equivalence relation in [1]. As a broad generalization Goldie extending modules We present a class of modules that is comparable to that of G^z-extending and G^r-extending modules that are shown in [5] and [6] respectively.

Definition 2.2: If for each cyclic submodule A of M, there exists a direct summand D of M such that $A\beta D$ then we call a module M is G^p-extending.
Observe that M is G-extending iff for all submodule A of M there is a direct summand D of M s.t. $A\beta D$. It's evident that the category of G^p-extending consists of both classes of G-extending and p-extending modules.

Now, we position the G^p-extending condition w.r.t. many well-known generalizations of the extending property.

Proposition 2.3: For any module M. Consider the following circumstances.
(i) M is CS.
(ii) M is G -extending.
(iii)M is G^p -extending.
(iv)M is p -extending.
Then (i) ⟹(ii) ⟹(iii) and (i)⟹ (iv)⟹(iii). In general, the converse implications do not hold.

Proof: (i) ⟹(ii) ⟹(iii) and (i) ⟹(iv)⟹(iii) are clear.
(ii)⇏(i) Let M be the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Q}$, where p is any prime integer. Then $M_{\mathbb{Z}}$ is G-extending by [1, corollary (3.3)]. However $M_{\mathbb{Z}}$ is not extending [7 , Example 10].
(iii)⇏(ii) Let $M_2(R)$be the ring as in [8, Example 13.8]. Then $M_2(R)$ is a von Neumann regular ring which is not a Baer ring. Hence it is neither right nor left

CS, by [9, example 2.7] , but it is well known that every von Neumann regular ring is nonsingular, therefore $M_2(R)$ is not is G-extending, see [1, Proposition 1.8]. Also, this is an example to show that (iv)$\not\Rightarrow$(i).

(iii)$\not\Rightarrow$(iv) Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$. Then $M_{\mathbb{Z}}$ is G^p-extending but not P-extending, see [1, Corollary 3.3].

The following statement identifies a circumstance in which G^p-extending and p-extending modules are equivalent.

Proposition 2.4: Let M be a module.

(i) If M is a UC- module. Then M is G^p-extending if and only if M is P-extending.

(ii) If M is a nonsingular module. Then M is G^p-extending if and only if M is P-extending.

(iii) If M is an indecomposable module. Then M is G^p-extending if and only if M is P-extending.

Proof:

(i) Assume that M is G^p-extending and let A be a cyclic submodule of M, then there is a direct D of M such that $A\beta D$. One can easily show that $(A \cap D)\alpha A$ and $(A \cap D)\alpha D$. But M is UC module, therefore α is transitive, hence $A\alpha D$. Thus M is P-extending. The converse is clear.

(ii) Let M be a G^p-extending and let A be a cyclic submodule of M, then there is a direct D of M such that $A\beta D$. It is sufficient to show that $A \leq D$. Since $\frac{A+D}{D} \cong \frac{A}{A \cap D}$ is singular and $\frac{A+D}{D} \leq \frac{M}{D} \cong D'$ is nonsingular, hence $A+D = D$ which implies that $A \leq D$. The converse is obvious.

(iii) Let M be a G^p-extending and let A be a cyclic submodule of M, then there is a direct D of M such that $A\beta D$. But M is indecomposable, therefore D=M. Thus M is P-extending module. The converse is clear.

Corollary 2.5: For any indecomposable M, the following statements are equivalent:

(i) M is uniform.

(ii) M is CS.

(iii) M is G -extending.

(iv) M is P -extending.

(v) M is G^p -extending.

Example 2.6:

Let F be a field and V be a vector space over F with dim (F_V)=2. Let R be the trivial extension of F with V, i.e,

$R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in F, v \in V \right\}$. Since R_R is indecomposable which is not CS, then R_R is not G^p-extending.

3. Characterizations of G^p-extending.

In this part, We offer equivalent conditions to G^p-extending property.
We begin by proving the next theorem.

Theorem 3.1: An R- module M is G^p-extending iff for every cyclic submodule A of M, there is a direct summand D of M s.t. $A\beta D$ and D' is a complement of A, where $M = D \oplus D'$.

Proof: Suppose that M is G^p-extending and take A to be a cyclic submodule of M, there is a direct summand D of M such that $A\beta D$. Take $M = D \oplus D'$, for some submodule D' of M. Since $A \cap D \leq_e A$, then $A \cap D' = 0$. Now, let B a submodule of M s.t. $D' \leq B$ and $A \cap B = 0$. For $A \cap D \leq_e D$, then $B \cap D = 0$. But D' is a complement of D, therefore B=D'. So, D' is a complement of A. The converse is clear.

The next result gives another characterization to G^p-extending modules.

Proposition 3.2: Let M be an R- module, the following conditions are equivalent:

(i) M is G^p-extending.

(ii)For each cyclic submodule A of M , there exists a submodule X of M and a direct summand D of M such that $X \leq_e A$ and $X \leq_e D$.

(iii) For every cyclic submodule A of M there exists a complement B of A and a complement C of B such that $A\beta C$ and every homomorphism $f: C \oplus B \to M$ extends to a homomorphism $g: M \to M$.

Proof: (i) $\Longrightarrow$(ii) Assume that M is G^p-extending and let A be a cyclic submodule of M, there is a direct summand D of M such that $A\beta D$, hence $A \cap D \leq_e A$ and $A \cap D \leq_e D$. Take $X = A \cap D$, we get the result.

(ii)⟹(iii) Let A be a cyclic submodule of M. By (ii), there exists a submodule X of M and a direct summand D of M such that $M = D \oplus D'$, $X \leq_e A$ and $X \leq_e D$. Take $D = C$ and $D' = B$.

(iii)⟹(i) Let A be a cyclic submodule of M. From (iii), there exists a complement B of A and a complement C of B such that $A\beta C$ and every homomorphism $f: C \oplus B \to M$ extends to a homomorphism $g: M \to M$and by [10, Lemma 3.97], D is a direct summand of M, hence M is G^p-extending.

Theorem 3.3 : An R- module M is G^p -extending iff for all direct summand A of the injective hull E(M) of M with $A \cap M$ is cyclic submodule of M, there is a direct summand D of M s.t. $(A \cap M)\beta D$.

Proof: take A a cyclic submodule of M and take B a complement of A, then $A \oplus B \leq_e M$. because $M \leq_e$ E(M), hence $A \oplus B \leq_e$ E(M) implies to $E(M) = E(A) \oplus E(B)$. It can be seen that $E(A) \cap M$ is cyclic submodule in M. By our assumption, there is a direct summand D of M such that $(E(A) \cap M)\beta D$. But we have$(A \cap M)\beta(E(A) \cap M)$, hence $A\beta D$. The converse implication is clear.

We finish this part by the following proposition.

Proposition 3.4: For a module M . M is G^P-extending module iff for each cyclic submodule A of M, there exists an idempotent $f \in$ End (M) so that A β f (M).

4. Decompositions.

There are nonsingular modules $M = M_1 \oplus M_2$ in which M_1 and M_2 are P-extending, but M is not P-extending (e.g, Let $R = \mathbb{Z}[x]$ be a polynomial ring of integers and let $M = \mathbb{Z}[x] \oplus \mathbb{Z}[x]$). Note that $\mathbb{Z}[x]$ is G −extending, by [1] and hence G^p-extending but M is not P-extending which is nonsingular, thus by proposition 2.4 M is not G^p-extending. Next, we give various conditions under which the direct sum of G^p-extending is G^p-extending.

Proposition 4.1: Let $M = M_1 \oplus M_2$ be a distributive module if M_1 and M_2 are G^p-extending modules, then M is G^p-extending.

Proof: Let A be a cyclic submodule of M. Since M is distributive, then $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. Since A is cyclic in M, then $A \cap M_1$ and $A \cap M_2$ are cyclic in M_1 and M_2 respectively. But M_1 and M_2 are G^p-extending modules, therefore there are direct summand D_1 of M_1 and D_2 of M_2 such that $(A \cap D_1)\beta D_1$ and $(A \cap D_2)\beta D_2$ hence A $\beta(D_1 \oplus D_2)$, by [11 , Proposition 1.4]. Thus, M is G^p-extending module.

The following statements are also easily proved by using a similar argument.

Proposition 4.2 : Let $M = M_1 \oplus M_2$ be a duo module if M_1 and M_2 are G^p-extending modules, then M is G^p-extending.

Proposition 4.3: Let M_1 and M_2 be G^p-extending modules such that $annM_1 + annM_2 = R$, then $M_1 \oplus M_2$ is G^p-extending module.

Proposition 4.4: Assume that for all cyclic submodule A of M, $A \cap M_1$ is a direct summand of A, then M is G^p-extending, where $M = M_1 \oplus M_2$ be an R-module with M_1 being G^p-extending and M_2 is semisimple.

Proof: Take A be a cyclic submodule of M, so its simple to see that $A+M_1 = M_1 \oplus [(A+M_1) \cap M_2]$. Because M_2 is semisimple, then $(A+M_1) \cap M_2$ is a direct summand of M_2 and therefore $A+M_1$ is a direct summand of M. We assumption, $A \cap M_1$ is a direct summand of A, then $A = (A \cap M_1) \oplus A'$, for some submodule A' of A. One can easily show that $A \cap M_1$ is cyclic in M_1. But M_1 is G^p-extending, then there is a direct summand D of M_1 such that $(A \cap M_1)\ \beta\ D$ hence $A = ((A \cap M_1) \oplus A')\beta(M_1+A)$. Thus, M is G^p-extending.

Proposition 4.5: Let $M = M_1 \oplus M_2$ such that M_1 is G^p-extending and M_2 be injective module. Then for all cyclic submodule A of M s.t. $M_2 \cap A \neq 0$ there exist a direct summand D of M s.t. $A\beta D$ iff M is G^p-extending.

Proof: Assume that for all cyclic submodule A of M s.t. $M_2 \cap A \neq 0$ there exist a direct summand D of M s.t. $A\beta D$. Let A be a cyclic submodule of M such that $A \cap M_2 = 0$. By [2], there is a submodule M' of M containing A such that $M = M' \oplus M_2$. Because $M' \cong \frac{M}{M2} \cong M_1$ be G^p-extending and A is cyclic submodule of M', so there exist a direct summand K of M' s.t. $A\beta K$. Therefore, M be G^p-extending. The opposite is clear.

We now list some circumstances in which a direct summand of a G^p-extending module be G^p-extending.

Proposition 4.6: The intersection of A with each direct summand of M is a direct summand of A, then A be G^p-extending module. Where M is G^p -extending $\mathcal{R}$-module and A is a direct summand of M.

Proof: Take X be a cyclic in A, then X is cyclic in M. But M is G^p-extending, therefore there exists a direct summand D of M such that $X\beta D$. It can be seen that $X\beta$ $(A\cap D)$. We suppose $A\cap D$ be a direct summand of A. Thus, A is G^p-extending.

Proposition 4.7: Take M is a G^p-extending module and take A is a cyclic submodule of M.

(i) If for each $e^2 = e \in End(M_R)$, there exists $f^2 = f \in End(A_R)$ such that $A \cap eM \ \leq_e fA$, then A is G^p-extending.
(ii)If for each $e^2 = e \in End(M_R)$, there exists $f^2 = f \in End(A_R)$ such that $eM\beta fM$ and $fA \subseteq A$, then A is G^p-extending.
Proof:
(i) Let Y be a cyclic submodule of A. Hence Y be a cyclic submodule of M. By proposition 3.2, there exist $X\leq_e Y$ and $e^2 = e \in End(M_R)$ s.t. $X\leq_e eM$. Then $X\leq_e eM\cap A \leq_e fA$, for some $f^2 = f \in End(M_R)$.Thus, A is G^p-extending.
(ii)Let Y is a cyclic submodule of A, then Y is cyclic in M. Then there exists $e^2 = e \in End(M_R)$ such that $Y\beta eM$. Hence $Y\beta fM$, where $f^2 = f \in End(A_R)$, A is G^p-extending.
Proposition 4.8: If M is G^p-extending then there is K is G^p-extending and M_1 is smaller than M s.t. $M = M_1 \oplus K$. Where K be a projection invariant cyclic submodule of M.

Proof: $\exists\, e^2 = e \in End(M_R)$ s.t. $K\beta eM$. But $K = eK \oplus (1-e)K$, $eK = K \cap eM$, and $(1-e)K = K \cap (1-e)M$ where K is projection invariant, so $eK \leq_e eM$ and $eK \leq_e K$. Therefore, $K \cap (1-e)M = 0$. then $K = eK \leq_e eM$. Because K is cyclic in M, then K=eM. Let $M_1 = (1-e)M$. Therefore $M = M_1 \oplus K$. Observe that, by Proposition 4.7 (ii), K is G^p-extending.

Theorem 4.9: If M hold the C_3 condition or has SIP, then every cyclic direct summand of M is G^p-extending where M be a G^p-extending module.
Proof: Take $M = N \oplus N'$ for some submodules N', N of M where N is cyclic in M. Employing proposition 4.8(i) , where N is regarded as cyclic in M utilizing the SIP offers that N is a G^p-extending.
Consider that M hold the C_3 condition. take $\pi: M \to N$ be the canonical projection. take K be any cyclic submodule of N, then K is cyclic in M. By assumption, $\exists\, L$ (direct summand) of M such that $K \cap L \leq_e K$ and $\cap L \leq_e L$. Because M holds C_3 condition, $N' \oplus L$ is a direct summand of M. It may be seen that $N' \oplus L = N' \oplus \pi(L)$ (see [10]). Hence $\pi(L)$ be a direct summand of N. $\forall\, 0 \neq y \in \pi(L)$, $y = \pi(x)$ for some $0 \neq x$ belong to L. $\exists\, r \in R$ s.t. $0 \neq xr$

belong to $K \cap L$. So $xr = k = x_1$, where $k \in K$ and $x_1 \in L$. Now $0 \neq xr = \pi(x)r = k = \pi(x_1) \in K \cap \pi(L)$. that follows $K \cap \pi(L) \leq_e \pi(L)$. It's easy to see that $\pi(L) = N \cap \left(N' \oplus \pi(L)\right) = N \cap (N' \oplus L)$. Hence $K \cap \pi(L) = K \cap (N' \oplus L) \leq_e K$. Thus N is G^p-extending.

References

[1] E. Akalan, G. F. Birkenmeier, A. Tercan, Goldie extending modules, Comm. Algebra 37(2) (2009),:663-683.

[2] N. V. Dung, D. V. Huynh, P. F. Smith, Wisbauer, R., Extending Modules. Harlow: Longman, (1994).

[3] M. A. Kamal, O. A. Elmnophy, On P-extending Modules, Acta Math. Univ. Comenianae, (2005), 279-286.

[4] P. F. Smith, Modules for which every submodule has a unique closure, in Ring theory (Graville, OH, 1992), 302-313, World Sci. Publ., River Edge, NJ, 1993.

[5] A. Tercan, R. Yasar and C. C. Yucel, , Goldie extending modules on class of z-closed submodules. Bull. Korean Math. Soc. (2022), 453-468.

[6] A. Tercan, R. Yasar and C. C. Yucel, Goldie extending modules on class of exact submodules. Comm. Algebra. (2022), 1363-1371.

[7] P. F. Smith and A. Tercan, Continuous and quasi- continuous modules, Houston J. Math. 18 (1992), 339-348.

[8] K. R. Goodearl, Von Neumann Regular Rings, Pitman, London, 1976.

[9] G. F. Birkenmeier and A. Tercan, when some complement of a submodule is a direct summand, Comm. Algebra, 35 (2007), 597-611.

[10] A. Tercan and C.C. Yucel, Modules Theory, extending modules and generalizations, Frontiers in Mathematics, Birkhauser/Springer, 2016.

[11] E. M. Aljumaily and B. H. Albahraany, On Goldie extending modules, Iraqi Journal of Science, 2015, Vol 56, No.1B, pp: 492-498 .