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Bayesian Estimation of Spherical Distribution Parameters under DeGroot Loss Function

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ABSTRACT

There are many statistical methods and approaches to estimating the parameters of statistical models. These estimations are distinguished by important criteria to indicate the preference in the estimation, the most important criteria are the standard mean square error. The main goal of any estimation process is to reach the best or closest estimate of the unknown parameter among all possible estimates.

In this paper, Bayesian estimation of spherical distribution parameters was used. The default values of the three-dimensional spherical Dirichlet distribution were obtained experimentally by conducting many experiments and choosing the values at which Bayes estimates stabilized and gave the best results. Using $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1$ as an intial values. The results showed that the DeGroot loss function gave the best results when the sample size was greater than 400.

A sample of size 720 patient was selected randomly and used in the applied aspect for myopia variables for real data, and the study concluded that the values of the probability density function estimated by the Bayes method at the DeGroot loss function are consistent with the values of the true probability density function for the three-dimensional spherical Dirichlet distribution.

1. Introduction

The Bayes method in estimation in its general concept focuses on using prior information about the unknown parameter that is required to be estimated, considering that this parameter is a random variable and not a fixed value. The estimator Bayesian estimation for any parameter depends on two functions, the first is known as the posterior probability density function and the second is the loss function. The posterior probability density function can be defined as representing all the information about the parameter to be estimated after viewing the current sample information, in other words it is a function of the initial information expressed by the initial distribution function of the parameter to be estimated (the initial information density function) and the current sample data, and Bayesian estimation is an approach to statistical estimation used to determine the most likely value of the parameter to be estimated, Bayesian estimation is based on the principle of conditional probabilities in probability theory [1].

2. Research Problem

The research problem is represented by the lack of estimates of the parameters of the spherical distribution using Bayesian estimation methods under selected loss functions, and the need to estimate a probability distribution function for the factors that contribute to myopia in children.

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3. Research objective

The aim of this paper is to estimate the parameters of the spherical-Dirichlet distribution using the Bayesian method in estimation under an asymmetric loss function, which is the DeGroot Loss function, and to propose a probability distribution for the factors that lead to myopia in children (which is the PRIOR Spherical Gamma distribution as a primary distribution) by converting the traditional Gamma distribution from Cartesian coordinates to spherical coordinates.

4. Loss functions [1][4]

Loss functions that make the loss function prediction at its lowest value are used to estimate the expected model parameters when making a decision based on the available information and known or proposed probability distributions. Bayesian estimates differ according to the different types of loss functions in order to obtain a Bayesian estimator at which the subsequent expected loss is as low as possible for the purpose of testing the accuracy of the Bayesian method to obtain the estimated parameter vector.

The loss function resulting from estimating the parameter vector $\underline{\alpha}$ is represented by the estimated vector $\underline{\hat{\alpha}}$ as there is often a noticeable difference between the estimator and the parameter and the loss function is a measure of the difference $\underline{\hat{\alpha}} - \underline{\alpha}$ or the ratio $\underline{\hat{\alpha}}/\underline{\alpha}$ or both through which the accuracy of the estimate is identified. It can be $\underline{\hat{\alpha}} = \underline{\alpha}$ which indicates no loss. Or it can be $\underline{\hat{\alpha}} < \underline{\alpha}$ called under estimation. Or it can be $\underline{\hat{\alpha}} > \underline{\alpha}$ and it is called over estimation .

A loss function is defined as a positive real-valued function that satisfies the following:

1. $l\left(\underline{\hat{\alpha}},\underline{\alpha}\right) \geq 0 \quad ; \forall\, \underline{\hat{\alpha}}, \underline{\forall\alpha}$
2. $l\left(\underline{\hat{\alpha}},\underline{\alpha}\right) = 0 \quad ; \forall\, \underline{\hat{\alpha}} = \underline{\alpha}$

4.1 Asymmetric Loss Functions

Asymmetric loss functions are loss functions that deal differently with positive and negative errors in estimating the required value. These functions are useful when positive and negative errors have different effects or when there are specific preferences or restrictions in the estimation. Symmetric loss functions are built on the assumption that the loss is the same in any direction, but this assumption may not be met in many cases. In some cases, the positive error is more important than the negative error and vice versa. Therefore, the use of symmetric loss functions is sometimes inappropriate. Therefore, it is preferable to use asymmetric loss functions.

4.2 DeGroot Loss Function:

It is an asymmetric loss function proposed by DeGroot in (1970) and its formula is [6]:

$$l_D\left(\underline{\hat{\alpha}}\,,\underline{\alpha}\right) = \left(\frac{\underline{\alpha} - \underline{\hat{\alpha}}}{\underline{\hat{\alpha}}}\right)^2 = \left(\frac{\underline{\alpha}}{\underline{\hat{\alpha}}} - 1\right)^2 \qquad (1)$$

To find the Bayes estimator for the parameter vector ($\underline{\alpha}$), the risk function is as follows:

$$R_D\left(\underline{\hat{\alpha}},\underline{\alpha}\right) = E\left(\frac{\underline{\alpha}}{\underline{\hat{\alpha}}} - 1\right)^2$$

$$= \int_{\forall\underline{\alpha}} \left(\frac{\underline{\alpha}}{\underline{\hat{\alpha}}} - 1\right)^2 \pi\left(\underline{\alpha}\mid x\right) d\underline{\alpha}$$

$$= \underline{\hat{\alpha}}^{-2} E\left(\underline{\alpha}^{\,2}\mid t\right) - 2\underline{\hat{\alpha}}^{-1} E\left(\underline{\alpha}\mid x\right) + 1 \qquad (2)$$

By taking the partial derivative of equation (2) with respect to the estimated parameter vector $\underline{\hat{\alpha}}$ and setting the result equal to zero, we get:

$$2\,\underline{\hat{\alpha}}^{-2}\; E\left(\underline{\alpha}\mid x\right) - 2\underline{\hat{\alpha}}^{-3}\; E\left(\underline{\alpha}^{\,2}\mid x\right) = 0$$

Therefore:

$$\underline{\hat{\alpha}}_{BD} = \frac{E\left(\underline{\alpha}^{\,2}\mid t\right)}{E\left(\underline{\alpha}\mid t\right)} \qquad (3)$$

Equation(2) is the formula for finding a Bayesian estimator $\underline{\alpha}$ for the vector of parameters using the DeGroot loss function, it is clear that the denominator in it is the expected posterior probability density function for the vector of parameters $\underline{\alpha}$, i.e. the Bayesian estimator for the vector of parameters $\underline{\alpha}$ using the square error loss function [5].

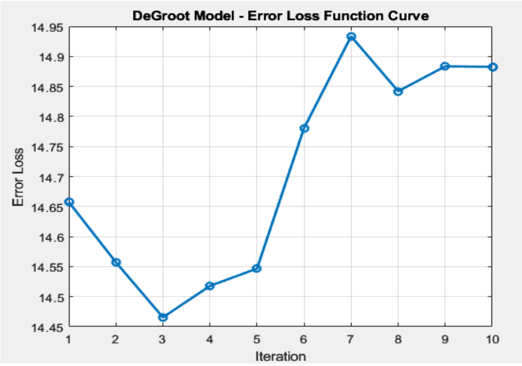

Figure (1) DeGroot loss function curve

5. Dirichlet distribution [6][2]

A probability distribution that is commonly used in statistical analysis and machine learning. This distribution is used to determine a probability distribution for the proportion of different classes in a set of samples.

It has many applications in different fields. It is the most suitable distribution for synthetic data and proportion modelling measures in Bayesian inference. It is used in deriving the distribution function in biology, calculating the probability of matching in forensics from several population groups, modelling player abilities in matches, and modelling consumer purchasing behaviour. The Dirichlet distribution plays a fundamental role in Bayesian theory and Bayesian estimation. It is a multivariate distribution that is commonly used to model a set of proportions that sum to one.

Let X_i be a random variable with a Gamma distribution

$$f(x) = \frac{e^{-xi} xi^{\alpha i-1}}{\Gamma \alpha i} \qquad (4)$$

where $X_i \sim G(\alpha_i, 1)$ and i=1,2,…..,k and X1,…..,Xk are independent, then the joint probability density function for (X1,…..,Xk) is written as follows [3]:

$$f(x_1, \dots \dots x_k) = \begin{cases} \prod_{i=1}^{k} \frac{1}{\Gamma(\alpha_i)} x_i^{\alpha_i-1} e^{-x_i} & if\ 0 < x_i < \infty \\ 0 & otherwise \end{cases} \qquad (5)$$

Let

$$P_i = \frac{X_i}{X_1+X_2+\cdots\dots+X_k} = \frac{X_i}{P_k} \quad ; i = 1,2, \dots\dots, k-1 \ ; \quad 0 < P_i < 1 \qquad (6)$$

There fore

$$P_k = X_1 + X_2 + \cdots + X_{k\,0}$$

Then

$$X_i = P_i P_k$$

And also

$$x_1 = P_1 P_k \,, x_2 = P_2 P_k, \dots, x_{k-1} = P_{k-1} P_k,$$
$$x_k = P_k(1 - P_1 - \cdots - P_{k-1}) \qquad (4)$$

A one could prove that the Jacobian determinate is

$$J = \begin{vmatrix} P_k & 0 & \cdots & 0 & P_1 \\ 0 & P_k & \cdots & 0 & P_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & P_k & P_{k-1} \\ -P_k & -P_k & \cdots & -P_k & 1 - P_1 - \cdots - P_{k-1} \end{vmatrix} = P_k^{k-1} \qquad (5)$$

Since this paper based on (3) variables then Then:

$$f(P_1, P_2, P_3) = f(x_1, x_2, x_3)\ |J|$$

Where,

$$f(x_1, x_2, x_3) = \frac{\prod_{i=1}^{3} x_{ij}^{\alpha_i-1}}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} e^{-\sum_{i=1}^{3} x_i j}\, p_3^{\alpha_1+\alpha_2+\alpha_3-1} \qquad (6)$$

$$= e^{-p_3} p_3^{\alpha_1+\alpha_2+\alpha_3-1} \frac{p_1^{\alpha_1-1} p_2^{\alpha_2-1} (1-p_1-p_2)^{\alpha_3-1}}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} \qquad (7)$$

where $J = p_3^2$, and therefore the joint probability density function will be as in the equation:

$$f(P_1, P_2, P_3) = \int_0^\infty f(P_1, P_2, P_3) d\,P_3 \qquad (8)$$

$$= \int_0^\infty e^{-p_3} p_3^{\alpha_1+\alpha_2+\alpha_3-1} \frac{p_1^{\alpha_1-1} p_2^{\alpha_2-1} (1-p_1-p_2)^{\alpha_3-1}}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} dP_3$$

$$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3}\ p_1^{\alpha_1-1} p_2^{\alpha_2-1} (1 - p_1 - p_2)^{\alpha_3-1} \qquad (9)$$

And equation (9) is the multivariate Dirichlet Distribution which is a Beta Distribution

$$P_i \sim Beta(\alpha_i, \textstyle\sum_{i=1}^{3} \alpha_i - \alpha_3);\ i = 1,2,3\,, 0 < Pi < 1$$

Let

$$y_1 = p_1, y_2 = p_2, y_3 = 1 - p_1 - p_2$$

then

$$f(y_1, y_2, y_3) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} \; y_1^{\alpha_1-1} y_2^{\alpha_2-1} y_3^{\alpha_3-1}$$

$$f(y_1, y_2, y_3) = \frac{\Gamma\alpha_0}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} \; y_1^{\alpha_1-1} y_2^{\alpha_2-1} y_3^{\alpha_3-1}$$

a_i are the parameters of the Dirichlet distribution, which are the number of variables (dimensions) that indicate the averages of the random variables in the distribution, and their number is (3).

6.Some properties of the Dirichlet distribution:

a. Mean [6]:

$$E(Y_i) = \frac{\alpha_i}{\alpha_0}, i = 1,2,3 \quad (10)$$

$$E(Y_i^2) = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_0+2)} \frac{\Gamma(\alpha_i+2)}{\Gamma(\alpha_i)} = \frac{(\alpha_i+1)\alpha_i}{(\alpha_0+1)\alpha_0} \quad (11)$$

b. $var(Y_i) = E(Y_i^2) - E(Y_i)^2$

$$= \frac{(\alpha_i+1)\alpha_i}{(\alpha_0+1)\alpha_0} - (\frac{\alpha_i}{\alpha_0})^2 = \frac{\alpha_i(\alpha_0-\alpha_i)}{\alpha_0^2(\alpha_0+1)} \quad (12)$$

$c. Cov(Y_i, Y_j) = E(Y_i, Y_j) - E(Y_i)E(Y_j)$

$$= \frac{\alpha_i\alpha_j}{\alpha_0(\alpha_0+1)} - \frac{\alpha_i}{\alpha_0}\frac{\alpha_j}{\alpha_0} = \frac{-\alpha_i\alpha_j}{\alpha_0^2\,(\alpha_0+1)}, \; i \neq \quad (13)$$

7. The Spherical-Dirichlet Distribution

This distribution is one of the useful tools in defining risk weights in automated classification systems [6], [7].

The spherical Dirichlet distribution is used in many applications, such as image and text classification, genetic data analysis, and weather forecasting. This distribution is used to determine a probability distribution for several different classes based on some characteristics related to the data.

The spherical Dirichlet distribution is characterized by being a multidimensional distribution, as it can be used to determine a probability distribution for several classes. It is also considered a continuous distribution, as it can take probability values at any point in the multidimensional space.

The spherical Dirichlet distribution is used in machine learning techniques, such as image and text classification, genetic data analysis, and weather forecasting, as it can be used to improve the accuracy of models and accurately identify classes.

We can obtain the spherical Dirichlet distribution by using the Dirichlet distribution transformation on the plane corresponding to the hyperbolic sphere as shown in Figure 2. If we have a probability density function for the distribution f_{Dir} in equation 1-5, it is transformed into a spherical one using the plane corresponding to the hyperbolic sphere through the following transformation:

$$x_i = \sqrt{y_i}, y_i = x_i^2, \frac{\partial y_i}{\partial x_i} = 2x_i, i = 1, \ldots . (k-1), x_k = \sqrt{y_k} \quad (14)$$

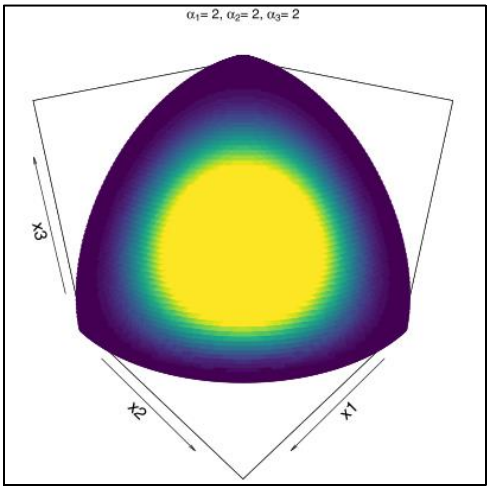

Fig (2) The plane corresponding to the hyperbolic sphere for three variables has a Dirichlet distribution.

The Jacobian is

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} = 2x_1 & 0 \\ 0 & \frac{\partial y_2}{\partial x_2} = 2x_2 \end{vmatrix} = 4x_1x_2 \quad (15)$$

Thus, the probability density function for the spherical Dirichlet distribution is as follows:

$$f_{SDir} = \frac{4\Gamma(\alpha_0)}{\Gamma\alpha_1\Gamma\alpha_2\Gamma\alpha_3} x_1^{2\alpha_1-1} x_2^{2\alpha_2-1} x_3^{2\alpha_3-2}$$

$$f_{SDir} = \frac{4\Gamma(\alpha_0)}{\Gamma\alpha_1\Gamma\alpha_2\Gamma\alpha_3} x_1^{2\alpha_1-1} x_2^{2\alpha_2-1} (x_3^2)^{\alpha_3-1}$$

$$f_{SDir} = \frac{4\Gamma(\alpha_0)}{\Gamma\alpha_1\Gamma\alpha_2\Gamma\alpha_3} x_1^{2\alpha_1-1} x_2^{2\alpha_2-1} (1 - x_1^2 - x_2^2)^{\alpha_3-1}$$

$\alpha_i \in R^+ , \alpha_0 = \sum_{i=1}^{3} \alpha_i$, $0 \le x_i \le 1$, $\sum_{i=1}^{3} {x_i}^2 = 1$

So the equation 22 is the formula for the spherical Dirichlet distribution, since x_i ~SDD (αi)

$$E(x_1) = \int \dots \int \frac{x_1 4\Gamma(\alpha_0)}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} x_1^{2\alpha_1-1} x_2^{2\alpha_2-1} (1 - x_1^2 - x_2^2)^{\alpha_3-1} dx_1 dx_2 dx_3$$

$$E(x_1) = \int \dots \int \frac{4\Gamma(\alpha_0)}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} x_1^{2\alpha_1} x_2^{2\alpha_2-1} (1 - x_1^2 - x_2^2)^{\alpha_3-1} dx_1 dx_2 dx_3$$

$$E(x_1) = \frac{4\Gamma(\alpha_0)}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} \frac{\Gamma\left(\alpha_1 + \frac{1}{2}\right) \Gamma\alpha_2 \Gamma\alpha_3}{4\,\Gamma\left(\alpha_0 + \frac{1}{2}\right)}$$

$$E(x_1) = \frac{\Gamma(\alpha_0)}{\Gamma\left(\alpha_0 + \frac{1}{2}\right)} \frac{\Gamma\left(\alpha_1 + \frac{1}{2}\right)}{\Gamma(\alpha_1)}$$

In general

$$E(x_i) = \frac{\Gamma(\alpha_0)}{\Gamma\left(\alpha_0 + \frac{1}{2}\right)} \frac{\Gamma\left(\alpha_i + \frac{1}{2}\right)}{\Gamma(\alpha_i)} \quad (16)$$

$$E(x_1^2) = \iiint \frac{4\Gamma(\alpha_0) x_1^2}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} x_1^{2\alpha_1-1} x_2^{2\alpha_2-1} (1 - x_1^2 - x_2^2)^{\alpha_3-1} dx_1 dx_2 dx_3$$

$$E(x_1^2) = \iiint \frac{4\Gamma(\alpha_0)}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} x_1^{2\alpha_1+1} x_2^{2\alpha_2-1} (1 - x_1^2 - x_2^2)^{\alpha_3-1} dx_1 dx_2 dx_3$$

$$E(x_1^2) = \frac{4\Gamma(\alpha_0)}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} \frac{\Gamma(\alpha_1 + 1) \Gamma\alpha_2 \Gamma\alpha_3}{4\,\Gamma(\alpha_0 + 1)}$$

$$E(x_1^2) = \frac{\Gamma(\alpha_0)}{\Gamma\alpha_1} \frac{\Gamma(\alpha_1 + 1)}{\Gamma(\alpha_0 + 1)} = \frac{\alpha_1}{\alpha_0}$$

therefore

$$E(x_i^2) = \frac{\alpha_i}{\alpha_0} \quad (17)$$

and

$$Var(xi) = \frac{(\alpha_i)}{(\alpha_0)} - \left[\frac{\Gamma(\alpha_0)}{\Gamma\left(\alpha_0 + \frac{1}{2}\right)} \frac{\Gamma\left(\alpha_i + \frac{1}{2}\right)}{\Gamma(\alpha_i)} \right]^2 \quad (18)$$

The covariance between any two variables is

$$E(x_1, x_2) = \iiint \frac{4\Gamma(\alpha_0) x_1 x_2}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} x_1^{2\alpha_1-1} x_2^{2\alpha_2-1} (1 - x_1^2 - x_2^2)^{\alpha_3-1} dx_1 dx_2 dx_3$$

$$E(x_1, x_2) = \iiint \frac{4\Gamma(\alpha_0)}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} x_1^{2\alpha_1} x_2^{2\alpha_2} (1 - x_1^2 - x_2^2)^{\alpha_3-1} dx_1 dx_2 dx_3$$

$$= \frac{\Gamma\left(\alpha 1 + \frac{1}{2}\right) \Gamma\left(\alpha 2 + \frac{1}{2}\right)}{\alpha_0 \Gamma(\alpha 1)\ \Gamma(\alpha 2)} \quad (19)$$

$$Cov(x_1, x_2) = E(x_1, x_2) - E(x_1)E(x_2)$$

$$Cov(x_1, x_2) = \frac{\Gamma\left(\alpha 1 + \frac{1}{2}\right) \Gamma\left(\alpha 2 + \frac{1}{2}\right)}{\alpha_0 \Gamma(\alpha 1)\ \Gamma(\alpha 2)} - \frac{\Gamma(\alpha_0)}{\Gamma\left(\alpha_0 + \frac{1}{2}\right)} \frac{\Gamma\left(\alpha_1 + \frac{1}{2}\right)}{\Gamma(\alpha_1)} \frac{\Gamma(\alpha_0)}{\Gamma\left(\alpha_0 + \frac{1}{2}\right)} \frac{\Gamma\left(\alpha_2 + \frac{1}{2}\right)}{\Gamma(\alpha_2)}$$

Let

$$\mu_i = \frac{\Gamma\left(\alpha_i + \frac{1}{2}\right)}{\Gamma(\alpha_i)}, \qquad i = 0, 1, 2, 3$$

therefore

$$E(x_i) = \frac{\mu_i}{\mu_0} \quad (20)$$

$$E(x_i\, x_j) = aij \frac{\alpha i}{\alpha_0} + \frac{\mu_i \mu_j}{\alpha_0} - aij \frac{\mu_i \mu_j}{\alpha_0}$$

$$i \ne j = 1,2,3 \quad (21)$$

$$Var(xi) = \frac{\alpha_i}{\alpha_0} - \left[\frac{\mu_i}{\mu_0} \right]^2$$

And the var-cov matrix is

$$\Sigma = \begin{bmatrix} \frac{\alpha_1}{\alpha_0} - \left[\frac{\mu_1}{\mu_0}\right]^2 & \frac{\mu_1\mu_2}{\alpha_0} - \frac{\mu_1\mu_2}{\mu_0^2} & \frac{\mu_1\mu_3}{\alpha_0} - \frac{\mu_1\mu_3}{\mu_0^2} \\ \frac{\mu_2\mu_1}{\alpha_0} - \frac{\mu_2\mu_1}{\mu_0^2} & \frac{\alpha_2}{\alpha_0} - \left[\frac{\mu_2}{\mu_0}\right]^2 & \frac{\mu_2\mu_3}{\alpha_0} - \frac{\mu_2\mu_3}{\mu_0^2} \\ \frac{\mu_3\mu_1}{\alpha_0} - \frac{\mu_3\mu_1}{\mu_0^2} & \frac{\mu_3\mu_2}{\alpha_0} - \frac{\mu_3\mu_2}{\mu_0^2} & \frac{\alpha_3}{\alpha_0} - \left[\frac{\mu_3}{\mu_0}\right]^2 \end{bmatrix} \quad (22)$$

8.Bayes Estimators for Vector of parameters of The Spherical-Dirichlet Distribution

We can obtain the estimators of the vector parameters of the spherical Dirichlet distribution using the Bayesian method by estimating under different loss functions as follows:

a - Informative Standard Bayesian Estimator for The Spherical-Dirichlet Distribution under DeGroot Loss function

The standard Bayesian estimator for the DeGroot function is as in the equation

$$\hat{\underline{\alpha}}_{INDSBSDD} = \frac{E(\underline{\alpha}^{\,2} \mid t)}{E(\underline{\alpha} \mid t)}$$

$$= \iiint \frac{\underline{\hat{\alpha}}^2 \prod_{i=}^{3} \boldsymbol{\alpha_i}^{2\alpha_i - 1} \boldsymbol{e}^{-b_i {\alpha_i}^2} [\boldsymbol{n} \ln \left(\frac{4\Gamma\alpha_0}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} \right) + \sum_{i=1}^{3} (2\alpha_i - 1) \sum_{j=1}^{n} \ln(x_{ij}) - \sum_{j=1}^{n} \ln(x_{i3})]}{\iiint \prod_{i=}^{3} \boldsymbol{\alpha_i}^{2\alpha_i - 1} \boldsymbol{e}^{-b_i {\alpha_i}^2} [\boldsymbol{n} \ln \left(\frac{4\Gamma\alpha_0}{\Gamma\alpha_1 \Gamma\alpha_2 \Gamma\alpha_3} \right) + \sum_{i=1}^{3} (2\alpha_i - 1) \sum_{j=1}^{n} \ln(x_{ij}) - \sum_{j=1}^{n} \ln(x_{i3})]\, d\alpha_1 d\alpha_2 d\alpha_3} \quad (23)$$

By solving equation 30, we obtain the parameters to be estimated, α_i , which is a nonlinear function that cannot be solved using conventional analytical methods. Therefore, the Monte-Carlo Macrocycle (MCMC) method will be used.

9. The applied aspect

This section includes a practical application of the Dirchleigh spherical distribution on real-world data obtained from Al-Karaawi Medical Center for Ophthalmology and Eye Surgery in Babylon Governorate for the years (2020-2022) that were measured using Optical Coherence Tomography (OCT), which is an imaging technique used to image structures inside the eye, including the retina and optic nerve. These variables are the direction of the eye structures and the direction of the optic disc and peripheral refraction, with the aim of applying them to the spherical distribution and extracting Bayesian estimators. In our study, we will discuss Bayesian estimators under the DeGroot loss function because it has proven to be superior to other loss functions. A program written in Matlab was used.

10. Myopia (near-sightedness)

It is a common visual disease, where those affected can see nearby objects clearly, while distant objects appear blurry. This happens when the shape of the eye, or rather the shape of certain parts of it, causes light rays to be refracted inaccurately. Because light rays that should be focused on the nerve tissue at the back of the eye (the retina) are focused in front of the retina instead, nearsightedness usually appears during childhood and adolescence and usually becomes more stable between the ages of 20 and 40. Nearsightedness often runs in families, a basic eye exam can confirm the extent of nearsightedness. You can compensate for blurry vision with glasses or contact lenses, or

with refractive surgery. Signs and symptoms of nearsightedness may include:

1. Blurred vision when looking at distant objects
2. Needing to squint or partially close the eyelids to see clearly
3. Headaches

11. Eye strain Children

May have difficulty seeing objects on whiteboards or classroom displays. Younger children may not express difficulty seeing, but they may exhibit the following behaviors that indicate difficulty seeing.

12. Applied Data Description

The applied data represent three variables to measure myopia for (720) patients at Al-Karaawi Medical Center for Ophthalmology and Eye Surgery in Babylon Governorate for the years (2020-2022) that were measured using Optical Coherence Tomography (OCT), which is an imaging technique used to image structures inside the eye, including the retina and optic nerve. It is one of the devices for diagnosing and monitoring conditions such as changes associated with myopia, macular degeneration, and glaucoma. These variables are:

a. Studies on the orientation of certain structures of the eye (such as blood vessels) in relation to myopia. This may include analyzing orientation angles using circular statistics.
b. Optic Disc Orientation (ODO): It is a measurement of the orientation of the optic disc from the point where the optic nerve exits the eye in degrees from the vertical meridian. This may be relevant to understanding how the orientation of the optic disc is related to the development of myopia.
c. Peripheral Refraction (PR): When assessing the progression of myopia, researchers sometimes examine how refraction changes in different parts of the visual field. This involves measuring the refractive error in different meridians.

- Orientation of Eye Structures) (OES)
- Optic Disc Orientation) (ODO)
- Peripheral Refraction (PR)

Table 1 samples of Variants of myopia

PR	ODO	OES	PR	ODO	OES	PR	ODO	OES	PR	ODO	OES
0.853	0.107	0.510	0.394	0.568	0.722	0.812	0.198	0.550	0.978	0.056	0.200
0.637	0.762	0.119	0.995	0.023	0.094	0.048	0.968	0.246	0.807	0.572	0.146
0.389	0.810	0.440	0.289	0.676	0.678	0.912	0.387	0.136	0.722	0.678	0.137
0.826	0.244	0.508	0.707	0.134	0.694	0.354	0.735	0.578	0.026	0.052	0.998
0.437	0.163	0.885	0.101	0.138	0.985	0.238	0.845	0.478	0.630	0.603	0.489
0.253	0.708	0.659	0.614	0.587	0.527	0.131	0.023	0.991	0.073	0.636	0.768
0.467	0.622	0.629	0.136	0.965	0.224	0.901	0.308	0.304	0.641	0.352	0.683
0.770	0.373	0.518	0.629	0.618	0.472	0.401	0.180	0.898	0.005	0.598	0.802
0.109	0.479	0.871	0.999	0.041	0.032	0.389	0.206	0.898	0.697	0.248	0.673
0.355	0.587	0.728	0.688	0.359	0.631	0.673	0.446	0.590	0.886	0.461	0.042
0.874	0.482	0.061	0.395	0.303	0.867	0.154	0.687	0.710	0.766	0.221	0.604
0.486	0.320	0.813	0.480	0.571	0.666	0.723	0.154	0.673	0.264	0.789	0.555
0.623	0.475	0.621	0.727	0.666	0.166	0.951	0.258	0.170	0.737	0.588	0.333

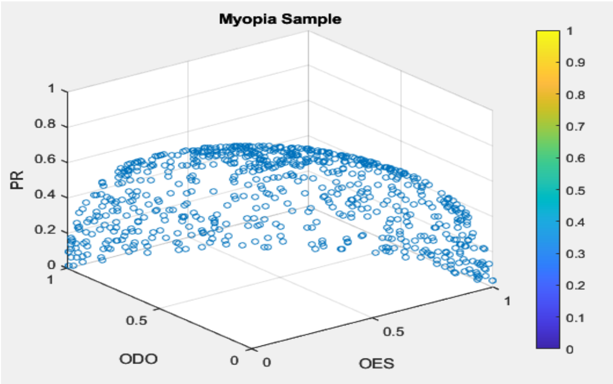

Figure (3) Three-dimensional (spherical) diffusion shape of real data

Table (2) shows the descriptive statistics of myopia data:

Criteria	OES	ODO	PR
Mean	**0.492**	**0.488**	**0.526**
Median	**0.489**	**0.488**	**0.541**
Mode	**0.930**	**0.260**	**0.290**
Std	**0.282**	**0.287**	**0.285**
Variance	**0.080**	**0.083**	**0.081**

We note from Table 1 that:

a. The mean direction of the Orientation of Eye Structures (OES) is (0.50) and the mean direction is (0.49) and the common angle is the angle (0.93).

b. The average direction of the optical disc orientation (ODO) is (0.49) and the common angle is (0.26)

c. The average direction of the peripheral refraction (PR) is (0.53) and the common angle is (0.29) .

13. Test of goodness of fit

Real data was tested at $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1$ using the Bayes method at the DeGroot loss function, which showed its superiority in the experimental aspect using the Kuiper (K) statistic and the Watson (W) statistic where Kuiper statistic formula is

$$K = \sqrt{n}\left\{\max_{1\leq i\leq n}\left(U_{(i)} - \frac{i-1}{n}\right) + \max_{1\leq i\leq n}\left(\frac{i}{n} - U_{(i)}\right)\right\}$$

The lowest value of any statistic gives the most appropriate distribution.

The test results were as shown in table 3 Real data test results

Distribution	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	K
Spherical Dirichlet	**0.975**	**0.998**	**0.897**	**0.036**
SE	**0.006**	**0.005**	**0.006**	
Dirichlet	**1.788**	**2.346**	**2.547**	**0.200**
SE	**0.235**	**0.895**	**0.459**	

It is clear from Table 2 that the spherical Dirichlet distribution was more suitable for the real data than the Dirichlet distribution with the lowest values according to the comparison criterion adopted in this paper, which is the Kuiper criterion.

14. Data analysis

We mentioned earlier that the results of the simulation experiments showed that the best method for estimating the parameters of the spherical Dirichlet distribution is the Bayes method at the DeGroot loss function, so this method will be applied to the real data to extract the values of the probability density function of the distribution for the purpose of finding the probabilities of myopia. shows the probability values of the real data and the estimated probability values corresponding to the simulation experiments at the values of the variables x, y, z.

Table (4) real data and the estimated probability values corresponding to the simulation experiments at the values of the variables x, y, z.

x	y	z	f(x)	$\hat{f}(x)$	x	y	z	f(x)	$\hat{f}(x)$
0.162	0.045	0.793	0.003	0.010	0.429	0.337	0.234	0.311	0.318
0.096	0.375	0.529	0.285	0.692	0.085	0.021	0.895	0.084	0.091
0.089	0.441	0.470	0.419	0.426	0.413	0.411	0.176	0.336	0.343
0.928	0.048	0.024	0.493	0.500	0.452	0.087	0.461	0.261	0.268
0.284	0.350	0.366	0.010	0.017	0.805	0.113	0.083	0.048	0.055
0.520	0.431	0.049	0.922	0.929	0.305	0.340	0.355	0.107	0.114
0.408	0.210	0.382	0.928	0.935	0.169	0.728	0.103	0.876	0.883
0.571	0.426	0.004	0.257	0.764	0.275	0.360	0.366	0.975	0.982
0.416	0.153	0.431	0.961	0.968	0.030	0.038	0.932	0.155	0.162
0.030	0.332	0.638	0.802	0.809	0.376	0.214	0.410	0.596	0.603
0.380	0.139	0.481	0.000	0.007	0.554	0.194	0.252	0.616	0.623
0.345	0.491	0.164	0.128	0.135	0.388	0.333	0.280	0.795	0.802
0.201	0.355	0.445	0.350	0.357	0.106	0.427	0.466	0.420	0.427

while Figure 4 represents the real probability density function estimated by the Bayes method under the DeGroot loss function for the three-dimensional spherical Dirichlet distribution.

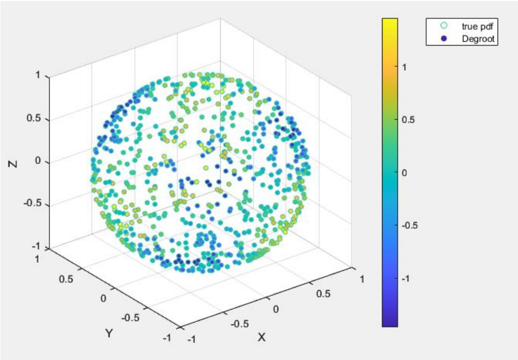

Fig. (4) The curve of the actual and estimated Bayesian probability density function under the DeGroot loss function for the three-dimensional spherical Dirichlet distribution

Where

X represents: the average direction of the orientation of the eye structures (Orientation of Eye Structures (OES))

Y represents: the average direction of the optic disc (Optic Disc Orientation (ODO)

Z represents: the average direction of the peripheral refraction (Peripheral Refraction (PR)

We note from Table 2 and Figure 4 that; the values of the probability density function estimated by the Bayes method under the DeGroot loss function are consistent with the values of the true probability density function for the three-dimensional spherical Dirichlet distribution.

a. The estimated probability density function values by Bayes method at the DeGroot loss function are consistent with the true probability density function values of the three-dimensional spherical Dirichlet distribution.
b. The variables Orientation of Eye Structures (OES), Optic Disc Orientation (ODO), and Peripheral Refraction (PR) contributed to the modeling of myopia represented by the probabilities resulting from the spherical Dirichlet distribution. Changing any of the three variables towards an increase or decrease affects the probability of developing myopia.
c. When the degree of the variable OES increases and each of the variable's ODO and PR decreases, the probability of developing myopia increases.
d. When the degree of the variable ODO increases and each of the variable's OES and PR decreases, the probability of developing myopia decreases.
e. When the degree of the variable PR increases and each of the variable's OES and ODO decreases, the probability of developing myopia decreases.
f. Increasing the degree of each of the variable's OES, ODO and PR leads to an increase in the probability of developing myopia.
g. Increasing the degree of each of the variable's OES and ODO and decreasing the degree of PR leads to a decrease in the probability of developing myopia.
h. When the degree of each of the variable's OES, ODO and PR approaches 0.50 on average, the probability of developing myopia approaches 0.50.

15. Conclusions

Through what was presented in the previous chapters, we can point out some of the conclusions that were reached and the recommendations that we believe are necessary.

a. The values of the probability density function estimated by the Bayes method at the DeGroot loss function are consistent with the values of the density function resulting from simulation experiments for the three-dimensional spherical Dirichlet distribution in terms of the positive and significant correlation coefficient between them.

b. When the average degree of infection for the variables OES, ODO and PR approaches 0.50 on average, the probability of developing myopia approaches 0.50.
c. Increasing the degree of the variables OES, ODO and PR leads to an increase in the probability of developing myopia.
d. Increasing the degree of the variables OES and ODO and decreasing the degree of PR leads to a decrease in the probability of developing myopia.

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