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Enhancing the Cosine-Lomax Distribution: A New Exponentiated Version with Improved Flexibility and Real-World Applications

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ABSTRACT

In this paper, we introduce a novel flexible probability distribution called the Exponentiated Cosine Lomax distribution (ECLD), developed by compounding the exponentiated family with the cosine Lomax distribution. The proposed model incorporates an additional shape parameter, enhancing its flexibility to model complex real-world data with heavy tails, skewness, and non-monotonic hazard rates. We derive key statistical properties of the ECLD, including moments, moment-generating function, quantile function, and hazard rate. The model parameters are estimated using the maximum likelihood estimation (MLE) and maximum product of spacings (MPS) methods. A comprehensive simulation study is conducted to assess the consistency and efficiency of the estimators. To demonstrate the practical applicability of the ECLD, we analyze two real-world datasets. Comparative studies with existing models, including the Odd Frechet Lomax, half logistic Lomax, cosine Lomax, Lomax and sine Lomax distributions, reveal that the proposed ECLD provides a significantly better fit based on goodness-of-fit criteria such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Kolmogorov-Smirnov (K-S) test. The findings suggest that the ECL distribution is a robust alternative for modeling skewed and heavy-tailed data in various fields.

1. Introduction

Probability distributions play a pivotal role in statistical modeling, providing the mathematical foundation for describing real-world phenomena across various disciplines, including engineering, finance, environmental science, and survival analysis. The choice of an appropriate probability distribution is crucial, as it directly influences the accuracy of data analysis, parameter estimation, and predictive performance. Over the years, researchers have developed numerous flexible distributions to capture complex data behaviors such as skewness, heavy tails, and multimodality, which classical distributions often fail to adequately model.

Recently, trigonometric transformations have gained significant attention in the statistical literature for their ability to introduce additional flexibility into existing distributions. By incorporating sine and cosine functions into the structure of traditional distributions, new flexible models have emerged, demonstrating superior fit in various applications. Examples include the sine half-logistic inverse Rayleigh distribution [1], the ArcTan Lomax distribution [2], the sine exponential distribution [3], the type-I cosine exponentiated Weibull distribution [4], the new sine inverse Rayleigh

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distribution [5], the Arc-tangent exponential distribution [6], the sine Lomax distribution [7], the sine Weibull distribution [8], the cosine pie-power odd Weibull distribution [9], the cosine Gompertz distribution [10], the sine power Rayleigh distribution [11], the sine type II Topp-Leone exponential distribution [12], the sine Lomax-exponential distribution [13], the sine inverted exponentiated Weibull distribution [14], the sine Topp-Leone exponentiated exponential distribution [15], and the cosine inverse Lomax exponentiated Weibull distribution [16]. However, many of these trigonometric based distributions still exhibit limitations in modeling datasets with extreme skewness or varying tail behaviors, often requiring additional parameters or structural modifications to improve their adaptability.

The cosine Lomax distribution [17] represents progress in this direction, offering improved modeling of heavy-tailed and right-skewed data in reliability and survival analysis. However, it still falls short when dealing with more complex data characteristics, especially in cases of strong asymmetry or heavier tails. To address this limitation, the exponentiated cosine Lomax distribution (ECLD) is proposed. By introducing an additional shape parameter through the exponentiated family, the ECLD offers greater flexibility, allowing it to accommodate a wider range of tail behaviors and skewed data patterns, while maintaining mathematical tractability.

In this paper, we derive key statistical properties of the ECLD, including its moments, moment-generating function, quantile function, and hazard rate function. We estimate the model parameters using both the maximum likelihood estimation (MLE) and maximum product of spacings (MPS) methods, followed by a simulation study to assess the consistency and efficiency of the estimators. To demonstrate the practical utility of the proposed model, we apply it to two real-world datasets and compare its performance with existing distributions, including the traditional Lomax, cosine Lomax, and other competing models. Our results show that the ECLD provides a superior fit based on goodness-of-fit criteria, reinforcing its potential as a valuable tool in statistical modeling.

2. Development of the Exponentiated Cosine Lomax Distribution

The cumulative distribution function (CDF) and probability density function (PDF) of the exponentiated-G family of distributions introduced by [18] are given by:

$$F(x) = [G(x)]^{\lambda} \tag{1}$$

$$f(x) = \lambda g(x)[G(x)]^{\lambda-1} \tag{2}$$

where $\lambda > 0$ is a shape parameter, and $g(x)\ and\ G(x)$ are the CDF and PDF of the baseline distribution, respectively.

The CDF and PDF of the cosine Lomax distribution are given by:

$$G(x) = 1 - cos\left[\frac{\pi}{2}\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]\right] \tag{3}$$

$$g(x) = \frac{\pi}{2}\frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)}\ sin\left[\frac{\pi}{2}\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]\right] \tag{4}$$

where $\alpha > 0$ is a shape parameter and $\beta > 0$ is a scale parameter. By substituting equations (3) and (4) into (1) and (2), respectively, we obtain the CDF and PDF of the new exponentiated cosine Lomax distribution (ECLD) as:

$$F(x) = \left[1 - cos\left[\frac{\pi}{2}\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{\lambda} \tag{5}$$

$$f(x) = \frac{\pi}{2}\frac{\alpha\lambda}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)}\ sin\left[\frac{\pi}{2}\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]\right]\ \left[1 - cos\left[\frac{\pi}{2}\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{\lambda-1} \tag{6}$$

Figure 1 displays the PDF plot of the ECLD, illustrating its right-skewed behaviour.

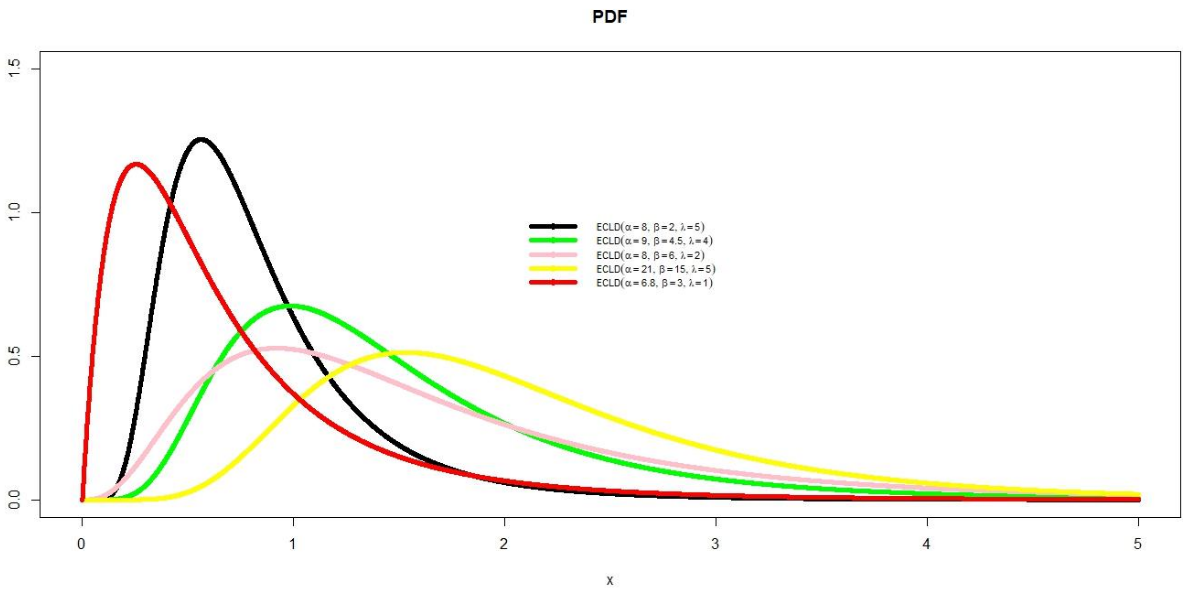

Figure 1: PDF plot of the ECLD

Survival Function

The survival function gives the probability that a random variable exceeds a specific value. For the ECLD, the survival function is given by:

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \left[1 - cos\left[\frac{\pi}{2}\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{\lambda} \tag{7}$$

Hazard Function

The hazard function is a fundamental concept in survival analysis that describes the instantaneous risk of an event occurring at time *x*, given that the event has not yet occurred up to that point. For the ECLD, the hazard function is obtained as:

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\frac{\pi}{2}\frac{\alpha\lambda}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} sin\left[\frac{\pi}{2}\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]\right] \left[1 - cos\left[\frac{\pi}{2}\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{\lambda-1}}{1 - \left[1 - cos\left[\frac{\pi}{2}\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{\lambda}} \tag{8}$$

Figure 2 displays the hazard function plot of the ECLD, revealing an upside-down bathtub-shaped (unimodal) failure rate pattern.

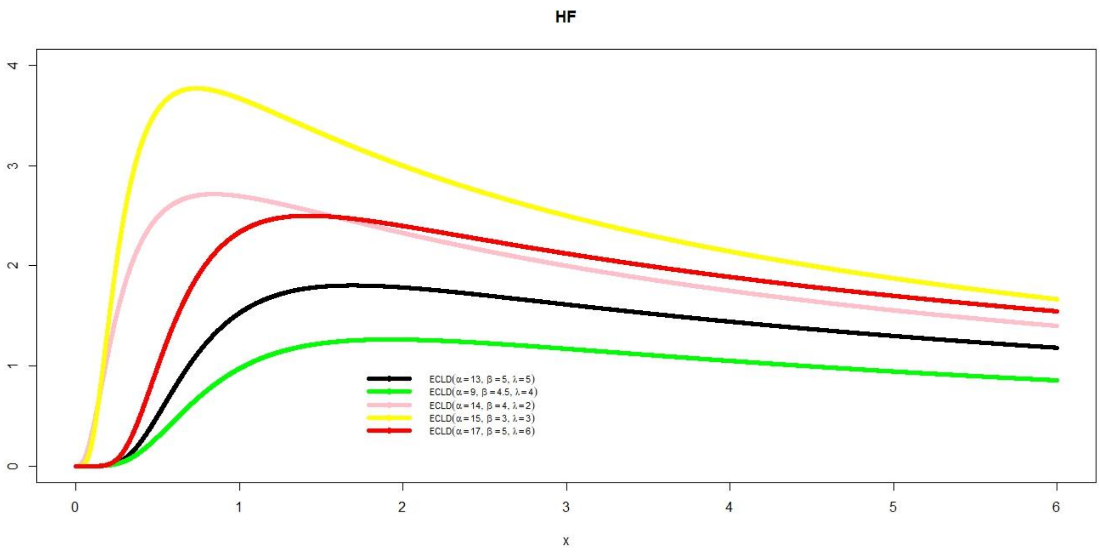

Figure 2: Hazard function plot of ECLD

Reverse Hazard Function

The reverse hazard function of ECLD is given by:

$$rh(x) = \frac{f(x)}{F(x)}$$

$$rh(x) = \frac{\pi}{2}\frac{\alpha\lambda}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} \; sin\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right] \; \left[1-cos\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{-1} \qquad (9)$$

Cumulative Hazard Function

The cumulative hazard function of the ECLD is given by:

$$H(x) = -ln\left\{1-\left[1-cos\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{\lambda}\right\} \qquad (10)$$

Quantile Function

The quantile function of the ECLD is given by:

$$\phi(u) = \beta\left\{\left[1-\frac{\cos^{-1}\left(1-u^{\frac{1}{\lambda}}\right)}{\pi/2}\right]^{-\left(\frac{1}{\alpha}\right)} - 1\right\} \qquad (11)$$

3. Mathematical Properties

3.1 Linear expansion of the PDF

The PDF of the ECLD cab be expanded using power series expansion as follows:

$$f(x) = \frac{\pi}{2}\frac{\alpha\lambda}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} \; sin\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right] \; \left[1-cos\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{\lambda-1}$$

Applying binomial expansion to the last term of the pdf gives:

$$\left[1-cos\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{\lambda-1} = \sum_{i=0}^{\infty}(-1)^{i}\binom{\lambda-1}{i}\left[cos\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^{i}$$

Therefore, equation (6) becomes:

$$f(x)=\frac{\pi}{2}\frac{\alpha\lambda}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} sin\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\sum_{i=0}^{\infty}(-1)^i\binom{\lambda-1}{i}\left[cos\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^i \quad (12)$$

The last part of equation (12) can be expanded using Taylor Series expansion as follows:

$$\left[cos\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\right]^i=\sum_{j=0}^{\infty}\frac{(-1)^j}{2j!}\left(\frac{\pi}{2}\right)^{2ij}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{2ji}$$

Therefore, equation (12) becomes:

$$f(x)=\frac{\pi}{2}\frac{\alpha\lambda}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} sin\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{2j!}\left(\frac{\pi}{2}\right)^{2ij}\binom{\lambda-1}{i}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{2ji} \quad (13)$$

Applying Taylor series expansion to the sine term gives;

$$sin\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{2k+1}$$

Therefore, equation (13) becomes:

$$f(x)=\frac{\pi}{2}\frac{\alpha\lambda}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}}{2j!\,(2k+1)!}\left(\frac{\pi}{2}\right)^{2ij+2k+1}\binom{\lambda-1}{i}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{2ji+2k+1} \quad (14)$$

Applying binomial expansion to the last term of equation (14);

$$\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{2ji+2k+1}=\sum_{l=0}^{\infty}(-1)^l\binom{2ji+2k+1}{l}\left(1+\frac{x}{\beta}\right)^{-\alpha l}$$

Therefore, equation (14) becomes:

$$f(x)=\sum_{i,j,k,l=0}^{\infty}\frac{\alpha\lambda}{\beta}\frac{(-1)^{i+j+k+l}}{2j!\,(2k+1)!}\left(\frac{\pi}{2}\right)^{2ij+2k+2}\binom{\lambda-1}{i}\binom{2k+2ji+1}{l}\left(1+\frac{x}{\beta}\right)^{-(\alpha l+\alpha+1)}$$

$$Let\ \Psi_{i,j,k,l}=\frac{\alpha\lambda}{\beta}\frac{(-1)^{i+j+k+l}}{2j!\,(2k+1)!}\left(\frac{\pi}{2}\right)^{2(ij+k+1)}\binom{\lambda-1}{i}\binom{2ij+2k+1}{l}$$

Therefore,

$$f(x)=\sum_{i,j,k,l=0}^{\infty}\Psi_{i,j,k,l}\frac{\alpha\lambda}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha l+\alpha+1)} \quad (15)$$

The expression in equation (15) is the reduced form of the PDF of the ECLD.

3.2 Moments

The r^{th} raw moment μ_r' of the ECLD is defined as:

$$\mu_r'=E(x^r)=\int_{-\infty}^{\infty}x^r f(x)dx$$

$$\mu_r'=\sum_{i,j,k,l=0}^{\infty}\Psi_{i,j,k,l}\frac{\alpha\lambda}{\beta}\int_0^{\infty}x^r\left(1+\frac{x}{\beta}\right)^{-(\alpha l+\alpha+1)}dx$$

$Let\ p=\alpha l+\alpha+1$

And $y=\frac{x}{\beta}\implies\frac{dy}{dx}=\frac{1}{\beta}\ and\ \ dx=\beta dy$

Substituting back into the integral we have:

$$\int_0^{\infty}(\beta y)^r(1+y)^{-p}\beta dy=\beta^{r+1}\int_0^{\infty}(y)^r(1+y)^{-p}dy$$

$$=\beta^{r+1}\int_0^{\infty}\frac{y^{(r+1)-1}}{(1+y)^{(r+1)+p-r-1}}dy$$

$$=\beta^{r+1}B(r+1,p-r-1)$$

$$\therefore\ \mu_r'=\sum_{i,j,k,l=0}^{\infty}\Psi_{i,j,k,l}\,\alpha\lambda\beta^r\,B(r+1,p-r-1) \quad (16)$$

3.3 Moment Generating Function (MGF)

The moment generating function is defined as the expected value of the exponential function of a random variable. It provides a summary of the distribution and can be used to obtain all the moments of the distribution by differentiation. For the ECLD, the mgf is given by:

$$M_x(x) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$M_x(x) = \sum_{i,j,k,l=0}^{\infty} \Psi_{i,j,k,l} \frac{\alpha\lambda}{\beta} \int_0^{\infty} e^{tx}\left(1+\frac{x}{\beta}\right)^{-p} dx$$

$$Let\ u = 1 + \frac{x}{\beta} \Rightarrow x = \beta(u-1)$$

As $x \longrightarrow 0,\ \ u \longrightarrow 1$

As $x \longrightarrow \infty,\ \ u \longrightarrow \infty$

$$\frac{du}{dx} = \frac{1}{\beta} \Rightarrow dx = \beta du$$

$$\int_{-\infty}^{\infty} e^{t\lambda(u-1)} u^{-p} \beta du$$

For the integral to converge, we set t<0, that is $t\lambda = -v$ so that, the integral becomes:

$$\beta e^{-t\beta} \int_{-\infty}^{\infty} e^{-vu} u^{-p} du = \beta e^{-t\beta} v^{p-1} \Gamma(1-p, v)$$

Substituting $-t\beta = v$, we have

$$\beta e^{-t\beta} (-t\beta)^{p-1} \Gamma(1-p, v)$$

Therefore, the moment generating function of the ECLD is given by:

$$M_x(t) = \sum_{i,j,k,l=0}^{\infty} \Psi_{i,j,k,l} \alpha\lambda e^{-t\beta} \left(-t\beta\right)^{p-1} \Gamma(1-p,v) \quad (17)$$

4. Parameter Estimation

This section discusses the two methods used to estimate the parameters of the ECLD.

4.1 Maximum Likelihood (ML) Method

Let $x_1, x_2, \dots, x_n$ be a random sample of size n from the ECLD. The log-likelihood function is given as:

$$log\ell = nlog\lambda + nlog\frac{\pi}{2} + nlog\alpha + nlog\frac{1}{\beta} - \alpha\sum_{i=1}^{n} log\left(1+\frac{x}{\beta}\right) + \sum_{i=1}^{n} log\, sin\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right] + (\lambda - 1)\sum_{i=1}^{n} log\left[1 - cos\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\right] \quad (18)$$

The partial derivatives of the log-likelihood function with respect to the parameters are:

$$\frac{\partial log\ell}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} log\left[1 - cos\left[\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\right]\right] \quad (19)$$

$$\frac{\partial log\ell}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} log\left(1+\frac{x}{\beta}\right) + \sum_{i=1}^{n} cot\left\{\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\frac{\pi}{2}\alpha\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\right\} + (\lambda-1)\sum_{i=1}^{n}\frac{\pi}{2}\alpha\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} \quad (20)$$

$$\frac{\partial log\ell}{\partial\beta} = \sum_{i=1}^{n} \frac{\frac{\alpha x}{\beta^2}}{\left(1+\frac{x}{\beta}\right)} - \frac{n}{\beta} - \sum_{i=1}^{n} cot\left\{\frac{\pi}{2}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]\frac{\pi}{2}\frac{\alpha x}{\beta^2}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\right\} - (\lambda-1)\sum_{i=1}^{n}\frac{\pi}{2}\frac{\alpha x}{\beta^2}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} \quad (21)$$

The maximum likelihood estimates (MLEs) of the parameters (λ, α, β) can be obtained by solving the system of equations formed by setting the above derivatives to zero.

4.2 Maximum Product of Spacings Method

The Maximum Product of Spacings (MPS) method provides an alternative to the MLE, particularly effective when dealing with small samples or heavy-tailed distributions. The MPS estimation is based on maximizing the geometric mean of the spacings between successive cumulative distribution function (cdf) values. The MPS method has been successfully applied in parameter estimation by several authors. For instance, it has been employed by [19], [20], [21], and [22] in estimating the parameters of different statistical models. To obtain the estimates of the parameters of the ECLD using this technique, the following function must be optimized:

$$MPS = \frac{1}{n+1}\sum_{i=1}^{n+1}\log I_i(x_i)$$

where,

$$I_i(x_i) = F(x_{i:n}) - F(x_{i-1:n})$$

$F(x_{0:n})=0$ and $F(x_{n+1:n})=1$.

5. Simulation Study

This section presents a simulation study conducted to assess the performance of the Maximum Likelihood Estimation (MLE) and Maximum Product of Spacings (MPS) methods in estimating the parameters of the Exponentiated Cosine Lomax Distribution (ECLD). Two parameter settings were considered: ($\beta = 2, \alpha = 2, \lambda = 1.5$) and ($\beta = 0.5, \alpha = 0.7, \lambda = 0.5$). For each setting, samples were generated for various sample sizes, specifically n = 20, 50, 70, 100, 150, 200, 250, 300, 350 and 400.

For each combination of parameter values and sample size, 10,000 random samples were generated. The performance of the estimators was evaluated based on the mean estimates, bias, and mean squared error (MSE) of the estimated parameters. This simulation aims to investigate the consistency and accuracy of the MLE and MPS methods as the sample size increases. Tables 1 and 2 present the results for the first parameter set ($\beta = 2, \alpha = 2, \lambda = 1.5$) using the MLE and MPS methods, respectively. Tables 3 and 4 report the results for the second parameter set ($\beta = 0.5, \alpha = 0.7, \lambda = 0.5$), also using MLE and MPS, respectively.

Table 1: Simulation results using MLE for $\beta = 2, \alpha = 2, \lambda = 1.5$

n	Properties	MLE		
		β	α	λ
20	**Mean**	2.3854	2.4143	2.1058
	Bias	1.3240	0.7908	0.8620
	MSE	2.1480	0.9836	1.0681
50	**Mean**	2.2371	2.1929	1.8973
	Bias	1.1576	0.5695	0.6307
	MSE	1.7513	0.5094	0.6874
70	**Mean**	2.2463	2.1520	1.8020
	Bias	1.0587	0.4792	0.5550
	MSE	1.5304	0.3716	0.5493
100	**Mean**	2.2689	2.1487	1.7251
	Bias	1.0005	0.4489	0.4741
	MSE	1.4114	0.3249	0.4291
150	**Mean**	2.1853	2.0941	1.6583
	Bias	0.8490	0.3743	0.3810
	MSE	1.0797	0.2290	0.2833
200	**Mean**	2.1407	2.0814	1.6506
	Bias	0.7578	0.3287	0.3535
	MSE	0.8975	0.1770	0.2462
250	**Mean**	2.1225	2.0619	1.6108
	Bias	0.6935	0.3024	0.3020
	MSE	0.7580	0.1476	0.1781
300	**Mean**	2.1366	2.0751	1.5861
	Bias	0.6288	0.2752	0.2636
	MSE	0.6649	0.1319	0.1346
350	**Mean**	2.1556	2.0774	1.5583
	Bias	0.5971	0.2603	0.2396
	MSE	0.6008	0.1196	0.1071
400	**Mean**	2.0954	2.0506	1.5718
	Bias	0.5489	0.2402	0.2287
	MSE	0.5070	0.0973	0.0976

Table 2: Simulation results using MPS for $\beta = 2, \alpha = 2, \lambda = 1.5$

n	Properties	MPS		
		β	α	λ
20	**Mean**	2.2122	1.9591	1.7829
	Bias	1.3022	0.6510	0.7932
	MSE	2.1420	0.6314	0.9020
50	**Mean**	2.1589	1.9611	1.6669
	Bias	1.0611	0.5037	0.5601
	MSE	1.6248	0.3668	0.5322
70	**Mean**	2.1788	1.9699	1.6105
	Bias	0.9497	0.4299	0.4877
	MSE	1.3830	0.2770	0.4141
100	**Mean**	2.2234	2.0112	1.5761
	Bias	0.8869	0.4029	0.4319
	MSE	1.2696	0.2462	0.3377
150	**Mean**	2.1494	1.9946	1.5424
	Bias	0.7315	0.3392	0.3392
	MSE	0.9656	0.1848	0.2116
200	**Mean**	2.1119	2.0022	1.5543
	Bias	0.6382	0.2982	0.3100
	MSE	0.7878	0.1428	0.1845
250	**Mean**	2.1099	1.9999	1.5284
	Bias	0.5759	0.2744	0.2725
	MSE	0.6657	0.1195	0.1384
300	**Mean**	2.1166	2.0180	1.5165
	Bias	0.4913	0.2433	0.2319
	MSE	0.5750	0.1065	0.1025
350	**Mean**	2.1234	2.0213	1.5028
	Bias	0.4640	0.2313	0.2141
	MSE	0.5155	0.0968	0.0828
400	**Mean**	2.0735	2.0036	1.5183
	Bias	0.4146	0.2118	0.2035
	MSE	0.4247	0.0786	0.0750

Table 3: Simulation results using MLE for $\beta = 0.5, \alpha = 0.7, \lambda = 0.5$

n	Properties	MLE		
		β	α	λ
20	**Mean**	1.1035	1.0766	1.1806
	Bias	0.9738	0.4815	0.7828
	MSE	2.2555	0.7211	1.6195
50	**Mean**	0.7224	0.8153	0.7386
	Bias	0.4945	0.2174	0.3189
	MSE	0.6736	0.1294	0.4631
70	**Mean**	0.6404	0.7668	0.6396
	Bias	0.3644	0.1551	0.2152
	MSE	0.3331	0.0586	0.2373
100	**Mean**	0.5983	0.7462	0.5816
	Bias	0.2899	0.2899	0.1518
	MSE	0.1785	0.0327	0.1216
150	**Mean**	0.5590	0.7266	0.5350
	Bias	0.2205	0.0971	0.0936
	MSE	0.0949	0.0184	0.0343
200	**Mean**	0.5338	0.7208	0.5266
	Bias	0.1801	0.0800	0.0778
	MSE	0.0586	0.0111	0.0124
250	**Mean**	0.5246	0.7127	0.5196
	Bias	0.1617	0.0716	0.0667
	MSE	0.0449	0.0086	0.0081

300	**Mean**	0.5289	0.7177	0.5155
	Bias	0.1491	0.0664	0.0591
	MSE	0.0409	0.0079	0.0061
350	**Mean**	0.5314	0.7158	0.5096
	Bias	0.1367	0.0626	0.0526
	MSE	0.0331	0.0073	0.0047
400	**Mean**	0.5195	0.7112	0.5123
	Bias	0.1252	0.0566	0.0504
	MSE	0.0263	0.0052	0.0043

Table 4: Simulation results using MPS for $\beta = 0.5, \alpha = 0.7, \lambda = 0.5$

$\boldsymbol{n}$	**Properties**	**MPS**		
		$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\lambda}$
20	**Mean**	1.0303	0.8249	0.8207
	Bias	0.8447	0.3346	0.4987
	MSE	1.8089	0.3308	0.8499
50	**Mean**	0.7113	0.7190	0.5789
	Bias	0.4394	0.1817	0.2053
	MSE	0.5276	0.0734	0.2155
70	**Mean**	0.6413	0.6986	0.5334
	Bias	0.3297	0.1346	0.1483
	MSE	0.2567	0.0358	0.1001
100	**Mean**	0.6083	0.6991	0.5124
	Bias	0.2724	0.1140	0.1140
	MSE	0.1535	0.0229	0.0518
150	**Mean**	0.5696	0.6938	0.4983
	Bias	0.2109	0.0923	0.0817
	MSE	0.0878	0.0148	0.0226
200	**Mean**	0.5434	0.6952	0.4991
	Bias	0.1730	0.0760	0.0704
	MSE	0.0542	0.0093	0.0084
250	**Mean**	0.5341	0.6921	0.4974
	Bias	0.1554	0.0699	0.0614
	MSE	0.0425	0.0076	0.0062
300	**Mean**	0.5377	0.7001	0.4967
	Bias	0.1439	0.0637	0.0548
	MSE	0.0393	0.0069	0.0049
350	**Mean**	0.5383	0.7008	0.4938
	Bias	0.1323	0.0594	0.0503
	MSE	0.0317	0.0060	0.0040
400	**Mean**	0.5267	0.6974	0.4977
	Bias	0.1210	0.0551	0.0477
	MSE	0.0254	0.0047	0.0036

As expected, both estimation methods improve with increasing sample size, with MPS generally yielding lower bias and MSE across both parameter settings. This confirms the consistency of the estimators and highlights the potential of the MPS method in providing more efficient estimates in small to moderate sample scenarios.

6 Real Life application

This section illustrates the practical utility of the proposed Exponentiated Cosine Lomax Distribution (ECLD) by applying it to two real-world datasets. To assess its empirical performance, the ECLD is compared with several existing models, including the Odd Fréchet Lomax Distribution (OFLD) [23], Half Logistic Lomax Distribution (HLD) [24], Cosine Lomax Distribution (CLD) [17], Lomax Distribution (LD) [25], and Sine Lomax Distribution (SLD) [7]. The evaluation is conducted using several well-known statistical criteria: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (CAIC), Hannan–Quinn Information Criterion (HQIC), Kolmogorov–Smirnov (KS) statistic, and the associated p-value. In this context, lower values of AIC, BIC, CAIC, HQIC, and KS indicate better model fit, while a higher KS p-value suggests stronger agreement between the empirical and theoretical distributions.

First Data Set:

The first dataset, originally presented by [26], contains 30 observations of March precipitation (in inches). The data values are: 0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05

The estimated parameters and goodness-of-fit measures for each competing model are summarized in Table 5. The fitted probability density function (PDF) and cumulative distribution function (CDF) plots of each model for the first dataset are shown in Figures 3 and 4, respectively.

Table 5: Goodness-of-fit statistics for the first dataset

MODEL	MLE	LL	AIC	BIC	CAIC	HQIC	KS	P value
ECLD	$\lambda = 1.83900$ $\alpha = 1101002$ $\beta = 1002747$	−38.1214	82.2428	86.4463	83.1658	83.5875	0.0709	0.9982
OFLD	$\alpha = 4603074$ $\theta = 1.104300$ $\lambda = 7080313$	−38.9625	83.9251	88.1286	84.8481	85.2698	0.1257	0.7303
HLD	$\alpha = 1719.632$ $\beta = 0.000505$	−42.5397	89.0794	91.8818	89.5238	89.9759	0.1894	0.2323
CLD	α= 15891091 β= 19928877	-39.8980	83.7960	86.5984	84.2404	84.6925	0.1247	0.7398
LD	$\alpha = 31566927$ $\beta = 52871827$	−45.4744	94.9487	97.7512	95.3932	95.8453	0.2352	0.0723
SLD	$\alpha = 15568963$ $\lambda = 45824487$	−44.3714	92.7428	95.5452	93.1873	93.6393	0.2204	0.1084

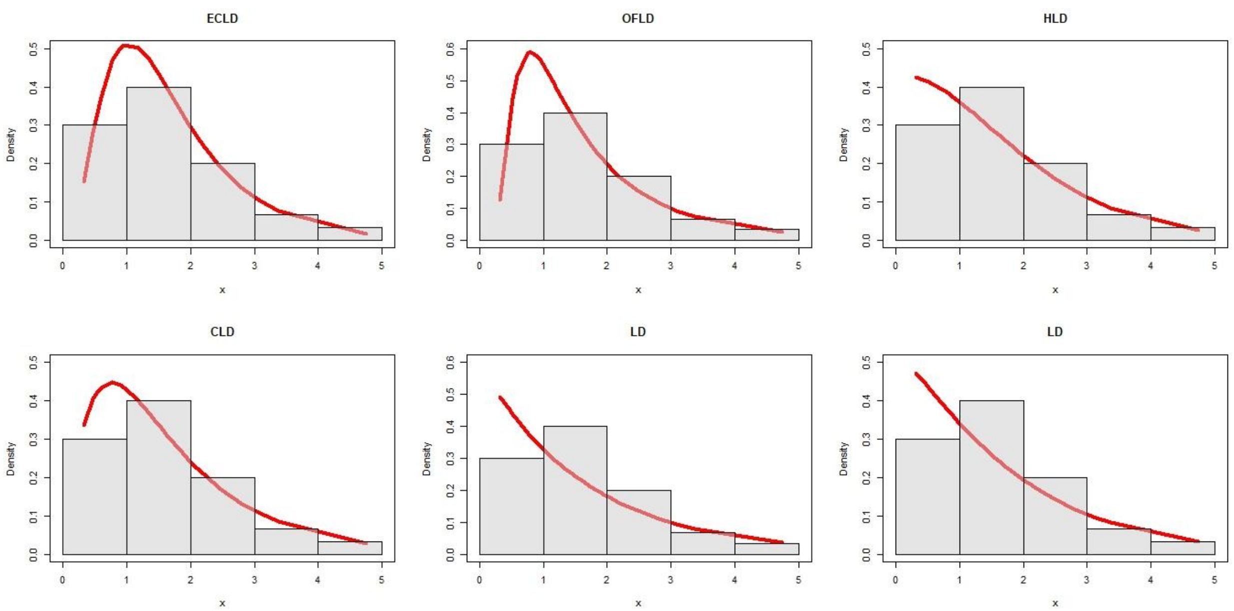

Figure 3: Fitted PDF plots of each model for the first dataset

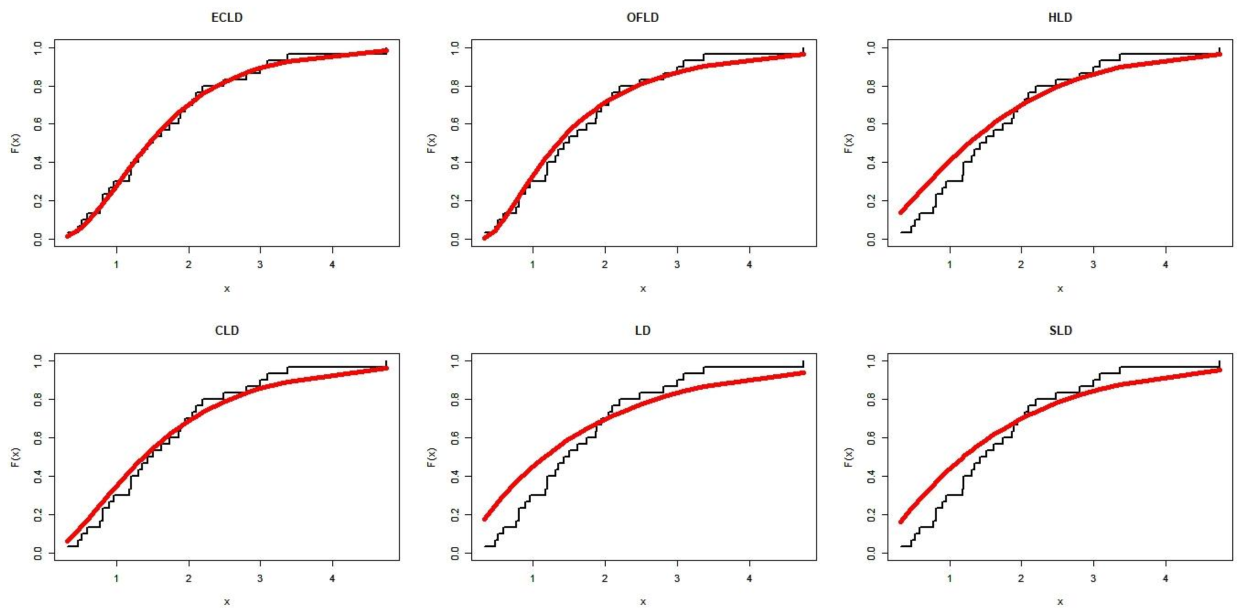

Figure 4: Fitted CDF plots of each model for the first dataset

Among all the considered models, the ECLD attained the lowest AIC, BIC, CAIC, and HQIC values, alongside the smallest KS statistic and highest p-value (0.9982), indicating an excellent fit to the precipitation data. Moreover, the fitted PDF and CDF plots clearly show that the ECLD aligns more closely with the empirical distribution than the competing models. This demonstrates the ECLD's superior ability to model the dataset compared to the other distributions.

Second Dataset:

The second dataset, discussed in [27], contains annual maximum flood discharges (in 1000 cubic feet per second) of the North Saskatchewan River at Edmonton over a period of 47 years. The data values are:

19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560

The estimated parameters and model selection criteria are presented in Table 6. The fitted probability density function (PDF) and cumulative distribution function (CDF) plots of each model for the second dataset are shown in Figures 5 and 6, respectively.

Table 6: Goodness-of-fit statistics for the second dataset

MODEL	MLE	LL	AIC	BIC	$CAIC$	$HQIC$	KS	$P\ value$
ECLD	$\lambda = 165.379$ $\alpha = 2.78576$ $\beta = 5.5953$	-215.216	436.433	442.046	436.978	438.554	0.0678	0.9799
OFLD	$\alpha = 0.3028$ $\theta = 4.5887$ $\lambda = 3.9169$	-215.327	436.6556	442.2693	437.2011	438.7770	0.0728	0.9610
HLD	$\alpha = 2492.052$ $\beta = 1.15e-05$	-232.251	468.5020	472.2444	468.7687	469.9163	0.2777	0.0012
CLD	$\alpha = 1230742$ $\beta = 47577708$	-226.3552	456.7105	460.4529	456.9771	458.1247	0.1929	0.0561

LD	$\alpha = 6153332$ $\beta = 316955309$	-237.1914	478.3829	482.1253	478.6496	479.7971	0.3202	0.0001
SLD	$\alpha = 2081078$ $\beta = 187327056$	-235.3074	474.6149	478.3573	474.8816	476.0292	0.3063	0.0002

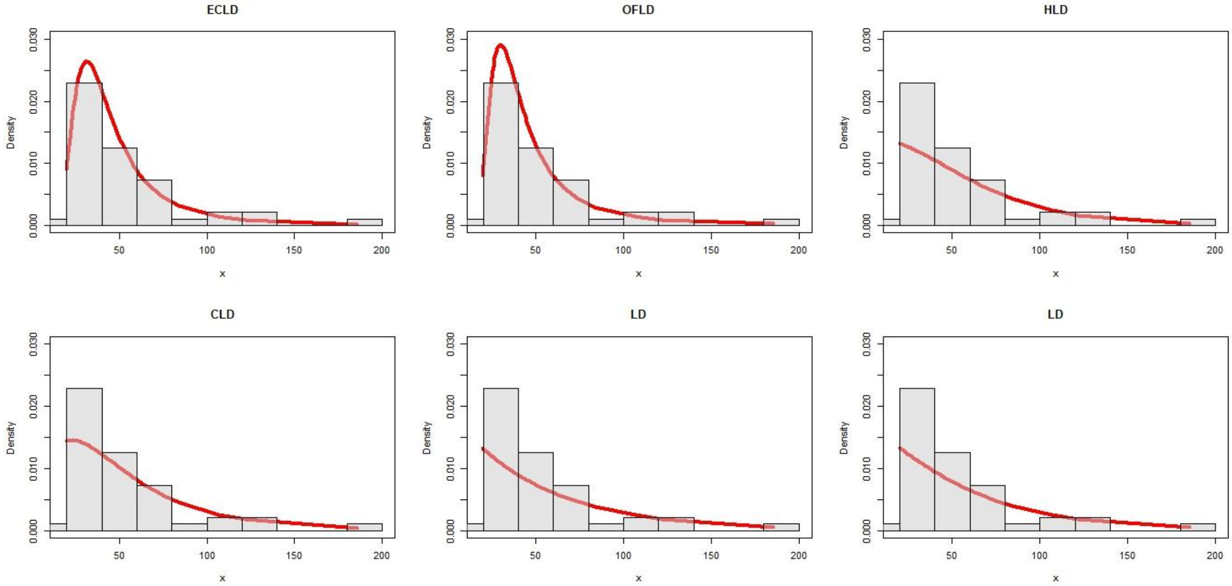

Figure 5: Fitted PDF plots of each model for the second dataset

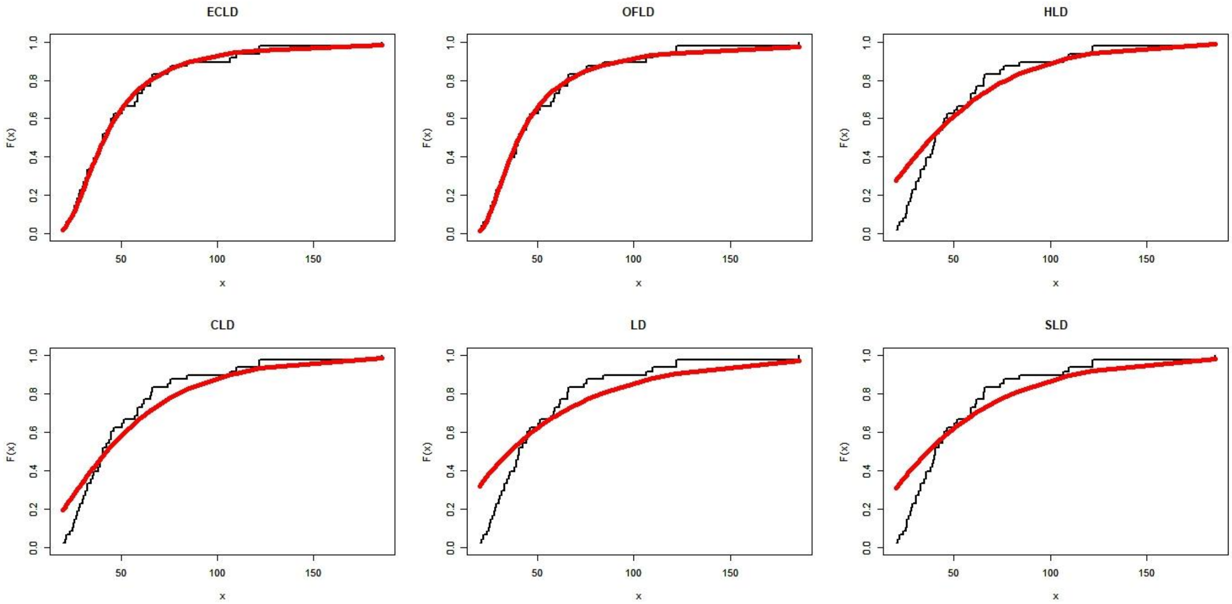

Figure 6 Fitted CDF plots of each model for the second dataset

In the case of the second dataset, the ECLD achieved the lowest values across all information criteria, the smallest KS statistic, and the highest p-value (0.9799), suggesting the best fit among all considered models. This conclusion is further supported by the PDF and CDF plots, where the ECLD shows the closest alignment with the observed data. Thus, the ECLD exhibits superior ability to model the dataset compared to the competing models.

7. Conclusion

In this study, we introduced a new probability distribution, the Exponentiated Cosine Lomax distribution (ECLD), by combining the exponentiated family with the cosine Lomax distribution. The proposed model offers greater flexibility in modeling skewed, heavy-tailed, and complex real-world data. We derived essential statistical properties, including moments, moment-generating function, and hazard rate, which are crucial for reliability analysis and risk assessment. The parameters of the ECLD were estimated using both the maximum likelihood estimation (MLE) and maximum product of spacings (MPS) methods, and a simulation study confirmed the consistency and efficiency of the estimators. The practical applicability of the ECLD was demonstrated through two real-world datasets, where it outperformed several competing models, including the standard Lomax, cosine Lomax, odd Fretchet Lomax, half logistic Lomax and sine Lomax distributions, based on goodness-of-fit measures. The superior performance of the ECLD suggests its potential as a valuable tool in actuarial science, survival analysis, and reliability engineering. Future research may explore Bayesian estimation methods, regression modeling based on the ECLD, and its application in other domains such as finance and engineering. Additionally, bivariate or multivariate extensions of the ECLD could be investigated to model dependent data structures.

Authors' Contributions:

Authors have worked equally to write and review the manuscript.

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