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A proposal to Employ the Laplace Estimator and the IRWL Algorithm as Robust Methods in Segmented Linear Regression and Compare them with the Maximum Likelihood Estimator using Simulation.

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ABSTRACT

Segmented linear regression is one of the important statistical tools that is used to model and explain the behavior of some events in which changes gets suddenly or the behavior pattern of the event goes through transitional stages. The importance of this research lies in studying the proposal to employ the robust Laplace estimator within the method of estimating the parameters of the segmented linear regression model to give robustness to the estimation when there are outliers in the data. Then some updates were made to this method according to the iterative algorithm (IRWL) to get better robust estimates. On the practical side, the simulation experiment was conducted with several different sample sizes, and assuming several cases of pollution rates in the data (outliers) (15%, 10%, 5%, 0%), Then After implementing the simulation experiment and comparing the proposed methods with Muggeo's Maximum likelihood (ML) method using the (MSE) criterion, The experimental results showed the efficiency of the proposed methods when the data contains pollution ratios, and the iterative algorithm (IRWL) has proven its efficiency in obtaining the best estimate of the parameters compared to other methods. If there is no pollution in the data, the Muggeo's maximum likelihood (ML) method is the best for estimation.

1. Introduction

The Regression analysis is one of the most important statistical methods used to study and interpret the relationship between variables and attempt to represent the phenomenon under study with a model that is as close as possible to reality. There are some phenomena in which changes happen suddenly. They may happen gradually or in the form of transitional stages, which can be graphically represented by a line segmented into several continuous sections. Therefore, segmented linear regression is used as a statistical tool to model and explain such cases. The concept of segmented linear regression is based primarily on the presence of specific points within the range of values of the explanatory variable (independent variable). These points represent the beginning of change in the behavior of the phenomenon or the beginning of the transition from one stage to another. Therefore, they are Shared, joint each transitional stage to the next stage. These points are called (join points[16]). So, these points are of great importance, as they represent the beginning of a transition or change in the pattern and behavior of the data.

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This model is called several names, including (segmented linear [9]), (multiple change points [16]), (multiphase [10]), (linear segment [15]).

Some authors in the literature have referred to the study and importance of using segmented linear regression models in several places, including (Quandt, 1958[12]) (Gbur and Dahm, 1985[8]), (Muggio, 2003[11]), (Kim, Fay, Yu, Barrett and Feuer, 2004[9]), and others. And some authors have interested on addressing the problem of data pollution, That is, the presence of outliers Values when using a segmented linear regression model, which can have an effect on the accuracy of estimating model parameters. So, many of them have tried to find robust methods to address this problem. Among them (Diniz, Milan, and Mazucheli, 2003[7]) studied Bayesian inference in segmented linear regression as a robust method when assuming heteroscedastic random error variance. (Chen, Chan, So, and Lee, 2011[6]) suggested a Bayesian approach to studying observation classification and estimating segmented linear regression model parameters and join points. They demonstrated the accuracy of this methodology in an experimental study using the suggested Markov Chain Monte Carlo algorithm (MCMC). (Ali and Abbas, 2019[4]) studied a suggested robust estimator (IRWm) to estimate the parameters of the segmented linear regression and find the join point. The idea of this suggestion lies in employing the robust M-estimator method within the classical (Muggo) method. They have proven the efficiency of this method to address data pollution. (Acitas and Senoglu, 2020[1]) presented an updated and robust alternative to the classical methods, in which they used the modified maximum likelihood (MML) methodology when the random error distribution of the model is symmetric (long-tailed). They demonstrated, using simulations, the strength and efficiency of the suggested methodology, and demonstrated in the applied aspect that using the ordinary least squares method is not appropriate due to the data containing anomalous (distant) observations, and that the suggested method gave more accurate and reliable results.

In this research, a proposal was studied to employ one of the well-known robust methods (Laplace estimator) within the classical methods for estimating the parameters and join points of the segmented linear regression model to add robustness to the estimation, and also the possibility of adding some improvements to this proposal to obtain more accurate estimators according to the proposed algorithm (IRWL), and then a comparison is made between the proposed methods and the classical (Muggo) maximum likelihood method using simulation to determine the efficiency of the estimation methods and the optimal method among them.

2. Theoretical Aspect

2.1. Segmented Linear Regression Model:

A Segmented linear regression model, which consists of continuous linear sections or stages, is often important for describing phenomena with multiple sections or directions. These sections are Connect by a Shared join point that joint each section to the next. These join points are important because they represent a change or transition in the behavior and direction of the data. Assuming the observations are:

$\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}, \;\; x_1 \leq \cdots \leq x_n$

and the response variable is [16]:

$$y_i = E(y_i|x_i) + e_i\,, \qquad i = 1, \ldots, \mathrm{n} \qquad \ldots (1)$$

Where:

$E(y_i|x_i)$: is the expected value of the response variable.

x_i : is the explanatory variable or (independent variable).

e_i : is the random error with mean E(e_i)=0 and variance V(e_i)=σ^2.

The segmented linear regression model imposes the adjective linearity on each section or segment of the model, That is:

$$E(y|x) = \beta_{j,0} + \beta_{j,1}x. \quad if \;\; \tau_{j-1} < x \leq \tau_j. \quad j = 1, \ldots, r-1 \qquad \ldots (2)$$

Where:

$\beta_{j,0}$:is the constant limit parameter for segment j. $\beta_{j,1}$: is the slope parameter for segment j.

r : is the number of sections or segments in the model, τ_j: is a value within the interval of values of x and it's the join point between each segment with the next one. Therefore, $E(y|x)$ is continuous within the interval $[x_0, x_n]$. That is, τ_j is the join point between segment j and j+1. The join point between the two segments, as in the following formula [16]:

$$\beta_{j,0} + \beta_{j,1}\tau_j = \beta_{j+1,0} + \beta_{j+1,1}\tau_j. \quad for \quad j = 1, \ldots, r-1 \qquad \ldots (3)$$

Since the response variable is continuous at the join points, the alternative formula for Model (2) is:

$$E(y|x) = \beta_{10} + \beta_{11}\, x + \Delta_1(x - \tau_1)_+ + \cdots + \Delta_{r-1}(x - \tau_{r-1})_+,$$
$$or \quad = \beta_{10} + \beta_{11}\, x + \sum_{j=1}^{r-1} \Delta_j\left(x - \tau_j\right)_+, \qquad \ldots (4)$$

Where:

Δ_j: is the difference between the two slope parameters ($\Delta_j = \beta_{j+1} - \beta_j$).

And: $\left(x - \tau_j\right)_+ = \begin{cases} x - \tau_j, & x > \tau_j \\ 0, & x \le \tau_j \end{cases}$

The determination of model parameters (4) implicitly satisfies the continuity condition at the join points τ_j.

2.2. Estimation Methods for the Model:

In this research, the maximum likelihood (ML) method was used to estimate the parameters of the Segmented linear regression model and also some proposed methods as robust methods in estimation.

2.2.1. Maximum likelihood using (Muggeo) Method:

The maximum likelihood method is an important method due to its wide applications in estimating parameters for statistical models, and because it has several properties such as consistency and Unbiased often.

The method (Muggeo, 2003[11]) was adopted to find the maximum likelihood (ML) estimator to the parameters of model (4), when Muggeo noticed non-differentiable the logarithm of the model's likelihood function at the join points $x_i = \tau_j$, he proposed a linearization processing technique to solve this problem, after which it is easy to find the estimation of the parameters using the maximum likelihood method in addition to estimating the join points using a simple iterative method (linear reparameterization).

The idea of using the linearization processing of the formula $\Delta_j\left(x - \tau_j\right)_+$in model (4) which is non-differentiable at the join points is lie in use the first order Taylor series to approximate the formula around the initial values $\tau_j^{(0)}$ which must be as close as possible to the real values, as in the following formula[15]:

$$\left(X - \tau_j\right)_+ \approx \left(X - \tau_j^{(0)}\right)_+ + (-1)I\left(X > \tau_j^{(0)}\right)\left(\tau_j - \tau_j^{(0)}\right) \qquad \ldots (5)$$

Where:

$(-1)I\left(X > \tau_j^{(0)}\right)$: is the first derivative of $\left(X - \tau_j^{(0)}\right)_+$with respect to $\tau_j^{(0)}$, $I\left(X > \tau_j^{(0)}\right)$: is equal to one if $X > \tau_j^{(0)}$ and zero otherwise.

This formula (5) is applied to all join points in the model, and the formula of the model (4) becomes after performing linearization processing according to the formula:

$$E(y|x) = \beta_{10} + \beta_{11}\, x + \sum_{j}^{r-1} \Delta_j \bar{U}_j + \sum_{j}^{r-1} \gamma_j \bar{V}_j , \qquad ...(6)$$

Where: $\gamma_j = \Delta_j\left(\tau_j - {\tau_j}^{(0)}\right)$, and

$$\bar{U}_j = \left(X - {\tau_j}^{(0)}\right)_+ , \qquad \bar{V}_j = (-1) I\left(X > {\tau_j}^{(0)}\right), \qquad ...(7)$$

After completing the linearization processing , the parameters of the Segmented linear regression model can be estimated using the maximum likelihood method, and when the random error follows a known probability distribution, according to the simple iterative algorithm of Muggeo[11]. as follows:

1- Set the initial values ${\tau_j}^{(s)}$. at repeating s=0.
2- Calculate ${\bar{U}_j}^{(s)}, {\bar{V}_j}^{(s)}$ from formula (7).
3- Calculate the maximum likelihood (ML) estimators for model (6) assuming that the random error follows the normal distribution, according to the formula:

$$\hat{\beta}^{(s)}_{(ML)} = (X'X)^{-1}X'Y \qquad ...(8)$$

Where:

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad X = \begin{pmatrix} 1 & x_1 & u_1^{(s)} \ldots u_j^{(s)} \ldots u_{r-1}^{(s)} & v_1^{(s)} \ldots v_j^{(s)} \ldots v_{r-1}^{(s)} \\ \vdots & \vdots & \vdots \;\cdots\; \vdots \;\cdots\; \vdots & \vdots \;\cdots\; \vdots \;\cdots\; \vdots \\ 1 & x_n & u_n^{(s)} \ldots u_{j.n}^{(s)} \ldots u_{r-1.n}^{(s)} & v_n^{(s)} \ldots v_{j.n}^{(s)} \ldots v_{r-1.n}^{(s)} \end{pmatrix}, \quad \hat{\beta}^{(s)}_{(ML)} = \begin{pmatrix} \hat{\beta}_0^{(s)} \\ \hat{\beta}_1^{(s)} \\ \hat{\Delta}_1^{(s)} \\ \vdots \\ \hat{\Delta}_{r-1}^{(s)} \\ \hat{\gamma}_1^{(s)} \\ \vdots \\ \hat{\gamma}_{r-1}^{(s)} \end{pmatrix}$$

4- Update and improve join points, according to the formula:

$$\hat{\tau}_j^{(s+1)} = \frac{\hat{\gamma}_j}{\hat{\Delta}_j} + {\tau_j}^{(s)} \qquad ...(9)$$

5- Repeat steps, (2) to (4).

After improving the join points and replacing them with the initial values to S from repetitions when the parameter values converge. That is, when ($\hat{\gamma}_j \approx 0$). That's means ($\tau_j^{(s)} - \tau_j^{(s-1)} \approx 0$) and then we get the best estimate (ML) of the parameters and join points τ_j.

2.2.2. Proposed employment of a Laplace estimator using the weighted least squares method:

The Laplace method is one of the alternative robust methods for estimation in the presence of outliers in the data under study. This method depends on the absolute value of the residuals and aims primarily to reduce the random error (e_i), according to the following formula[2],[14]:

$$Lap = \min \sum_{i=1}^{n} |e_i|^u , \quad 1 \le u \le 2 \qquad ...(10)$$

Where:

e_i: is residuals.

u: is the optimal value that gives robust estimators.

If the value of u = 1, the estimators are in the most robust direction, and if u = 2, they are equivalent to least squares estimators.

The estimates are obtained using the weighted least squares (WLS) Technique[3]. So, this method will be employed within the Muggeo method to obtain robust estimates for the parameters of the Segmented linear regression model, according to the following steps:

1- Initialize the initial values of $\hat{\tau}_j^{(s)}$. at repeating s=0.

2- vector calculation $\bar{U}_j^{(s)}, \bar{V}_j^{(s)}$ from formula (7).

3- Calculating the robust Laplace estimators for model (6), as follows[3]:

a- Calculate the initial parameter vector $\hat{\beta}^{(s)}_{(OLS)}$ using the ordinary least squares (OLS) method for the parameters of model (6) according to the same method used in formula (8).

b- Calculate the values of the residuals $e_i^{(s)}$, according to the formula:

$$e_i^{(s)} = y_i - \beta_{10}{}^{(s)}_{ols} + \beta_{11}{}^{(s)}_{ols}\, x + \sum_{j}^{r-1} \Delta_j{}^{(s)}_{ols} \bar{U}_j + \sum_{j}^{r-1} \gamma_j{}^{(s)}_{ols} \bar{V}_j \qquad \dots (11)$$

c- Calculate the Laplace estimators $\hat{\beta}^{(s)}_{(lap)}$ using the weighted least squares (WLS) Technique, according to the formula:

$$\hat{\beta}^{(s)}_{(lap)} = (X'WX)^{-1}X'WY\,, \qquad \dots (12)$$

Where:

X: The matrix of explanatory variables, whose elements are given in Formula (8).

Y: The observation vector (response variable).

W: The diagonal matrix of degree (n×n), whose elements are given by the formula:

$$w_i = \left|e_i^{(s)}\right|^{u-2}, \qquad \dots (13)$$

The values $(u_1 = 1.5\,, u_2 = 1.8)$ were used as the optimal value for (u) to give robust estimates[3],[13].

4- After calculating the Laplace estimators $\hat{\beta}^{(s)}_{(lap)}$, the join points are updated according to formula (9).

5- Replace the updated join points $\hat{\tau}_j^{(s+1)}$ with the initial $\hat{\tau}_j^{(s)}$ and repeat steps (2) to (4) to S from repetitions.

The robust Laplace estimator $\hat{\beta}^{(s)}_{(lap)}$ is obtained when the values of the parameters $\hat{\gamma}_j$ converge to zero, that is, when $(\tau_j^{(s)} - \tau_j^{(s-1)} \approx 0)$, and then we obtain the best estimate of the join points τ_j as well [11].

2.2.3. Proposed iterative algorithm for weighted Laplace (IRWL) estimator:

In this method, some improvements will be made to the previous method (2-2-2) for the robust Laplace estimator by adding an iterative technique to find the best robust estimator for the Laplace estimator. This improvement will be in step (3) when calculating the robust Laplace estimators $\hat{\beta}^{(s)}_{(\mathrm{lap})}$ for model (6), which are as follows:

a- Calculate the initial parameters vector $\hat{\beta}^{(m)}$ at repeating *m*=0 using the (OLS) method for the parameters of model (6) according to the same method used in formula (8).

b- Calculate the values of the residuals $e_i^{(m)}$ according to the same method used in formula (11).

c- Calculate the Laplace estimators $\hat{\beta}^{(m+1)}$ as in the previous method according to formula (12).

d- Repeat steps (a) to (c) for *m* repeats after replacing $\hat{\beta}^{(m+1)}$ with the initial parameters $\hat{\beta}^{(m)}$, until the estimated parameters converge according to formula[4]:

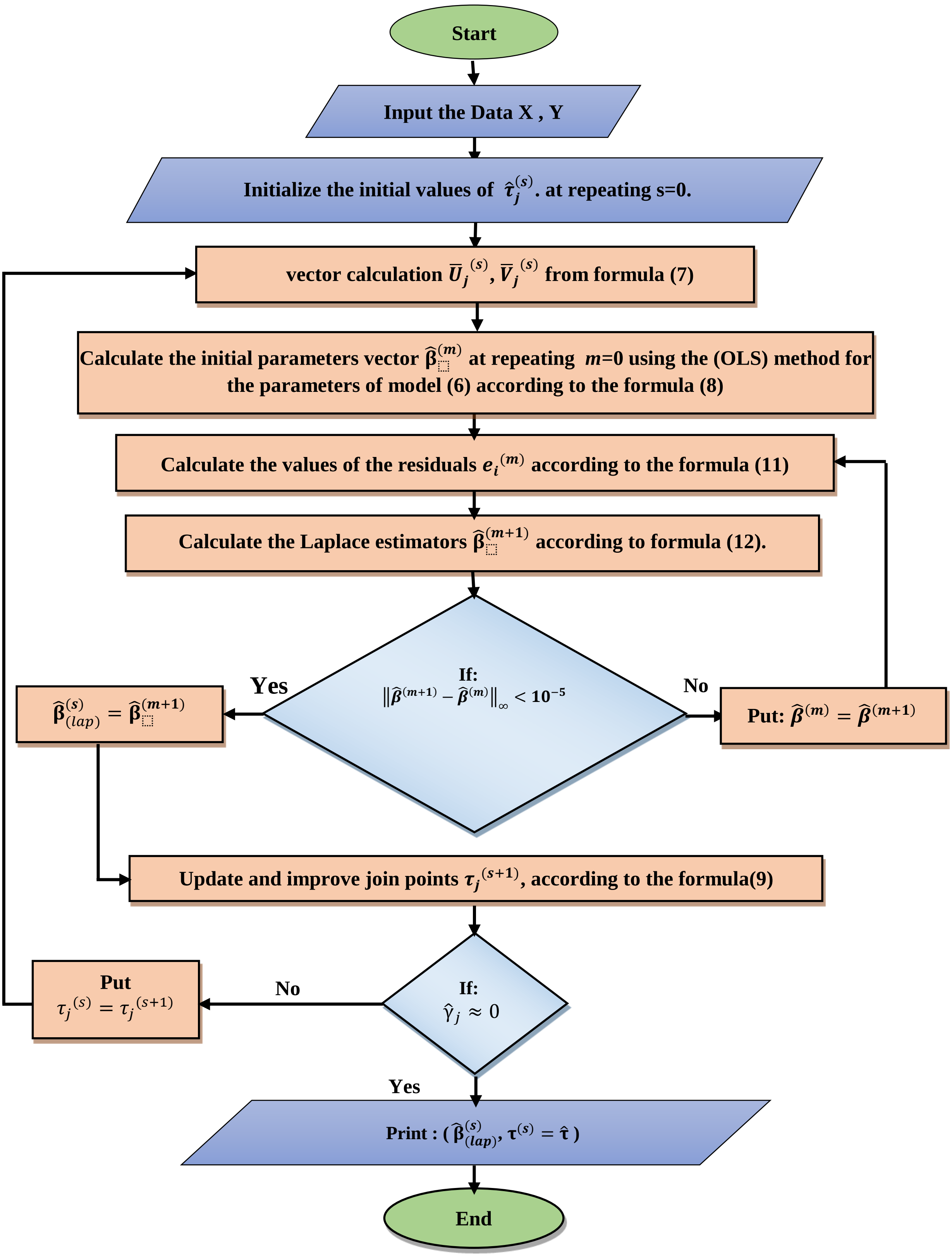
$$\left\|\hat{\beta}^{(m+1)} - \hat{\beta}^{(m)}\right\|_\infty < 10^{-5}, \qquad \dots (14)$$

Where:

$\|v\|_\infty = \max_j |v_j|$ for all $v \in R^q$.

After convergence in the *m*th step, Then $\hat{\beta}^{(s)}_{(lap)} = \hat{\beta}^{(m+1)}$.

Then, the remaining steps (4, 5) followed in the previous method (2-2-2) are completed. This method can be summarized according to the following Flowchart No.(1):



Flowchart No. (1) of the proposed (IRWL) method

❖ Flowchart is the author's work.

3. Practical experimental aspect:

Simulation is a process of representing the behavior of the real world under study by applying numerous experiments subject to certain conditions and assumptions, which are as close as possible to the real world. Its use saves researchers a lot of time and effort, especially when it is impossible to obtain accurate information data, or when the researcher needs to verify the results of the application of real data, or to Statement and knowledge of the best method for studying at the expense of other methods.

The simulation experiment for this study was conducted with a number of scenarios using programming in the (Matlab) language, version (MATLAB R2017a), to evaluate the performance of the estimation methods mentioned in this research in paragraph (2-2), with knowing the best method among them, by generating data with different sample sizes and in the bing or not of pollution rates (outliers) in the data.

3.1. Stages of the simulation experience:

The simulation experiments were conducted in this study as follows:

1- Determine the sample sizes. Three different sample sizes were determined (n= 30, 60, 120). The experiment was repeated for each of these sizes (R= 1000) repetitions.

2- Generate the explanatory (independent) variable x_i according to the uniform distribution for the interval (a,b) [15].

$$x_i \sim Uniform(a,b), i = 1,2,\cdots,n \quad \dots (15)$$

The values of x_i were generated within the interval (a=0, b=12).

3- Generate the random error e_i according to the normal distribution with mean ($\mu = 0$) and variance σ^2 using the method (Box and Muller, 1958[5]) as follows:

- Generating random variables (U_1,U_2) according to the standard uniform distribution (0,1).
- Perform a transformation of the variables (U_1, U_2) into independent random variables that follow the standard normal distribution according to the formula:

$$Z_1 = \sqrt{-2\log U_1} * \cos(2\pi U_2), \dots (16)$$

$$Z_2 = \sqrt{-2\log U_1} * \sin(2\pi U_2), \dots (17)$$

- Generate the random variable U_3 according to the standard uniform distribution $U_3 \sim Uniform(0,1)$.
- A random variable following the standard normal distribution $Z \sim N(0,1)$ is obtained according to the formula:

$$Z = \begin{cases} Z_1 & if\ U_3 \geq 0.5 \\ Z_2 & if\ U_3 < 0.5 \end{cases}, \quad \dots (18)$$

- Then we have the random variable e that follows the distribution $e \sim N(\mu, \sigma^2)$ using the formula:

$$e = \mu + \sigma * Z, \quad \dots (19)$$

This study addressed several assumptions for generating random error, which are:

- Assuming there are no outliers in the data, then: $e_i \sim N(0,1)$.
- If we assume that there are outliers in the data, the random error follows the distribution $e_i \sim N(\mu, \sigma^2)$ with a ratio of (1-p) and $e_i \sim N(\mu, \Omega^2\sigma^2)$ with a ratio of p. That is, p is the pollution ratio (the ratio of outliers in the data) and is (0≤p≤1).

4- Observation generation (response variable): The response variable y_i was obtained according to Model (6) based on the values of x_i and e_i generated in the previous two steps (2, 3).

This study conducted a simulation of the segmented linear regression model (6) according to two scenarios[15] of the default values of the parameters, which are:

Experiment (1):

$\boldsymbol{\beta} = (\beta_0 = 0.01, \beta_1 = 0.1, \Delta_1 = 2)$,

$\boldsymbol{\tau} = (\tau_1 = 5)$.

Experiment (2):

$\boldsymbol{\beta} = (\beta_0 = 0.01, \beta_1 = 0.1, \Delta_1 = 7, \Delta_2 = -9)$,

$\boldsymbol{\tau} = (\tau_1 = 4, \tau_2 = 9)$

5- Apply the estimation methods used in this study in paragraph (2-2) to the data generated in the previous steps, and then find the Mean and Mean Square Error (MSE) for the estimated model parameters.

6- A comparison of applied estimation methods is conducted using the MSE comparison criterion to evaluate the estimation methods and determine the best method among them. This criterion is one of the most common criterion, as it calculates the extent of convergence and deviation of the estimated model parameters from the real values. The optimal method is selected based on the one with the lowest MSE. This criterion is calculated using the formula [10]:

$$\boldsymbol{MSE_\beta} = \frac{1}{R}\sum_{i=1}^{R}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^2 , \quad \boldsymbol{MSE_\tau} = \frac{1}{R}\sum_{i=1}^{R}(\hat{\boldsymbol{\tau}} - \boldsymbol{\tau})^2 \quad \ldots.(20)$$

Where:

R: repeatability value of the simulation experiment.

$\widehat{\boldsymbol{\beta}}$: parameters estimated in the experiment.

$\boldsymbol{\beta}$: real parameters.

$\hat{\boldsymbol{\tau}}$: join points estimated in the experiment.

$\boldsymbol{\tau}$: real join points.

3.2. Simulation experiment results:

The simulation experiment was implemented using the computer program (Matlab R2017a), and the procedures were carried out according to the two cases:

Experiment (1):

Assume that the generated data has a one join point.

Experiment (2):

Assume that the generated data has two join points.

Several different sizes of data were generated (n= 30, 60, 120) with repetitions (R= 1000) , and with several assumptions for the random error values e_i for the generated data, which are:

- Assuming the data do not contain outliers (unpolluted), then:
  $e_i \sim N(0 , 1)$
- Assuming the data do contain outliers (polluted), then:
  The random error e_i is generated according to the distribution $e_i \sim N(0,\ 0.01)$ with a ratio of (1-p) and $e_i \sim N(0, 3)$ with a ratio of p. where p takes the following proportions: (5%, 10%, 15%).

After completing the data generation, the estimation methods for the segmented linear regression model that were discussed in paragraph (2-2) of the theoretical aspect were applied to know the performance of these methods, evaluate them, and determine the optimal method among them, after conducting the comparison process according to the (MSE). The results of this experiment were according to the following tables:

Table No. (1): Results of simulation experiment (1) with 1000 repetitions and an error distribution of $e_i \sim N(0, 1)$.

Methods	Parameter	τ	β_0	β_1	Δ
	True value	5	0.01	0.1	2
When the sample size is: n = 30					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.99616	0.05366	0.07918	2.02679
	MSE	**0.18129**	**0.41249**	**0.05402**	**0.07267**
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.99970	0.05506	0.07984	2.02670
	MSE	0.18227	0.42317	0.05458	0.07316
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.99961	0.05422	0.07979	2.02675
	MSE	0.18437	0.41727	0.05457	0.07292
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00098	0.05638	0.07932	2.02734
	MSE	0.18946	0.44243	0.05743	0.07687
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00031	0.05400	0.08003	2.02650
	MSE	0.18402	0.41952	0.05473	0.07299
When the sample size is: n = 60					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	5.00747	0.00573	0.09862	2.00543
	MSE	**0.07940**	**0.16799**	**0.02181**	**0.02904**
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00843	0.00523	0.09859	2.00549
	MSE	0.08334	0.17537	0.02292	0.02999
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00829	0.00500	0.09892	2.00510
	MSE	0.08051	0.17019	0.02205	0.02928
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00772	0.00585	0.09796	2.00580
	MSE	0.08970	0.18929	0.02466	0.03159
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00733	0.00597	0.09836	2.00549
	MSE	0.08134	0.17159	0.02232	0.02945
When the sample size is: n = 120					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	5.00453	0.01943	0.09855	2.00135
	MSE	0.04057	**0.08679**	0.01066	**0.01374**
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00336	0.01826	0.09861	2.00092
	MSE	0.04126	0.08772	0.01081	0.01406
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00340	0.01894	0.09846	2.00119
	MSE	**0.04053**	0.08691	**0.01064**	0.01381
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00138	0.01820	0.09806	2.00117
	MSE	0.04286	0.09176	0.01126	0.01482
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00348	0.01863	0.09856	2.00107
	MSE	0.04059	0.08716	0.01067	0.01385

Table No. (2): Results of simulation experiment (1) with 1000 repetitions and a polluted error distribution

(95% following $e_i \sim N(0,\ 0.01)$ and 5% following $e_i \sim N(0,3)$)

Methods	Parameter	τ	β_0	β_1	Δ
	True value	5	0.01	0.1	2
When the sample size is: n = 30					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	5.00423	0.01703	0.09331	2.00976
	MSE	0.09666	0.26956	0.03029	0.03955
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.99929	0.01675	0.09585	2.00473
	MSE	0.03089	0.10966	0.01219	0.01503
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00263	0.01534	0.09529	2.00653
	MSE	0.05763	0.18404	0.02034	0.02614
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.99908	0.01488	0.09785	2.00196
	MSE	**0.01289**	**0.05442**	**0.00590**	**0.00705**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00158	0.01667	0.09487	2.00680
	MSE	0.05205	0.16889	0.01883	0.02391
When the sample size is: n = 60					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.99509	0.02087	0.09211	2.01100
	MSE	0.03992	0.07563	0.01035	0.01354
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.99781	0.01350	0.09745	2.00344
	MSE	0.00744	0.01704	0.00216	0.00288
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.99811	0.01552	0.09593	2.00612
	MSE	0.01786	0.03777	0.00488	0.00650
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.99924	0.01033	0.09932	2.00104
	MSE	**0.00272**	**0.00671**	**0.00084**	**0.00111**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.99675	0.01590	0.09575	2.00584
	MSE	0.01553	0.03252	0.00434	0.00566
When the sample size is: n = 120					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	5.00516	0.00722	0.09985	2.00250
	MSE	0.01968	0.03984	0.00492	0.00616
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00252	0.00814	0.10053	2.00030
	MSE	0.00364	0.00830	0.00098	0.00128
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00246	0.00850	0.09998	2.00112
	MSE	0.00772	0.01714	0.00210	0.00265
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00082	0.00937	0.10033	1.99975
	MSE	**0.00114**	**0.00284**	**0.00033**	**0.00043**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00250	0.00844	0.10018	2.00080
	MSE	0.00614	0.01394	0.00170	0.00216

Table No. (3): Results of simulation experiment (1) with 1000 repetitions and a polluted error distribution

(90% following $e_i \sim N(0, 0.01)$and 10% following $e_i \sim N(0,3)$)

Methods	Parameter	τ	β_0	β_1	Δ
	True value	5	0.01	0.1	2
When the sample size is: n = 30					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	5.00678	0.01337	0.09289	2.01086
	MSE	0.14766	0.35135	0.04262	0.05393
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.99424	0.01298	0.09384	2.00620
	MSE	0.05180	0.13657	0.01834	0.02177
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00095	0.01107	0.09450	2.00690
	MSE	0.08747	0.23254	0.02857	0.03549
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.99740	0.01045	0.09770	2.00211
	MSE	**0.02142**	**0.06735**	**0.00803**	**0.00968**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00011	0.01059	0.09495	2.00612
	MSE	0.07740	0.21091	0.02587	0.03211
When the sample size is: n = 60					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.99907	0.03197	0.08919	2.01450
	MSE	0.07638	0.16027	0.01976	0.02490
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.99974	0.01535	0.09699	2.00433
	MSE	0.01534	0.03976	0.00444	0.00571
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.99917	0.02191	0.09369	2.00868
	MSE	0.03784	0.08676	0.01040	0.01307
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.99883	0.01080	0.09886	2.00143
	MSE	**0.00543**	**0.01443**	**0.00164**	**0.00205**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.99935	0.01969	0.09463	2.00756
	MSE	0.03132	0.07496	0.00883	0.01116
When the sample size is: n = 120					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	5.00070	0.01235	0.09651	2.00527
	MSE	0.03891	0.07437	0.00990	0.01292
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00123	0.00808	0.10018	2.00029
	MSE	0.00752	0.01564	0.00202	0.00268
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.99943	0.01000	0.09835	2.00231
	MSE	0.01790	0.03469	0.00474	0.00612
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00068	0.00853	0.10045	1.99964
	MSE	**0.00233**	**0.00509**	**0.00067**	**0.00089**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.99953	0.00957	0.09874	2.00178
	MSE	0.01460	0.02903	0.00397	0.00514

Table No. (4): Results of simulation experiment (1) with 1000 repetitions and a polluted error distribution

(85% following $e_i \sim N(0,\ 0.01)$ and 15% following $e_i \sim N(0,3)$)

Methods	Parameter	τ	β_0	β_1	Δ
	True value	5	0.01	0.1	2
When the sample size is: n = 30					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	5.02734	0.04771	0.08134	2.03084
	MSE	0.27286	0.59086	0.08399	0.10055
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00410	0.04430	0.08310	2.02133
	MSE	0.11224	0.33486	0.04700	0.05267
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.01848	0.04383	0.08356	2.02502
	MSE	0.17936	0.45420	0.06244	0.07360
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00119	0.03446	0.08808	2.01425
	MSE	**0.05890**	**0.21712**	**0.03109**	**0.03414**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.01269	0.04553	0.08294	2.02380
	MSE	0.15893	0.42764	0.05872	0.06885
When the sample size is: n = 60					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	5.01428	-0.00723	0.10058	2.00482
	MSE	0.11584	0.25323	0.03030	0.04028
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00279	0.00289	0.09939	2.00233
	MSE	0.03176	0.07871	0.00989	0.01243
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00945	-0.00327	0.10065	2.00295
	MSE	0.06398	0.14859	0.01823	0.02376
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00004	0.00512	0.09958	2.00124
	MSE	**0.01261**	**0.03202**	**0.00426**	**0.00518**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00642	-0.00101	0.09955	2.00350
	MSE	0.05664	0.13192	0.01668	0.02131
When the sample size is: n = 120					
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	5.00900	0.02486	0.09537	2.00683
	MSE	0.06543	0.12270	0.01618	0.02061
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00244	0.01543	0.09902	2.00094
	MSE	0.01240	0.02640	0.00339	0.00443
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00575	0.01962	0.09757	2.00351
	MSE	0.03094	0.06006	0.00787	0.01000
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	5.00058	0.01312	0.09955	2.00017
	MSE	**0.00362**	**0.00847**	**0.00106**	**0.00142**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	5.00557	0.01857	0.09809	2.00286
	MSE	0.02523	0.05046	0.00658	0.00842

Table No. (5): Results of simulation experiment (2) with 1000 repetitions and an error distribution of $e_i \sim N(0, 1)$.

Methods	Parameter	τ_1	τ_2	β_0	β_1	Δ_1	Δ_2
	True value	4	9	0.01	0.1	7	-9
When the sample size is: n = 30							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.03846	8.95930	-0.03045	0.13291	7.03542	-9.00508
	MSE	**0.03583**	**0.05218**	**0.62561**	**0.12985**	**0.20464**	**0.69631**
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.03936	8.95736	-0.03091	0.13510	7.03201	-8.99555
	MSE	0.03691	0.05445	0.65298	0.13411	0.21352	0.72358
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.03952	8.95677	-0.03268	0.13475	7.03400	-8.99458
	MSE	0.03692	0.05344	0.64330	0.13340	0.21006	0.71766
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.03977	8.95536	-0.03223	0.13587	7.03141	-8.98925
	MSE	0.03933	0.05638	0.70260	0.14361	0.22449	0.75351
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.03895	8.95671	-0.03117	0.13405	7.03413	-8.99457
	MSE	0.03621	0.05390	0.64227	0.13140	0.20874	0.72360
When the sample size is: n = 60							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.00888	8.99585	-0.02618	0.11735	6.99244	-9.00413
	MSE	0.00990	**0.00771**	**0.24125**	**0.04870**	0.06766	**0.13951**
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00703	8.99497	-0.02004	0.11293	6.99695	-9.00304
	MSE	0.01012	0.00783	0.24975	0.05019	0.06894	0.14344
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00833	8.99466	-0.02379	0.11570	6.99491	-9.00233
	MSE	0.00989	0.00784	0.24378	0.04881	**0.06759**	0.14034
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00695	8.99297	-0.01828	0.11170	6.99972	-9.00066
	MSE	0.01049	0.00859	0.26131	0.05215	0.07156	0.15414
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00839	8.99483	-0.02393	0.11579	6.99470	-9.00282
	MSE	**0.00986**	0.00781	0.24446	0.04894	0.06767	0.14095
When the sample size is: n = 120							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.00173	8.99666	0.01384	0.10143	7.00236	-9.00371
	MSE	**0.00438**	**0.00306**	**0.10871**	**0.02084**	**0.02919**	**0.05762**
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00174	8.99649	0.01172	0.10166	7.00265	-9.00404
	MSE	0.00443	0.00309	0.11081	0.02113	0.02937	0.05884
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00177	8.99648	0.01266	0.10164	7.00244	-9.00355
	MSE	0.00438	0.00308	0.10949	0.02094	0.02919	0.05787
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00209	8.99541	0.01033	0.10183	7.00383	-9.00260
	MSE	0.00470	0.00341	0.11700	0.02240	0.03095	0.06226
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00181	8.99644	0.01226	0.10174	7.00243	-9.00359
	MSE	0.00439	0.00309	0.10964	0.02097	0.02925	0.05815

Table No. (6): Results of simulation experiment (2) with 1000 repetitions and a polluted error distribution

(95% following $e_i \sim N(0,\ 0.01)$ and 5% following $e_i \sim N(0,3)$)

Methods	Parameter	τ_1	τ_2	β_0	β_1	Δ_1	Δ_2
	True value	4	9	0.01	0.1	7	-9
When the sample size is: n = 30							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.01528	8.98151	-0.02310	0.11583	7.01746	-9.00500
	MSE	0.01585	0.02048	0.40711	0.07113	0.11389	0.29246
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00904	8.98555	-0.00880	0.10972	7.01209	-8.98960
	MSE	0.00661	0.01400	0.19963	0.03319	0.06193	0.24029
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.01302	8.98256	-0.01807	0.11349	7.01627	-8.99695
	MSE	0.01135	0.01792	0.30279	0.05174	0.08826	0.30187
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00631	8.98691	-0.00404	0.10773	7.00801	-8.97847
	MSE	**0.00413**	**0.01138**	**0.12261**	**0.01963**	**0.04135**	**0.18614**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.01292	8.98273	-0.01773	0.11365	7.01549	-8.99602
	MSE	0.01061	0.01744	0.28469	0.04843	0.08328	0.29591
When the sample size is: n = 60							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.00584	8.99620	0.00461	0.10234	7.01009	-9.01108
	MSE	0.00421	0.00339	0.09209	0.01865	0.02804	0.05808
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00189	8.99978	0.00815	0.10084	7.00297	-9.00729
	MSE	0.00102	0.00088	0.02441	0.00512	0.00746	0.01618
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00294	8.99873	0.00813	0.10034	7.00665	-9.01000
	MSE	0.00221	0.00177	0.05011	0.01043	0.01525	0.03348
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00103	8.99997	0.00783	0.10097	7.00087	-9.00402
	MSE	**0.00040**	**0.00033**	**0.01017**	**0.00209**	**0.00299**	**0.00615**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00266	8.99890	0.00826	0.10034	7.00600	-9.00939
	MSE	0.00191	0.00156	0.04370	0.00916	0.01332	0.02966
When the sample size is: n = 120							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.00035	8.99931	0.00501	0.10107	6.99866	-8.99469
	MSE	0.00205	0.00140	0.05173	0.01046	0.01425	0.02847
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00000	8.99963	0.00875	0.10040	6.99916	-8.99668
	MSE	0.00041	0.00027	0.01050	0.00213	0.00288	0.00573
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	3.99997	8.99954	0.00763	0.10046	6.99896	-8.99571
	MSE	0.00090	0.00059	0.02332	0.00480	0.00638	0.01315
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	3.99973	8.99973	0.00983	0.09989	6.99971	-8.99802
	MSE	**0.00013**	**0.00009**	**0.00371**	**0.00070**	**0.00098**	**0.00202**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	3.99991	8.99961	0.00803	0.10037	6.99901	-8.99604
	MSE	0.00074	0.00049	0.01926	0.00399	0.00531	0.01101

Table No. (7): Results of simulation experiment (2) with 1000 repetitions and a polluted error distribution

(90% following $e_i \sim N(0,\ 0.01)$ and 10% following $e_i \sim N(0,3)$)

Methods	Parameter	τ_1	τ_2	β_0	β_1	Δ_1	Δ_2
	True value	4	9	0.01	0.1	7	-9
When the sample size is: n = 30							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.03737	8.95763	-0.05223	0.13556	7.05394	-9.01563
	MSE	0.03756	0.03532	0.55488	0.11614	0.31318	0.43836
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.02525	8.96951	-0.03077	0.12365	7.04037	-9.00811
	MSE	0.02325	0.02457	0.29350	0.06421	0.24374	0.33374
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.03128	8.96398	-0.03936	0.12903	7.04733	-9.01305
	MSE	0.02982	0.02969	0.42155	0.09057	0.27990	0.37880
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.01743	8.97709	-0.01462	0.11446	7.03431	-9.00528
	MSE	**0.01714**	**0.01860**	**0.17559**	**0.03912**	**0.20534**	**0.28190**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.03072	8.96470	-0.03853	0.12887	7.04578	-9.01216
	MSE	0.02847	0.02891	0.39525	0.08510	0.27265	0.37025
When the sample size is: n = 60							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.01005	8.99017	0.01140	0.10290	7.01640	-8.99967
	MSE	0.00976	0.00641	0.21522	0.04207	0.05607	0.12210
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00224	8.99645	0.01316	0.09843	7.00805	-8.99965
	MSE	0.00206	0.00200	0.06177	0.01108	0.01539	0.04317
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00454	8.99378	0.01450	0.09872	7.01246	-8.99910
	MSE	0.00459	0.00384	0.12422	0.02322	0.03181	0.07840
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00055	8.99848	0.01136	0.09876	7.00405	-9.00057
	MSE	**0.00077**	**0.00084**	**0.02506**	**0.00443**	**0.00595**	**0.02071**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00397	8.99428	0.01441	0.09857	7.01159	-8.99880
	MSE	0.00397	0.00342	0.10924	0.02037	0.02783	0.07043
When the sample size is: n = 120							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	3.99990	8.99609	0.01404	0.09479	7.01172	-9.00265
	MSE	0.00386	0.00306	0.09455	0.01891	0.02913	0.05594
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	3.99911	8.99929	0.01185	0.09756	7.00333	-9.00043
	MSE	0.00077	0.00070	0.01998	0.00409	0.00611	0.01334
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	3.99925	8.99798	0.01231	0.09643	7.00655	-9.00110
	MSE	0.00181	0.00152	0.04550	0.00925	0.01406	0.02920
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	3.99931	8.99955	0.01116	0.09856	7.00181	-8.99992
	MSE	**0.00026**	**0.00023**	**0.00692**	**0.00143**	**0.00203**	**0.00442**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	3.99919	8.99828	0.01202	0.09673	7.00574	-9.00092
	MSE	0.00152	0.00128	0.03847	0.00786	0.01183	0.02503

Table No. (8): Results of simulation experiment (2) with 1000 repetitions and a polluted error distribution

(85% following $e_i \sim N(0,\ 0.01)$ and 15% following $e_i \sim N(0,3)$)

Methods	Parameter	τ_1	τ_2	β_0	β_1	Δ_1	Δ_2
	True value	4	9	0.01	0.1	7	-9
When the sample size is: n = 30							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.05127	8.94827	-0.04608	0.14124	7.06645	-9.02264
	MSE	0.06225	0.05687	0.93816	0.20594	0.43366	0.79813
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.03141	8.96317	-0.02116	0.12285	7.05197	-9.00940
	MSE	0.03825	0.03749	0.58677	0.12801	0.32472	0.60309
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.04061	8.95928	-0.03011	0.13112	7.05701	-9.02297
	MSE	0.04919	0.04449	0.74184	0.16567	0.37294	0.68085
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.02124	8.97209	-0.01264	0.11564	7.03795	-8.99929
	MSE	**0.02278**	**0.02768**	**0.42444**	**0.08702**	**0.27057**	**0.48733**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.03911	8.95978	-0.02893	0.12984	7.05610	-9.01967
	MSE	0.04727	0.04339	0.70834	0.15791	0.36186	0.66708
When the sample size is: n = 60							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.01126	8.98979	-0.02447	0.11200	7.01042	-9.01386
	MSE	0.01414	0.01039	0.34412	0.06790	0.09812	0.19167
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00405	8.99784	-0.00970	0.10604	7.00156	-9.01025
	MSE	0.00386	0.00355	0.12139	0.02273	0.03278	0.08013
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00633	8.99408	-0.01409	0.10664	7.00785	-9.01196
	MSE	0.00826	0.00663	0.21432	0.04110	0.06126	0.13233
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00195	8.99928	-0.00277	0.10384	6.99961	-9.00525
	MSE	**0.00150**	**0.00175**	**0.05377**	**0.00950**	**0.01379**	**0.04026**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00625	8.99455	-0.01406	0.10717	7.00640	-9.01126
	MSE	0.00718	0.00605	0.18950	0.03562	0.05436	0.12169
When the sample size is: n = 120							
Muggeo (ML)	$\hat{\tau}, \hat{\beta}$	4.00607	8.99592	0.00874	0.10581	7.00041	-8.99914
	MSE	0.00615	0.00482	0.14110	0.02706	0.03985	0.07982
Lap1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00142	8.99892	0.01208	0.10147	6.99890	-8.99710
	MSE	0.00132	0.00121	0.03280	0.00623	0.00945	0.02152
Lap2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00338	8.99752	0.01137	0.10318	6.99956	-8.99684
	MSE	0.00300	0.00244	0.07157	0.01375	0.02056	0.04383
IRWL1(U=1.5)	$\hat{\tau}, \hat{\beta}$	4.00032	8.99955	0.01204	0.10035	6.99923	-8.99743
	MSE	**0.00043**	**0.00040**	**0.01099**	**0.00210**	**0.00314**	**0.00760**
IRWL2(U=1.8)	$\hat{\tau}, \hat{\beta}$	4.00282	8.99829	0.01194	0.10269	6.99913	-8.99782
	MSE	0.00251	0.00215	0.06077	0.01170	0.01752	0.03847

3.3. Analysis of simulation results:

From the previous tables, the following was noted:

1- The results, as shown in Table No. (1), (5), that in the absence of outliers in the data (unpolluted), that is, when the error follows the distribution $e_i \sim N(0,1)$, the (ML) method of (Muggeo) obtained the lowest value for the (MSE) for estimating the parameters and join points in the two experiments (1, 2) and at all different sample sizes.

2- The results in Tables No. (2), (3), (4), (6), (7), (8) also showed that in the case of the presence of outliers in the data (pollution), the proposed method of employing the Laplace estimator (Lap) and the proposed iterative algorithm (LRWL) and at the optimal values (u1, u2) obtained (MSE) values lower than the (ML) method of (Muggeo) in experiments (1, 2) and at all different sample sizes.
The proposed iterative algorithm (LRWL) at the optimal value (u1=1.5) obtained the lowest value of (MSE) compared to the Other methods, followed by the proposed method of employing the Laplace estimator (Lap) at the optimal value (u1=1.5).

4. Conclusions and Recommendations:

4.1. Conclusions:

Based on the above and the results achieved after application in the practical experimental aspect using simulation, the following conclusions were drawn:

1- Muggeo's (ML) method excelled the methods proposed in this study when there are no outliers in the data (unpolluted), at all different sample sizes considered in this study, and at all cases of multiple join points applied in the experiment (1, 2).

2- The proposed iterative algorithm for the weighted Laplace (IRWL) estimator excelled the other methods applied in this study at the optimal value (u1=1.5) when there are outliers in the data (pollution) at a rate of (5%, 10%, 15%) and at all different sample sizes and at all cases of multiple join points applied in the experiment (1, 2).

4.2. Recommendations:

Based on the conclusions reached from the numerical results of this study, some recommendations can be made:

- Using Muggeo's (ML) method to estimate the parameters of a segmented linear regression model when the data are unpolluted (no outliers).
- The iterative weighted Laplace (IRWL) estimator algorithm can be adopted at the optimal value (u1=1.5) as a robust method for estimating the parameters of a segmented linear regression model when the data are pollution (presence of outliers).

References:

[1] Acitas, S., Senoglu, B.(2020),”Robust change point estimation in two phase linear regression models:An application to metabolic pathway data”,Journal of Computational & Applied Mathematics,Vol.363,p. 337–349.

[2] Ahmed, A.D, Abdulwahhab, B.I, Abdulah, E.K.(2020),” A comparison among Different Methods for Estimating Regression Parameters with Autocorrelation Problem under Exponentially Distributed Error”, *Baghdad Science Journal,* Vol. 17, no. 3, p. 980-987.

[3] Ahmed. I.H. Shehab. D.H.(2017),”A Comparison between Methods of Laplace Estimators and the Robust Huber for Estimate parameters logistic regression model”, *journal of Economics & Administrative Sciences*, Vol. 23, no. 101, p. 524-547.

[4] Ali, O.A., Abbas, M.A. (2019),”New Robust Estimator Of Change Point In Segmented Regression Model For Bes-Load Of Rivers”, *J. Mech. Cont.& Math. Sci*., Vol.-14, No.-6, p. 384-402.

[5] Box, G.E., Muller, M.E.(1958),”A note On The Generation Of Random Normal Deviates”, *Ann. Math. Stat*., 29, p. 610-611.

[6] Chen, C.W.S., Chan, J.S.K., So, M.K.P. and Lee, K.K.M.(2011), “Classification in segmented regression problems”, Computational Statistics and Data Analysis, 55, P. 2276–2287.

[7] Diniz, C.A.R., Milan, L.A. and Mazucheli, J.(2003),"Bayesian inference of a linear segmented regression model",Brazilian Journal of Probability & Statistics,17, p. 1–16.
[8] Gbur, E.E., Dahm, P.F.(1985),"Estimation of the linear-linear segmented regression model in the presence of measurement error", Communications in Statistics-Theory and Methods,14(4),p. 809-826.
[9] Kim, H.J., Fay, M.P., Yu, B., Barrett, M.J. and Feuer, E.J.(2004), "Comparability of segmented line regression models", Biometrics, 60,P. 1005–1014.
[10] Lu, KP., Chang, ST.(2020),"Robust algorithms for multiphase regression models", Applied Mathematical Modelling,77, p.1643–1661.
[11] Muggeo, V. M.R.(2003),"Estimating regression models with unknown breakpoints", Statistics in Medicine, 22, P. 3055–3071.
[12] Quandt, R.E.(1958),"The estimation of the parameters of a linear regression system obeying two separate regimes", J. Amer. Statist. Assoc,53,P. 873–880.
[13] Ramsy, J.O.(1977),"A Comparative Study of Several Robust Estimates of Slope, Intercept, and Scale in Linear Regression", JASA, Vol. 72, no. 359, p. 608-615.
[14] Rousseeuw, P.J. and Leroy, A.(1987),"Robust regression and Outlier Detection", John Wiley and Sons.
[15] Shi, S., Li, Y. and Wan, C.(2018),"Robust continuous piecewise linear regression model with multiple change points",The Journal Of Supercomputing.
[16] Yu, B., Barrett, M. J., Kim, H.-J. and Feuer, E.J.(2007),"Estimating join points in continuous time scale for multiple change-point models", Computational Statistics and Data Analysis, 51, p. 2420–2427.