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Evaluating the Performance of Bayesian Estimates for Burr XII Inverse Rayleigh Parameters using MCMC

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ABSTRACT

This work applied multiple Bayesian approaches to estimate the parameters of the Burr XII Inverse Rayleigh (BXII-IR) Distribution. For Bayesian estimators, needed priors distribution for the parameters and certain loss functions such squared error, general entropy, and linear-exponential, owing to the unavailability of closed-form solutions for Bayesian estimates with these loss functions. Bayesian estimate employing the Markov Chain Monte Carlo (MCMC) approach were assessed for performance. The simulation results for Bayesian methods indicate that all methods consistently estimate the parameters, with the LN2 loss function estimator demonstrating the highest efficiency as assessed by mean squared error (MSE), root mean squared error (RMSE), bias, Highest Posterior Density Intervals (HPD), average interval length (AIL), and coverage probability (CP).

1. Introduction

Bayesian inference has emerged as a powerful framework in statistics, allowing researchers to update their beliefs about uncertain parameters based on observed data. Unlike traditional frequentist approaches, Bayesian methods incorporate prior knowledge and beliefs into the analysis, offering a more flexible and coherent way to make statistical inferences. This adaptability is particularly valuable in complex modelling situations where data may be sparse or noisy.

Probability distributions are a crucial element of statistics, as they are used to formulate models that reflect real-world phenomena. These distributions illustrate the risks and uncertain events we encounter in the natural world. Given the variety of natural events, it has become necessary to design a diverse range of probability distributions.

However, in many cases, well-established probability distributions fail to effectively represent the data. This has led to the development of generalized probability distributions, which have experienced growth and alteration in response to the widespread use of new elements. By including specific factors, we have been able to more accurately describe the tail shape of the distribution and make known probability distributions more compatible with data from natural occurrences.

In recent years, several extensions of the inverse Rayleigh (IR) distribution have been

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developed using various mathematical techniques, often drawn from broader distribution families. The following distribution has been identified: the beta IR distribution [1], the transmuted IR distribution [2], the weighted IR distribution [3], the odd Fr´echet IR distribution [4], the type II Topp-Leone IR distribution [5], the exponentiated IR distribution [6], the half-logistic IR distribution [7], the alpha-power exponentiated IR distribution [8], extended odd Weibull IR distribution [9], and a new IR distribution [10]. We examined Bayesian techniques for parameter estimation of the BXII-IR distribution. When it comes to statistical inference and data modelling, parameter estimation is king. In ambiguous situations, Bayesian methods provide adaptability and robustness by incorporating previous knowledge into parameter estimations. As an alternative, Parameter estimate methodologies have been explored and researched by numerous authors in this topic, as demonstrated in ref. [11], [12], [13], [14], [15], [16], [17].

The remainder of the document is structured as follows: Section 2 delineates the BXII-IR distribution. The Maximum Likelihood Estimation methods presented in Section 3. Section 4 examines Bayesian estimation methodologies that use several loss functions, including Squared Error (SE), LINEX, and Generalized Error (GE), while supposing independent priors. Section 5 simulation and Section 6 presents the Discussion.

2. The Burr XII Inverse Rayleigh (BXII-IR) Distribution

According to the study conducted by Khalaf [18], we have formulated this Burr XII – G family of distributions (BXII-G).

$$K(x)_{BXII-G} = 1 - \left[1 + \left[\frac{F(x;\xi)^2}{1-F(x;\xi)}\right]^{\rho}\right]^{-\lambda} \quad (1)$$

The pdf of the BXII –G Family is

$$k(x)_{BXII-G} = \lambda\rho F(x;\xi) f(x;\xi)\left(2 - F(x;\xi)\right) \times \left(1-F(x;\xi)\right)^{-2}\left[\frac{F(x;\xi)^2}{1-F(x;\xi)}\right]^{\rho-1} \times \left[1 + \left[\frac{F(x;\xi)^2}{1-F(x;\xi)}\right]^{\rho}\right]^{-(\lambda+1)} \quad (2)$$

Let X be a random variable characterized by an Inverse Rayleigh (IR) distribution with a scaling parameter. β is greater than zero. The cumulative distribution function (cdf) and probability density function (pdf) are as follows:

$$J(x)_{IR} = e^{-\frac{\beta}{x^2}} \quad (3)$$

and

$$j(x)_{IR} = \frac{2\beta}{x^3} e^{-\frac{\beta}{x^2}} \qquad , x > 0, \beta > 0 \quad (4)$$

By inserting equations (1) and (2) into equations (3) and (4), one may derive a new distribution known as the Burr XII Inverse Rayleigh (BXII-IR) distribution. The cdf and pdf of the OBXII-IR distribution have been derived.[19]

$$K(x)_{\mathrm{OBXII-IR}} = 1 - \left(1 + \left(\frac{e^{-\frac{2\beta}{x^2}}}{1-e^{-\frac{\beta}{x^2}}}\right)^{\rho}\right)^{-\lambda} \quad (5)$$

$$k(x)_{\mathrm{OBXII-IR}} = \frac{2\lambda\rho\beta}{x^3} e^{-\frac{2\beta}{x^2}}\left(2 - e^{-\frac{\beta}{x^2}}\right) \times \left(1 - e^{-\frac{\beta}{x^2}}\right)^{-2}\left(\frac{e^{-\frac{2\beta}{x^2}}}{1-e^{-\frac{\beta}{x^2}}}\right)^{\rho-1} \times \left(1 + \left(\frac{e^{-\frac{2\beta}{x^2}}}{1-e^{-\frac{\beta}{x^2}}}\right)^{\rho}\right)^{-(\lambda+1)} \quad (6)$$

where x ≥ 0, both λ, ρ > 0 shape parameters, and β > 0 is the scale parameter.

The following figures (1) and (2) represent the pdf and cdf of the studied distribution. Researchers have utilized the R programming language to effectively visualize these functions. By employing libraries such as ggplot2, they can create clear and informative plots that illustrate the behavior of the distribution, facilitating a deeper understanding of its properties and applications in statistical analysis. These visualizations are crucial for interpreting the underlying patterns and trends in the data, enabling researchers to make informed decisions based on their findings.

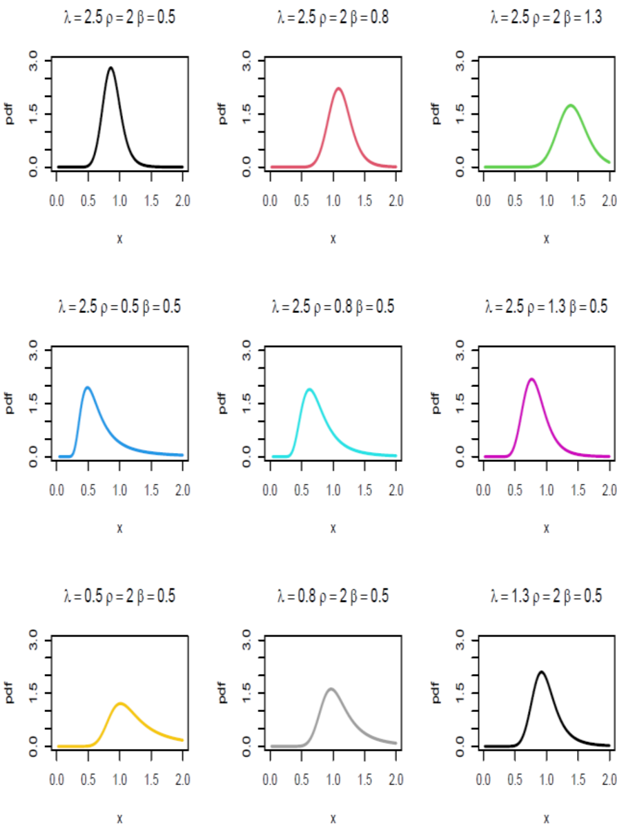

Figure 1: Plots of the BXII-IR distribution pdf.

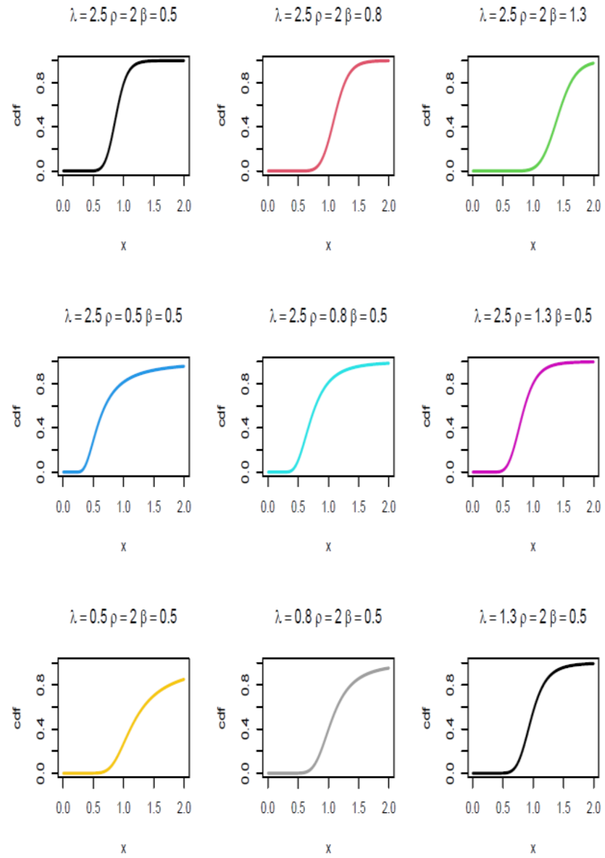

Figure 2: Plots of the BXII-IR distribution cdf.

3. Maximum Likelihood Estimation

Let $x_1, x_2, \dots, x_n$ represent a random sample of size n obtained from the BXII-IR distribution; hence, the likelihood function L is articulated as: [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32]:

$$L(\Theta|X) = \prod_{v=1}^{n} f(x_v;\Theta)$$

$$= \prod_{v=1}^{n} \left(\frac{2\lambda\rho\beta}{x_v^3} e^{-\frac{2\beta}{x_v^2}} \left(2 - e^{-\frac{\beta}{x_v^2}}\right)\left(1 - e^{-\frac{\beta}{x_v^2}}\right)^{-2} \left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)^{\rho-1} \left(1 + \left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)^{\rho}\right)^{-(\lambda+1)} \right) \quad (7)$$

the derivation of the log-likelihood function for Θ is as follows:

$$\ell(\Theta) = nlog2 + nlog\lambda + nlog\,\rho + nlog\,\beta - 3\sum_{v=0}^{n} log\, x_v - 3\beta \sum_{v=0}^{n} \frac{1}{x_v^2} + \sum_{v=0}^{n} log\left(2 - e^{-\frac{\beta}{x_v^2}}\right) - 2\sum_{v=0}^{n} log\left(1 - e^{-\frac{\beta}{x_v^2}}\right) + (\rho-1)\sum_{v=0}^{n} log\left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right) - (\lambda+1)\sum_{v=0}^{n} log \left(1 + \left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)^{\rho}\right) \quad (8)$$

use the first partial derivative of the log-likelihood function to estimate the probability of parameters λ, ρ, and β.

$$\frac{\partial\ell(\Theta)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{v=0}^{n} log\left(1 + \left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{--\frac{\beta}{x_v^2}}}\right)^{\rho}\right) \quad (9)$$

$$\frac{\partial\ell(\Theta)}{\partial\rho} = \frac{n}{\rho} + \sum_{v=0}^{n} log\left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right) - (\lambda+1)\sum_{i=0}^{n} \frac{\left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)^{\rho} \ln\left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)}{1 + \left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)^{\rho}} \quad (10)$$

$$\frac{\partial \ell(\Theta)}{\partial \beta}=\frac{n}{\beta}-3\sum_{v=0}^{n}\frac{1}{x_v^2}+\sum_{v=0}^{n}\frac{\frac{1}{x_v^2}e^{-\frac{\beta}{x_v^2}}}{\left(2-e^{-\frac{\beta}{x_v^2}}\right)}-2\sum_{v=0}^{n}\frac{\frac{1}{x_v^2}e^{-\frac{\beta}{x_v^2}}}{\left(1-e^{-\frac{\beta}{x_v^2}}\right)}$$
$$+(\rho-1)\sum_{v=0}^{n}e^{-\frac{2\beta}{x_v^2}}\left(1-e^{-\frac{\beta}{x_v^2}}\right)$$
$$\left(-\frac{\frac{2}{x_i^2}e^{-\frac{2\beta}{x_v^2}}}{\left(1-e^{-\frac{\beta}{x_v^2}}\right)}-\frac{\frac{1}{x_i^2}e^{-\frac{3\beta}{x_v^2}}}{\left(1-e^{-\frac{\beta}{x_v^2}}\right)^2}\right)-(\lambda+1)$$
$$\times\sum_{i=0}^{n}\frac{\rho e^{-\frac{2\beta}{x_v^2}}\left(1-e^{-\frac{\beta}{x_v^2}}\right)\left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)^{\rho}\left(-\frac{\frac{2}{x_i^2}e^{-\frac{2\beta}{x_v^2}}}{\left(1-e^{-\frac{\beta}{x_v^2}}\right)}-\frac{\frac{1}{x_i^2}e^{-\frac{3\beta}{x_v^2}}}{\left(1-e^{-\frac{\beta}{x_v^2}}\right)^2}\right)}{1+\left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)^{\rho}} \quad (11)$$

to estimate λ, ρ, and β, solve three simultaneous equations. $\frac{\partial(l)}{\partial\lambda}=\frac{\partial(l)}{\partial\rho}=\frac{\partial(l)}{\partial\beta}=0$, The needed findings may be obtained using statistical software such as R.

4. Bayesian Estimation of BXII-IR Parameters

4.1 prior distribution

The parameters λ, ρ, and β. are assumed to be independent random variables that follow gamma prior distributions:

$$\pi_1(\lambda)\propto\lambda^{s_1-1}e^{-k_1\lambda}\quad \lambda,s_1,k_1>0$$

$$\pi_2(\rho)\propto\rho^{s_2-1}e^{-k_2\rho}\quad \rho,s_2,k_2>0$$

$$\pi_3(\beta)\propto\beta^{s_3-1}e^{-k_3\beta}\quad \beta,s_3,k_3>0.$$

Hyper-parameters s_j, m_j, j = 1, 2, and 3 are selected to reflect the known information about the unknown parameters .

Here is the joint prior density of $\Theta=(\lambda,\rho,\beta)$

$$\pi(\Theta)=\pi_1(\lambda)\pi_1(\rho)\pi_1(\beta)$$

$$\pi(\Theta)\propto\lambda^{s_1-1}\rho^{s_2-1}\beta^{s_3-1}e^{-k_1\lambda-k_2\rho-k_3\beta}\quad (12)$$

4.2 Posterior distribution

The posterior distribution may be articulated as the joint posterior density function of Θ, derived from the amalgamation of equations (7) and (8).

$$\pi(\Theta\backslash x=x_1,x_2,\dots,x_n)=\frac{\pi(\Theta)L(\Theta/\mathrm{x})}{\int_0^\infty\int_0^\infty\pi(\Theta)L(\Theta/\mathrm{x})d\Theta}\quad (13)$$

so

$$\pi(\Theta\backslash x)=\lambda^{s_1-1}\rho^{s_2-1}\beta^{s_3-1}e^{-m_1\lambda-m_2\rho-m_3\beta}$$
$$\times\prod_{v=1}^{n}\left(\frac{\frac{2\lambda\rho\beta}{x_v^3}e^{-\frac{2\beta}{x_v^2}}\left(2-e^{-\frac{\beta}{x_v^2}}\right)\left(1-e^{-\frac{\beta}{x_v^2}}\right)^{-2}\left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)^{\rho-1}}{\left(1+\left(\frac{e^{-\frac{2\beta}{x_v^2}}}{1-e^{-\frac{\beta}{x_v^2}}}\right)^{\rho}\right)^{-(\lambda+1)}}\right)\quad (14)$$

4.3 Loss Functions

The loss functions are used to quantify the cost of errors in parameter estimates. Here are the three methods:

4.3.1 Squared Error Loss Function

It is a well-liked loss function. A symmetric function is defined as [33]

$$L_{SE}\left(l(\Theta),\hat{l}(\Theta)\right)=\left(l(\Theta)-\hat{l}(\Theta)\right)^2\quad (15)$$

where $\hat{l}(\Theta)$ is a function that estimates $l(\Theta)$.

The Bayes estimate of $l(\Theta)$ is the posterior mean expressed as

$$\hat{l}_{SEL}(\Theta)=E(l(\Theta)\backslash x)=\int_{\Theta}l(\Theta)\,\pi(\Theta\backslash x)d\Theta\quad (16)$$

furthermore, using the SE loss function inside a Bayesian framework incurs an identical penalty for under- and over-estimation. The use of the LINEX and GE loss functions offers a resolution.

4.3.2 LINEX Loss Function

This loss function is common in Bayes calculations. An asymmetric function. It is presented by [34]

$$L_{\text{LINEX}}\left(l(\Theta),\hat{l}(\Theta)\right)=e^{v\left(l(\Theta)-\hat{l}(\Theta)\right)}-v\left(l(\Theta),\hat{l}(\Theta)\right)-1\quad ,\ v\neq 0\quad (17)$$

where v affects the degree of asymmetry. The Bayes estimate is shown as

$$\hat{l}_{LINEX}(\Theta)=-\frac{1}{v}\ln\left(E_\Theta\left[e^{-vl(\Theta)}\backslash x\right]\right)$$
$$=-\frac{1}{v}ln\int_{\Theta}e^{-vl(\Theta)}\,\pi(\Theta\backslash x)d\Theta\quad (18)$$

4.3.3 General Entropy Loss Function

Investigated a novel loss function called the global entropy loss function, which is described as [35], [36]

$$L_{\text{GE}}\left(l(\Theta),\hat{l}(\Theta)\right)=\left(\frac{\hat{l}(\Theta)}{l(\Theta)}\right)^{\omega}-\omega\ln\left(\frac{\hat{l}(\Theta)}{l(\Theta)}\right)-1\,,\omega\neq 0 \qquad (19)$$

this is also an asymmetric loss function. The Bayes estimate of $l(\Theta)$ is

$$\hat{l}_{GE}(\Theta)=(E_{\Theta}[(l(\Theta))^{-\omega}\backslash x])^{-\frac{1}{\omega}}$$

$$=\left[\int_{\Theta}(l(\Theta))^{-\omega}\,\pi(\Theta\backslash x)d\Theta\right]^{-\frac{1}{\omega}} \qquad (20)$$

4.3.4 Markov Chain Monte Carlo (MCMC)

We used MCMC in R to estimate unknown parameters due to the analytical complexity of Equations (18), (19), and (20). Subsequently, we produced posterior samples and obtained satisfactory Bayesian estimates by MCMC. MCMC simulation is advantageous for predicting critical parameters and sampling posterior distributions. Bayesian estimates may be derived using MCMC and three functions. The Bayesian estimates of $\Theta^{(i)}=\left(\lambda^{(i)},\rho^{(i)},\beta^{(i)}\right)$ were calculated using MCMC under SEL, LINEX, and GEL functions.

$$\widehat{\Theta}_{SEL}=\frac{1}{Q-l_B}\sum_{i=l_B}^{Q}\Theta^{(i)} \qquad (21)$$

$$\widehat{\Theta}_{LINEX}=-\frac{1}{\eta}\ln\left(\frac{1}{Q-l_B}\sum_{i=l_B}^{Q}e^{-\eta\Theta^{(i)}}\right) \qquad (22)$$

and

$$\widehat{\Theta}_{GE}=\left(\frac{1}{Q-l_B}\sum_{i=l_B}^{Q}\left[\Theta^{(i)}\right]^{-\omega}\right)^{-\frac{1}{\omega}} \qquad (23)$$

where l_B is the burn-in duration of the MCMC.

5. Simulation

This section calculates the distribution parameters of BXII-IR using a Bayesian estimator. The Bayesian method was executed via MCMC and the Metropolis-Hastings (MH) algorithm with a substantive prior. In calculating the informative prior, we posited that all hyperparameters of the gamma distribution were equal to double the parameter values. The Bayesian estimate used the SE, GE, and LINEX loss functions. The simulation analyzed many scenarios with distinct values for:

- λ= 0.6, ρ=3.6, β=1.3
- λ= 0.5, ρ=3.3, β=1.1

They evaluated the two ρ values of the LINEX loss function, -0.5 and 0.5. Similarly, we tested θ = −0.5 and 0.5 using the GE loss function with s1=s2=s3=p1=p2=p3=0.5. Each sample of N=1000 comprised of sizes 35, 70, 120, 200, and 400.

The estimator's effectiveness was assessed using mean, mean squared error (MSE), root mean squared error (RMSE), bias, Highest Posterior Density Intervals (HPD), average interval length (AIL), and coverage probability (CP).

The simulation results indicate the following observations:

- All Bayesian estimating techniques include the consistency characteristic, which guarantees that, as the sample size develops the estimates will converge to the real values.
- All of our comparative criteria diminish as sample size grows, with the exception of CP.
- Tables 1-4 demonstrate that all estimation strategies exhibit strong performance for the OBXII-IR model parameters.
- The Bayesian simulation results reveal that the LN2 approach exhibits higher performance relative to other estimating approaches in most cases, with the GE method following closely behind.

  Tables and Figures are presented in center as shown in Table 1 and Figure 3.

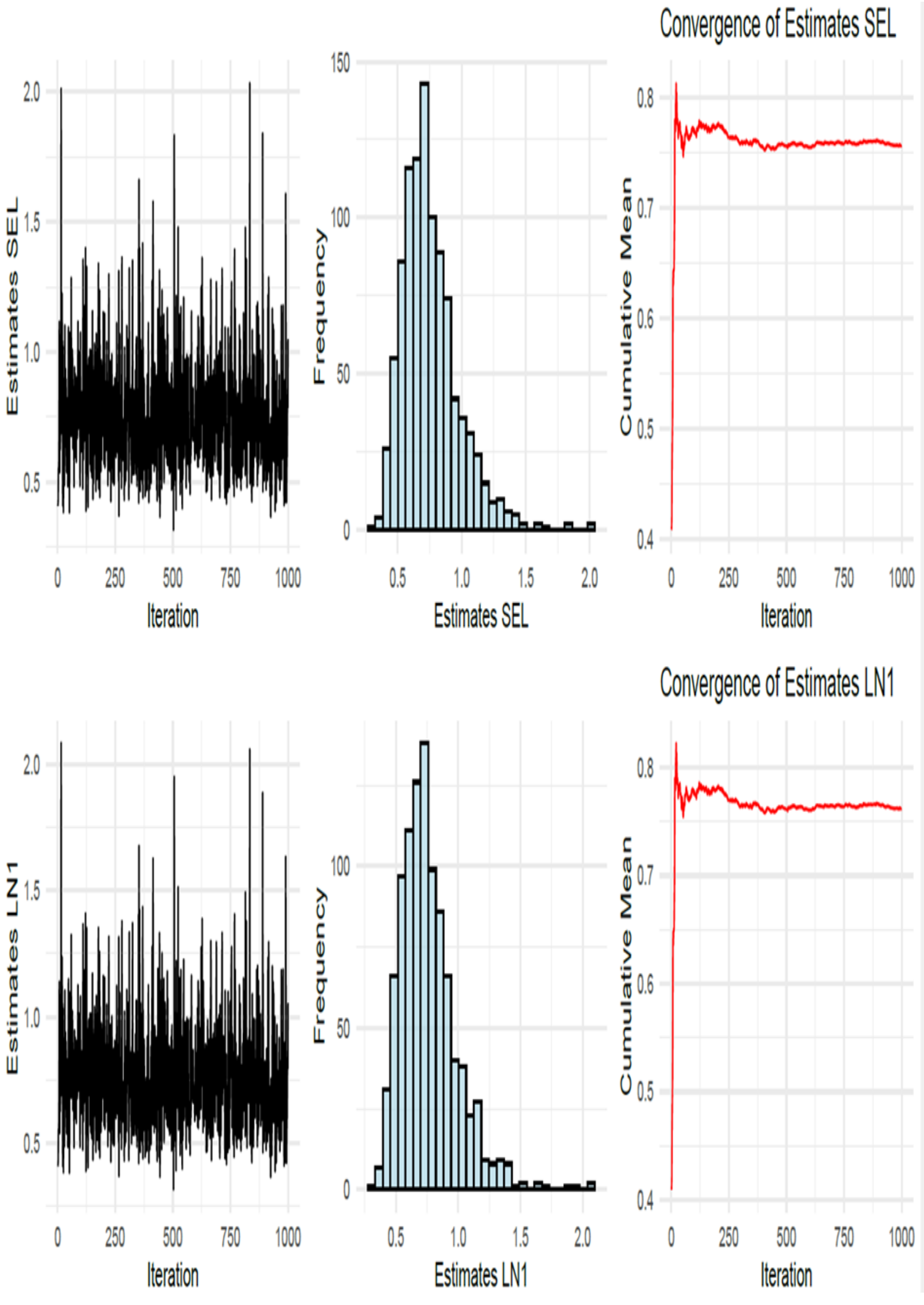

Figure 3: MCMC plots of the BXII-IR distribution with parameters λ= 0.6, ρ=3.6, β=1.3 for SEL, and LN1, loss functions.

Table 1: The MSE, RMSE, and Bias are computed using the Bayesian BXII-IR model with parameters λ= 0.6, ρ=3.6, β=1.3

n		Est. Par.	SEL	LN1 -0.5	LN2 0.5	GE1 -0.5	GE2 0.5
35	MSE	$\hat{\lambda}$	6.110070{4}	8.415795{5}	3.943897{1}	5.454733{3}	4.521906{2}
		$\hat{\rho}$	12.13299{4}	40.53164{5}	5.522070{1}	10.89569{3}	8.996492{2}
		$\hat{\beta}$	1.042125{4}	1.762598{5}	0.720570{1}	0.949799{3}	0.797936{2}
	RMSE	$\hat{\lambda}$	2.471855{4}	2.900999{5}	1.985924{1}	2.335537{3}	2.126477{2}
		$\hat{\rho}$	3.483244{5}	6.366447{4}	2.349908{3}	3.300862{2}	2.999415{1}
		$\hat{\beta}$	1.020845{4}	1.327628{5}	0.848864{1}	0.974576{3}	0.893277{2}
	Bias	$\hat{\lambda}$	1.120200{4}	1.256976{5}	0.894944{1}	1.042105{3}	0.910327{2}
		$\hat{\rho}$	0.977840{4}	1.660580{5}	0.550289{1}	0.894932{3}	0.739540{2}
		$\hat{\beta}$	0.438105{4}	0.529706{5}	0.379665{1}	0.419408{3}	0.385663{2}
	$\sum$ ***Ranks***		37{5}	34{4}	11{1}	24{3}	17{2}
70	MSE	$\hat{\lambda}$	3.477345{5}	2.662075{2}	2.033752{1}	3.233073{4}	2.675967{3}
		$\hat{\rho}$	5.993521{4}	6.395947{5}	3.111940{1}	5.420946{3}	4.528560{2}
		$\hat{\beta}$	0.548940{4}	0.864720{5}	0.383680{1}	0.504250{3}	0.427780{2}
	RMSE	$\hat{\lambda}$	1.864764{5}	1.631586{2}	1.426097{1}	1.798074{4}	1.635841{3}
		$\hat{\rho}$	2.448167{4}	2.529021{5}	1.764069{1}	2.328292{3}	2.128041{2}
		$\hat{\beta}$	0.740905{4}	0.929903{5}	0.619419{1}	0.710105{3}	0.654048{2}
	Bias	$\hat{\lambda}$	0.675975{5}	0.568589{3}	0.550075{1}	0.642981{4}	0.567784{2}
		$\hat{\rho}$	0.824902{4}	0.993145{5}	0.599722{1}	0.779649{3}	0.692229{2}
		$\hat{\beta}$	0.274100{4}	0.315047{5}	0.245707{1}	0.265124{3}	0.248478{2}
	$\sum$ ***Ranks***		39{5}	37{4}	9{1}	30{3}	20{2}
120	MSE	$\hat{\lambda}$	1.004037{2}	1.109127{3}	1.199568{4}	1.043261{2}	1.234615{5}
		$\hat{\rho}$	2.063999{4}	2.516117{5}	1.724403{1}	2.006391{3}	1.897875{2}
		$\hat{\beta}$	0.409182{4}	0.639325{5}	0.300376{1}	0.382135{3}	0.336577{2}
	RMSE	$\hat{\lambda}$	1.002016{1}	1.053151{3}	1.095247{4}	1.021401{2}	1.111132{5}
		$\hat{\rho}$	1.436662{4}	1.586227{5}	1.313165{1}	1.416471{3}	1.377633{2}
		$\hat{\beta}$	0.639673{4}	0.799578{5}	0.548066{1}	0.618171{3}	0.580153{2}
	Bias	$\hat{\lambda}$	0.355974{3}	0.364983{4}	0.368798{5}	0.346949{2}	0.339716{1}
		$\hat{\rho}$	0.657578{4}	0.780975{5}	0.546003{1}	0.632842{3}	0.583836{2}
		$\hat{\beta}$	0.204188{4}	0.227487{5}	0.188494{1}	0.199430{3}	0.190553{2}
	$\sum$ ***Ranks***		30{3}	40{5}	19{1}	34{4}	23{2}
200	MSE	$\hat{\lambda}$	0.361046{5}	0.278072{2}	0.263599{1}	0.329848{4}	0.277839{3}
		$\hat{\rho}$	1.338734{4}	1.527633{5}	1.181994{1}	1.308575{3}	1.251104{2}
		$\hat{\beta}$	0.112033{4}	0.226066{5}	0.079353{1}	0.104139{3}	0.091555{2}
	RMSE	$\hat{\lambda}$	0.600871{5}	0.527325{3}	0.513419{1}	0.574323{4}	0.527104{2}
		$\hat{\rho}$	1.157036{4}	1.235974{5}	1.087195{1}	1.143929{3}	1.118527{2}
		$\hat{\beta}$	0.334714{4}	0.475464{5}	0.281698{1}	0.322707{3}	0.302581{2}
	Bias	$\hat{\lambda}$	0.246196{4}	0.246213{5}	0.227603{2}	0.233757{3}	0.209882{1}
		$\hat{\rho}$	0.650295{4}	0.729530{5}	0.576211{1}	0.633290{3}	0.599510{2}
		$\hat{\beta}$	0.126591{4}	0.134362{5}	0.122088{2}	0.124813{3}	0.121451{1}
	$\sum$ ***Ranks***		38{4}	40{5}	11{1}	29{3}	17{2}
400	MSE	$\hat{\lambda}$	0.074255{4}	0.077949{5}	0.070852{2}	0.071302{3}	0.065833{1}
		$\hat{\rho}$	0.826828{4}	0.912027{5}	0.752030{1}	0.810994{3}	0.780364{2}
		$\hat{\beta}$	0.018221{4}	0.018479{5}	0.017954{2}	0.018045{3}	0.017715{1}
	RMSE	$\hat{\lambda}$	0.272499{4}	0.279195{5}	0.266180{2}	0.267025{3}	0.256579{1}
		$\hat{\rho}$	0.909301{4}	0.955001{5}	0.867196{1}	0.900552{3}	0.883382{2}
		$\hat{\beta}$	0.134959{4}	0.135939{5}	0.133993{2}	0.134334{3}	0.133098{1}
	Bias	$\hat{\lambda}$	0.157077{4}	0.162225{5}	0.152082{3}	0.151277{2}	0.139890{1}
		$\hat{\rho}$	0.634374{4}	0.679714{5}	0.590833{1}	0.624260{3}	0.604118{2}
		$\hat{\beta}$	0.091742{4}	0.092624{5}	0.090867{2}	0.091136{3}	0.089931{1}
	$\sum$ ***Ranks***		36{4}	45{5}	16{2}	26{3}	12{1}
Over Ranks			**21{4}**	**23{5}**	**6{1}**	**16{3}**	**9{2}**

Table 2: The HPD, AIL, and CP are computed using the Bayesian BXII-IR model with parameters λ= 0.6, ρ=3.6, β=1.3.

n		Est. Par.	SEL	LN1 -0.5	LN2 0.5	GE1 -0.5	GE2 0.5
35	HPD	$\hat{\lambda}$	0.037427{4} 7.549165{4}	0.037512{3} 9.371830{5}	0.059576{1} 5.733666{1}	0.045599{2} 7.014039{3}	0.030631{5} 6.218598{2}
		$\hat{\rho}$	0.614069{3} 10.35810{4}	0.569677{4} 11.92185{5}	0.541307{5} 8.734096{1}	0.749808{1} 8.934022{2}	0.672411{2} 10.05630{3}
		$\hat{\beta}$	0.950352{4} 3.831182{5}	0.950719{3} 4.365884{1}	0.949986{5} 3.460050{2}	0.973648{1} 3.646608{3}	0.972968{2} 3.697648{4}
	AIL	$\hat{\lambda}$	7.511737{4}	9.334318{5}	5.674090{1}	6.968440{3}	6.187966{2}
		$\hat{\rho}$	9.744035{4}	11.35217{5}	8.192789{2}	8.184213{1}	9.383893{3}
		$\hat{\beta}$	2.880830{4}	3.415165{5}	2.510064{1}	2.672959{2}	2.724680{3}
	CP	$\hat{\lambda}$	96.10{2}	96.60{1}	95.60{4}	95.70{3}	95.20{5}
		$\hat{\rho}$	96.70{2.5}	96.50{5}	96.70{2.5}	95.90{4}	97.10{1}
		$\hat{\beta}$	95.80{2.5}	95.40{5}	95.80{2.5}	95.60{4}	97.00{1}
	$\sum$ ***Ranks***		43{4}	47{5}	28{1}	29{2}	33{3}
70	HPD	$\hat{\lambda}$	0.055330{2} 5.633320{5}	0.055591{1} 3.790540{1}	0.055071{3} 5.024039{3}	0.049986{4} 5.056437{4}	0.038183{5} 4.843163{2}
		$\hat{\rho}$	1.219268{1} 7.776007{4}	1.185444{2} 8.204072{5}	1.047113{5} 7.002795{1}	1.106169{3} 7.726221{3}	1.053896{4} 7.390620{2}
		$\hat{\beta}$	0.956935{3} 3.551622{4}	1.034800{1} 3.722842{5}	1.034399{2} 3.077993{1}	0.956850{4} 3.542016{3}	0.956681{5} 3.078648{2}
	AIL	$\hat{\lambda}$	5.577989{5}	3.734949{1}	4.968968{3}	5.006450{4}	4.804979{2}
		$\hat{\rho}$	6.556739{4}	7.018627{5}	5.955681{1}	6.620052{3}	6.336723{2}
		$\hat{\beta}$	2.594686{4}	2.688041{5}	2.043531{1}	2.585165{3}	2.121966{2}
	CP	$\hat{\lambda}$	95.60{4}	96.70{1}	96.50{2}	95.50{5}	95.80{3}
		$\hat{\rho}$	96.90{2}	96.30{3}	95.90{5}	97.00{1}	96.30{4}
		$\hat{\beta}$	96.80{2}	96.60{3}	96.30{4}	96.90{1}	96.20{5}
	$\sum$ ***Ranks***		40{5}	33{2}	31{1}	38{3.5}	38{3.5}
120	HPD	$\hat{\lambda}$	0.193473{4} 2.433005{5}	0.195545{3} 1.940622{2}	0.272451{1} 2.299096{4}	0.184015{5} 1.867234{1}	0.266736{2} 2.233986{3}
		$\hat{\rho}$	2.228137{3} 7.789427{4}	2.208415{4} 7.902191{5}	2.234549{2} 7.032290{2}	2.245546{1} 7.522744{3}	2.206744{5} 6.966401{1}
		$\hat{\beta}$	1.102588{3} 2.031199{2}	1.102695{2} 2.127802{5}	1.102481{4} 2.116536{4}	1.105230{1} 2.027432{1}	1.102297{5} 2.050323{3}
	AIL	$\hat{\lambda}$	2.149531{5}	1.745077{2}	2.026645{4}	1.683210{1}	1.967250{3}
		$\hat{\rho}$	5.561289{4}	5.693776{5}	4.797741{2}	5.277198{3}	4.759656{1}
		$\hat{\beta}$	0.928610{2}	1.025107{5}	1.014054{4}	0.922201{1}	0.948025{3}
	CP	$\hat{\lambda}$	97.00{1}	95.50{4}	96.80{2.5}	95.40{5}	96.80{2.5}
		$\hat{\rho}$	99.10{1}	98.50{3}	98.10{4}	98.80{2}	97.70{5}
		$\hat{\beta}$	96.60{4.5}	97.10{2}	97.20{1}	96.60{4.5}	96.90{3}
	$\sum$ ***Ranks***		38.5{4}	42{5}	34.5{2}	28.5{1}	36.5{3}
200	HPD	$\hat{\lambda}$	0.299802{1} 1.585624{1}	0.238514{5} 1.697002{4}	0.299597{2} 1.602405{3}	0.299111{3} 1.747130{5}	0.275939{4} 1.593538{2}
		$\hat{\rho}$	2.340151{4} 6.17361{7}	2.433860{1} 6.566146{5}	2.411503{3} 6.489029{4}	2.336729{5} 6.398365{3}	2.413825{2} 6.182083{2}
		$\hat{\beta}$	1.130886{5} 1.783693{4}	1.127099{3} 1.789702{5}	1.141026{2} 1.718688{2}	1.141053{1} 1.724985{3}	1.126298{4} 1.711545{1}
	AIL	$\hat{\lambda}$	1.285821{1}	1.458488{5}	1.302808{2}	1.448319{4}	1.317599{3}
		$\hat{\rho}$	3.833466{2}	4.132286{5}	4.077526{4}	4.061635{3}	3.768257{1}
		$\hat{\beta}$	0.625806{4}	0.662603{5}	0.577662{1}	0.583932{2}	0.618115{3}
	CP	$\hat{\lambda}$	95.90{5}	96.70{2.5}	96.50{4}	97.30{1}	96.70{2.5}
		$\hat{\rho}$	96.50{5}	97.30{3}	98.20{1}	97.60{2}	97.00{4}
		$\hat{\beta}$	97.00{1.5}	97.00{1.5}	96.00{5}	96.10{4}	96.50{3}
	$\sum$ ***Ranks***		40.5{4}	45{5}	33{2}	36{3}	31.5{1}
400	HPD	$\hat{\lambda}$	0.367295{2} 1.178246{1}	0.384975{1} 1.185863{2}	0.365198{3} 1.190741{3}	0.362900{5} 1.217937{5}	0.364924{4} 1.209932{4}
		$\hat{\rho}$	3.044700{1} 5.530277{2}	3.012961{4} 5.785793{5}	3.016605{3} 5.642622{4}	3.027745{2} 5.544400{3}	2.695808{5} 5.529270{1}
		$\hat{\beta}$	1.225579{1} 1.586774{3}	1.220255{4} 1.587386{5}	1.216803{5} 1.584986{1}	1.220487{3} 1.586388{2}	1.221887{2} 1.587092{4}
	AIL	$\hat{\lambda}$	0.810951{2}	0.800887{1}	0.825543{3}	0.855036{5}	0.845007{4}
		$\hat{\rho}$	2.485577{1}	2.772832{5}	2.626016{4}	2.516655{2}	2.563461{3}

		$\hat{\beta}$	0.361195{1}	0.367131{4}	0.368183{5}	0.364339{2}	0.365204{3}
	CP	$\hat{\lambda}$	95.90{4}	95.60{5}	96.10{3}	96.60{2}	97.00{1}
		$\hat{\rho}$	96.80{5}	97.90{2}	98.20{1}	97.10{3.5}	97.10{3.5}
		$\hat{\beta}$	96.70{3}	96.80{2.5}	96.60{4}	96.80{2.5}	96.90{1}
	$\sum$ ***Ranks***		26{1}	40.5{5}	39{4}	37{3}	35.5{2}
Over Ranks			**18{4}**	**22{5}**	**10{1}**	**12.5{2.5}**	**12.5{2.5}**

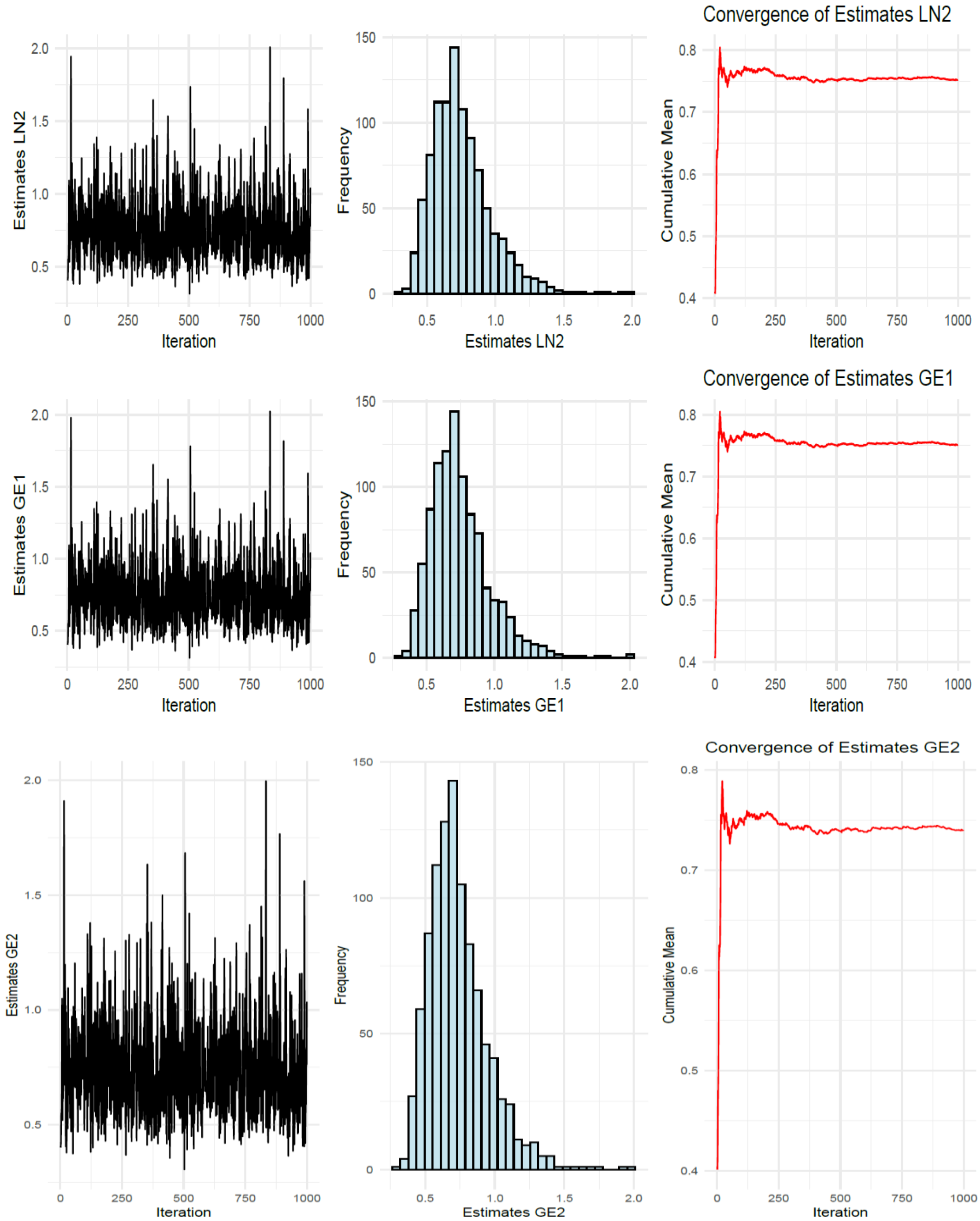

Figure 4: MCMC plots of the BXII-IR distribution with parameters λ= 0.6, ρ=3.6, β=1.3 for LN2, GE1 and GE2 loss functions.

Table 3: The MSE, RMSE, and Bias are computed using the Bayesian BXII-IR model with parameters λ= 0.5, ρ=3.3, β=1.1

n		Est. Par.	SEL	LN1 -0.5	LN2 0.5	GE1 -0.5	GE2 0.5
35	MSE	$\hat{\lambda}$	3.308107{4}	4.720729{5}	2.518438{1}	3.012347{3}	2.664968{2}
		$\hat{\rho}$	15.28463{4}	17.46780{5}	6.411173{1}	13.56394{3}	10.90914{2}
		$\hat{\beta}$	0.913080{4}	1.366317{5}	0.672755{1}	0.841094{3}	0.717845{2}
	RMSE	$\hat{\lambda}$	1.818820{4}	2.172723{5}	1.586958{1}	1.735611{3}	1.632473{2}
		$\hat{\rho}$	3.909556{4}	7.514506{5}	2.532029{1}	3.682926{3}	3.302899{2}
		$\hat{\beta}$	0.955552{4}	1.168895{5}	0.820216{1}	0.917112{3}	0.847257{2}
	Bias	$\hat{\lambda}$	0.772964{4}	0.906336{5}	0.669097{2}	0.723766{3}	0.647735{1}
		$\hat{\rho}$	1.244475{4}	2.196795{5}	0.718169{1}	1.143528{3}	0.956681{2}
		$\hat{\beta}$	0.404533{4}	0.470372{5}	0.358138{1}	0.388565{3}	0.358928{2}
	$\sum$ ***Ranks***		36{4}	45{5}	10{1}	27{3}	17{2}
70	MSE	$\hat{\lambda}$	2.169307{5}	1.997624{3}	1.675796{1}	2.008607{4}	1.805568{2}
		$\hat{\rho}$	6.056701{4}	12.63636{5}	3.413122{1}	5.521526{3}	4.700350{2}
		$\hat{\beta}$	0.608895{4}	0.891544{5}	0.442757{1}	0.564226{3}	0.485564{2}
	RMSE	$\hat{\lambda}$	1.472857{5}	1.413373{3}	1.294796{1}	1.417253{4}	1.343714{2}
		$\hat{\rho}$	2.461036{4}	3.554765{5}	1.847463{1}	2.349792{3}	2.168029{2}
		$\hat{\beta}$	0.780317{4}	0.944216{5}	0.665400{1}	0.751150{3}	0.696824{2}
	Bias	$\hat{\lambda}$	0.498031{5}	0.472908{4}	0.445785{2}	0.469198{3}	0.422342{1}
		$\hat{\rho}$	0.893212{4}	1.175313{5}	0.687469{1}	0.843714{3}	0.749172{2}
		$\hat{\beta}$	0.276032{4}	0.311785{5}	0.248795{1}	0.267222{3}	0.250491{2}
	$\sum$ ***Ranks***		39{4}	40{5}	10{1}	29{3}	17{2}
120	MSE	$\hat{\lambda}$	0.810862{4}	0.525408{1}	0.754998{3}	0.850309{5}	0.754764{2}
		$\hat{\rho}$	2.226745{4}	2.833104{5}	1.810783{1}	2.152525{3}	2.013833{2}
		$\hat{\beta}$	0.360546{4}	0.542691{5}	0.262555{1}	0.338121{3}	0.298274{2}
	RMSE	$\hat{\lambda}$	0.900479{4}	0.724850{1}	0.868906{3}	0.922122{5}	0.868771{2}
		$\hat{\rho}$	1.492228{4}	1.683182{5}	1.345653{1}	1.467148{3}	1.419096{2}
		$\hat{\beta}$	0.600455{4}	0.736675{5}	0.512401{1}	0.581481{3}	0.546144{2}
	Bias	$\hat{\lambda}$	0.272727{5}	0.245338{2}	0.262985{3}	0.266892{4}	0.237423{1}
		$\hat{\rho}$	0.788911{3}	0.855347{5}	0.668047{1}	0.790897{4}	0.735482{2}
		$\hat{\beta}$	0.183854{4}	0.201512{5}	0.170797{1}	0.179710{3}	0.171766{2}
	$\sum$ ***Ranks***		36{5}	34{4}	15{1}	33{3}	17{2}
200	MSE	$\hat{\lambda}$	0.139245{1}	0.174079{2}	0.175491{3}	0.214205{5}	0.186445{4}
		$\hat{\rho}$	1.525619{4}	1.783466{5}	1.321803{1}	1.484360{3}	1.406150{2}
		$\hat{\beta}$	0.107811{4}	0.143315{5}	0.090604{1}	0.104091{3}	0.097388{2}
	RMSE	$\hat{\lambda}$	0.373156{1}	0.417227{1}	0.418916{2}	0.462823{4}	0.431793{3}
		$\hat{\rho}$	1.235159{4}	1.335464{5}	1.149697{1}	1.218345{3}	1.185812{2}
		$\hat{\beta}$	0.328346{4}	0.378570{5}	0.301005{1}	0.322632{3}	0.320717{2}
	Bias	$\hat{\lambda}$	0.145342{4}	0.156028{5}	0.143378{2}	0.145084{3}	0.126338{1}
		$\hat{\rho}$	0.733442{4}	0.825211{5}	0.649063{1}	0.713160{3}	0.682957{2}
		$\hat{\beta}$	0.112112{4}	0.116524{5}	0.108791{2}	0.110475{3}	0.107289{1}
	$\sum$ ***Ranks***		30{3.5}	38{5}	14{1}	30{3.5}	19{2}
400	MSE	$\hat{\lambda}$	0.034219{4}	0.035728{5}	0.032819{3}	0.032729{2}	0.030029{1}
		$\hat{\rho}$	0.964520{4}	1.070668{5}	0.871460{1}	0.943924{3}	0.904159{2}
		$\hat{\beta}$	0.013464{4}	0.013678{5}	0.013255{2}	0.013307{3}	0.013000{1}
	RMSE	$\hat{\lambda}$	0.184985{4}	0.189021{5}	0.181160{3}	0.180912{2}	0.173290{1}
		$\hat{\rho}$	0.982099{4}	1.034731{5}	0.933520{1}	0.971559{3}	0.950872{2}
		$\hat{\beta}$	0.116037{4}	0.116956{5}	0.115133{2}	0.115358{3}	0.114019{1}
	Bias	$\hat{\lambda}$	0.086299{3}	0.089486{4}	0.093189{5}	0.081634{2}	0.072461{1}
		$\hat{\rho}$	0.709206{4}	0.760458{5}	0.610270{1}	0.697198{3}	0.673319{2}
		$\hat{\beta}$	0.076878{4}	0.077659{5}	0.076103{2}	0.076248{3}	0.074997{1}
	$\sum$ ***Ranks***		36{4}	44{5}	20{2}	24{3}	12{1}
Over Ranks			**20.5{4}**	**24{5}**	**6{1}**	**15.5{3}**	**9{2}**

Table 4: The HPD, AIL, and CP are computed using the Bayesian BXII-IR model with parameters λ= 0.5, ρ=3.3, β=1.1

n		Est. Par.	SEL	LN1 -0.5	LN2 0.5	GE1 -0.5	GE2 0.5
35	HPD	$\hat{\lambda}$	0.036285{2} 5.771879{4}	0.036355{1} 7.401329{5}	0.036215{3} 4.761244{1}	0.034271{4} 5.434909{3}	0.028025{5} 4.927689{2}
		$\hat{\rho}$	0.561452{4} 12.60619{4}	0.571787{3} 15.87803{5}	0.653518{1} 9.763789{1}	0.572929{2} 10.31191{2}	0.503041{5} 12.06369{3}
		$\hat{\beta}$	0.803419{3} 3.622872{4}	0.803731{2} 3.944790{5}	0.806952{1} 3.341832{1}	0.803027{4} 3.509273{2}	0.802241{5} 3.557339{3}
	AIL	$\hat{\lambda}$	5.735594{4}	7.364974{5}	4.725028{1}	5.400637{3}	4.899664{2}
		$\hat{\rho}$	12.04474{4}	15.30624{5}	9.110250{1}	9.746263{2}	11.56065{3}
		$\hat{\beta}$	2.819453{4}	3.141059{5}	2.534879{1}	2.706245{2}	2.755098{3}
	CP	$\hat{\lambda}$	96.10{2}	96.70{1}	95.40{4}	95.60{3}	95.20{5}
		$\hat{\rho}$	96.60{3}	96.50{4}	96.90{1.5}	95.80{5}	96.90{1.5}
		$\hat{\beta}$	95.80{3}	95.40{5}	96.00{2}	95.50{4}	96.90{1}
	$\sum$ ***Ranks***		41{4}	46{5}	18.5{1}	36{2}	38.5{3}
70	HPD	$\hat{\lambda}$	0.052857{2} 3.922076{3}	0.053043{1} 4.813617{5}	0.052671{3} 4.166999{4}	0.048907{4} 3.527437{1}	0.043601{5} 3.596780{2}
		$\hat{\rho}$	0.855965{2} 7.865363{3}	0.874874{1} 8.381983{5}	0.838707{3} 7.070810{1}	0.836761{4} 7.978860{4}	0.801728{5} 7.501301{2}
		$\hat{\beta}$	0.815743{2} 3.542553{5}	0.815868{1} 3.441164{3}	0.815619{3} 2.860545{1}	0.815592{4} 3.488299{4}	0.815294{5} 2.999918{2}
	AIL	$\hat{\lambda}$	3.869218{3}	4.760573{5}	4.114328{4}	3.478529{1}	3.553178{2}
		$\hat{\rho}$	7.009398{3}	7.507109{5}	6.232102{1}	7.142099{4}	6.699572{2}
		$\hat{\beta}$	2.726809{5}	2.625295{3}	2.044926{1}	2.672706{4}	2.184624{2}
	CP	$\hat{\lambda}$	95.60{4}	96.70{1}	96.50{2}	95.50{5}	95.90{3}
		$\hat{\rho}$	96.50{2}	96.20{4}	96.00{5}	97.00{1}	96.20{3}
		$\hat{\beta}$	96.80{2}	96.30{3}	96.00{5}	96.90{1}	96.20{4}
	$\sum$ ***Ranks***		36{2}	37{3.5}	33{1}	37{3.5}	37{3.5}
120	HPD	$\hat{\lambda}$	0.134871{3} 1.902675{5}	0.198619{1} 1.577531{3}	0.134030{4} 1.610912{4}	0.193841{2} 1.537776{1}	0.120687{5} 1.573543{2}
		$\hat{\rho}$	0.732954{5} 6.655785{1}	1.176195{4} 7.179702{4}	1.866461{1} 6.868132{3}	1.717562{2} 7.328221{5}	1.687474{3} 6.660842{2}
		$\hat{\beta}$	0.915648{2} 1.779807{2}	0.916160{1} 1.855633{5}	0.910062{3} 1.845585{4}	0.910043{4} 1.760222{1}	0.892081{5} 1.786615{3}
	AIL	$\hat{\lambda}$	1.767803{5}	1.378911{2}	1.476881{4}	1.343934{1}	1.452856{3}
		$\hat{\rho}$	5.922831{5}	6.000512{3}	5.001670{2}	5.610658{4}	4.973368{1}
		$\hat{\beta}$	0.864158{2}	0.939473{5}	0.935522{4}	0.850179{1}	0.894533{3}
	CP	$\hat{\lambda}$	96.90{1}	95.90{4}	96.10{3}	95.70{5}	96.20{2}
		$\hat{\rho}$	96.80{5}	97.30{3}	97.80{2}	98.10{1}	97.10{4}
		$\hat{\beta}$	96.60{4}	97.10{1.5}	97.10{1.5}	96.40{5}	96.70{3}
	$\sum$ ***Ranks***		40{5}	36.5{4}	34.5{2}	32{1}	36{3}
200	HPD	$\hat{\lambda}$	0.233806{1} 1.166498{1}	0.211691{5} 1.277049{4}	0.233407{2} 1.196276{2}	0.230046{3} 1.314613{5}	0.220062{4} 1.197572{3}
		$\hat{\rho}$	2.364060{1} 6.269040{3}	2.162985{3} 6.396989{5}	2.159527{4} 6.300975{4}	2.102627{5} 6.263620{2}	2.303016{2} 6.145110{1}
		$\hat{\beta}$	0.962238{5} 1.520237{4}	0.967614{3} 1.547866{5}	0.979167{1} 1.486482{1}	0.979162{2} 1.488595{2}	0.966273{4} 1.489710{3}
	AIL	$\hat{\lambda}$	0.932691{1}	1.065357{4}	0.962868{2}	1.084567{5}	0.977510{3}
		$\hat{\rho}$	3.904980{2}	4.234004{5}	4.141448{3}	4.16099{4}	3.839095{1}
		$\hat{\beta}$	0.557999{4}	0.580251{5}	0.507300{1}	0.509433{2}	0.520437{3}
	CP	$\hat{\lambda}$	95.70{5}	96.80{2}	96.30{4}	97.10{1}	96.60{3}
		$\hat{\rho}$	97.70{3}	97.40{5}	98.40{1}	97.80{2}	97.50{4}
		$\hat{\beta}$	97.00{2}	97.30{1}	96.60{5}	96.70{4}	96.80{3}
	$\sum$ ***Ranks***		32{2}	47{5}	30{1}	37{4}	34{3}
400	HPD	$\hat{\lambda}$	0.318833{2} 0.954266{4}	0.317734{3} 0.942945{2}	0.313565{4} 0.952399{3}	0.312378{5} 0.970740{5}	0.624661{1} 0.913752{1}
		$\hat{\rho}$	2.876261{5} 5.478235{4}	2.633392{2} 5.586789{5}	2.602185{1} 5.377707{1}	2.822011{4} 5.452318{3}	2.720840{3} 5.409999{2}
		$\hat{\beta}$	1.021425{5} 1.349866{2}	1.028323{2} 1.364330{5}	1.025142{4} 1.360463{4}	1.060490{1} 1.347296{1}	1.027326{3} 1.358226{3}
	AIL	$\hat{\lambda}$	0.635432{2}	0.625211{1}	0.638834{3}	0.658361{5}	0.649090{4}
		$\hat{\rho}$	2.601973{1}	2.953397{5}	2.775521{4}	2.630307{2}	2.689152{3}

		$\hat{\beta}$	0.328440{1}	0.336007{5}	0.335320{4}	0.331247{3}	0.330899{2}
	CP	$\hat{\lambda}$	97.20{3}	96.60{5}	97.30{2}	97.80{1}	97.00{4}
		$\hat{\rho}$	97.40{1}	97.20{3.5}	97.10{5}	97.30{2}	97.20{3.5}
		$\hat{\beta}$	96.30{4}	97.50{1}	97.40{2}	96.20{5}	97.30{3}
	$\sum$ ***Ranks***		34{2}	39.5{5}	37{3.5}	37{3.5}	32.5{1}
Over Ranks			**13{2}**	**22.5{5}**	**8.5{1}**	**14{4}**	**13.5{3}**

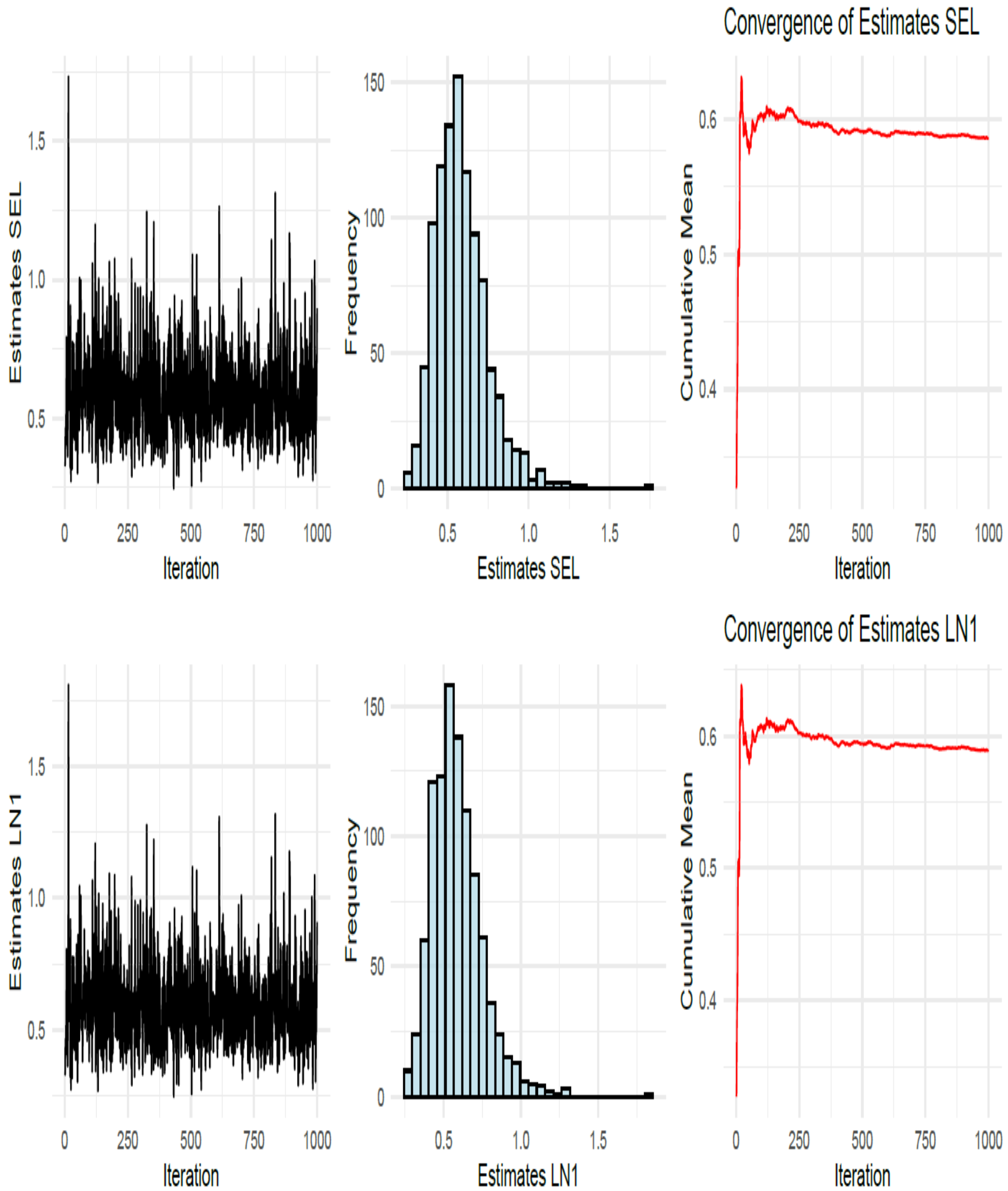

Figure 5: MCMC plots of the BXII-IR distribution with parameters λ= 0.5, ρ=3.3, β=1.1 for SEL, LN1, loss functions.

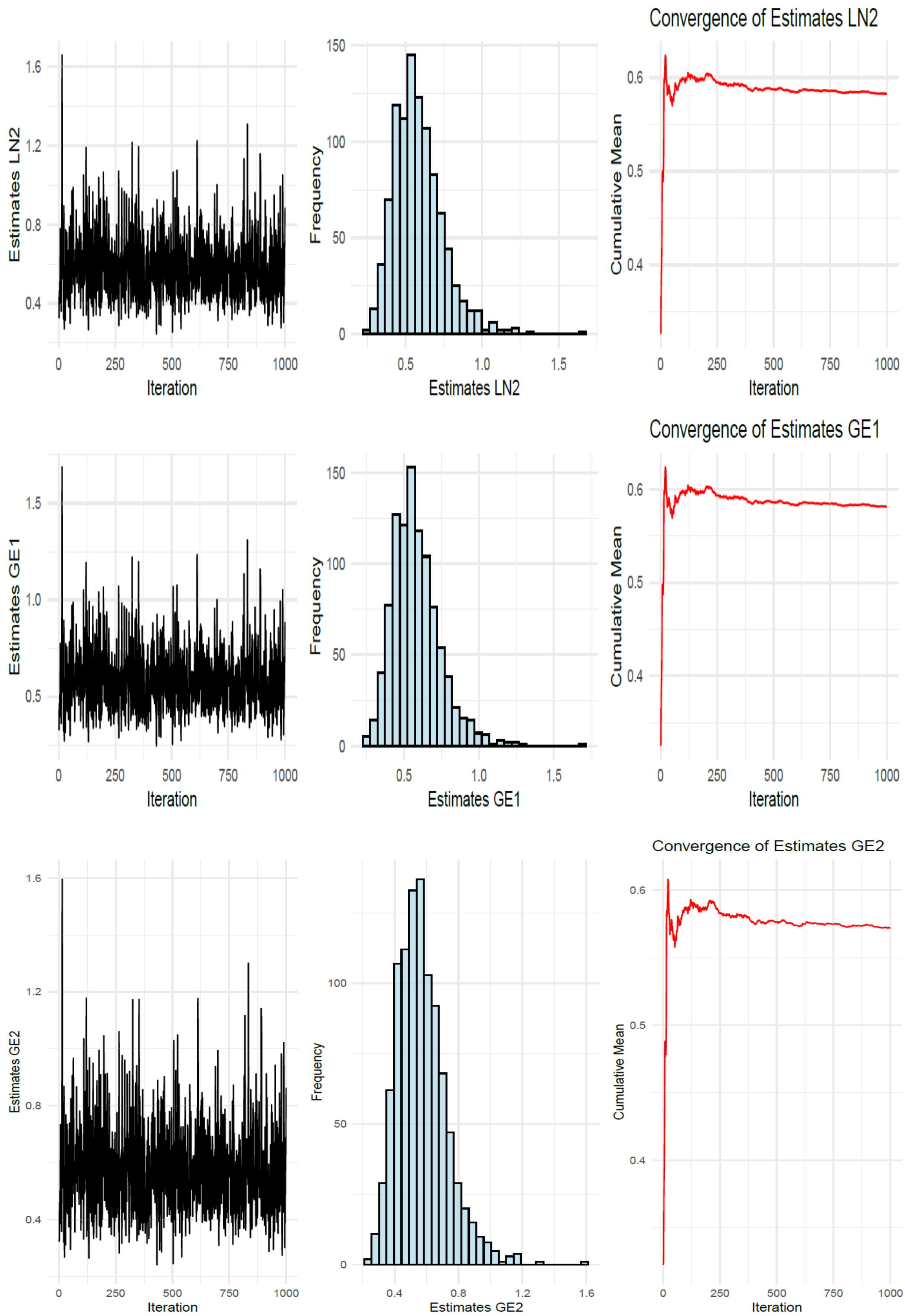

Figure 6: MCMC plots of the BXII-IR distribution with parameters λ= 0.5, ρ=3.3, β=1.1 for LN2, GE1 and GE2 loss functions.

6. Results and discussion

This paper focuses on estimating parameters for the BXII-IR distribution by employing various estimators Bayesian, encompassing both symmetric and asymmetric loss functions. The Bayesian estimators were produced using the squared error (SE), general entropy (GE), and LINEX loss functions while utilizing suitable prior distributions for the parameters. Given the unavailability of closed-form solutions for Bayesian estimates under these loss functions, the research employed the Markov Chain Monte Carlo (MCMC) approach. Through detailed simulation experiments, we investigated the performance of the estimate techniques under the stated loss functions. from Tables 1 to 4 The simulation results for Bayesian methods indicate that all methods consistently estimate the parameters, with the LN2 loss function estimator demonstrating the highest efficiency as assessed by MSE, RMSE, bias, HPD, AIL, and CP.

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