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Degree of Best Approximation of Unbounded Functions by Operators

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Abstract
In this work, we identified two operators of type $\underset{-}{\underset{h}{H}}(f,x)$ and $\underset{+}{\underset{h}{H}}(f,x)$, where $\underset{-}{\underset{h}{H}}(f,x)$ is below the function f and $\underset{+}{\underset{h}{H}}(f,x)$ is the above the function f. In this paper, the aim of it is to study and for a few unbounded functions, find the level of the most accurate approximation in the multiplier space $L_{P,\alpha_n}(x)$ and we obtained results.

Keywords: Multiplier Modulus of Smoothness, Multiplier Averaged, Degree of Best Multiplier Approximation of a Function,

1 Introduction

Approximation theory is a deep theoretical study of methods that use numerical approximation for the problems of mathematical analysis[1]. The concept of approximation was one of the most basic concepts that contribute effectively in the development of mathematical in frastructure in the past century [2].Many researchers and mathematicians are still researching the problems of approximation theory and its applications, such as:

In (2014) [3] introduced the best one-sided approximation of unbounded functions in $1 \leqslant P < \infty$ by positive linear operator and entire functions. In (2012) [4] studied the magnitude of the constants in the equivalence between the first and second order Ditzian-Totik moduli of smoothness and related k-functional, continuous functions in topological spaces can also be used in the theory of function approximation in the future [8, 9, 10, 11]. In [12] investigated the Levitan and Bebutov approaches to the metrical approximations by trigonometric polynomials and ρ-periodic type functions. In [13] introduce two notions of closure in the category of proximity spaces which satisfy (weak) hereditariness, productivity, and idempotency.

2 Definitions and Concepts

Definition 2.1 [5]

The k-th difference of every function f of order k with step h at point x is defined by $\Delta_h^k f(x) = \sum_{m=0}^{k} (-1)^{m+k} \binom{k}{m} f(x+mh)$, where:
$\binom{k}{m} = \frac{k!}{m!(k-m)!}$ is called the binomial coefficient.
Definition 2.2 [6], [7]

(1) For $f \in L_p(X), X = [a,b]$, we define the local modulus of smoothness of the function f of order k at point $x \in [a,b], 0 \leq \delta \leq \frac{b-a}{k}$ by:

$$\omega_k(f,x,\delta) = \sup\{|\Delta_h^k f(t)| : t, t+kh \in [a,b] \cap \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right]\}. \quad (1)$$

(2) The degree of best one-sided approximation is defined by:
$\tilde{E}_n(f) = \inf\{\|P_n - q_n\| : P_n, q_n \in \mathbb{P}_n\}$
Where $P_n(x) \leqslant f(x) \leqslant q_n(x)$.
(2)

$\mathbb{P}_n$ is the set of all algebraic or trigonometric polynomials of degree of n.
Definition 2.3
For $L_{p,\alpha_n}(X)$,where $X = \{f : f$ is unbounded function on $[0,\infty)$, if there is a sequence $\{\alpha_n\}_{n=0}^{\infty}$ of real numbers such that $\int_0^{\infty} f(x)\, \alpha_n\, dx < \infty\}$,with under the norm:
$\|f\|_{p,\alpha_n} = \left(\int_X |f \alpha_n(x)|^p\, dx\right)^{1/p}, p \geq 1.$
(3)
Definition 2.4

The degree of best multiplier approximation of f and the degree of best one-sided multiplier approximation of f with respect to the algebraic or trigonometric polynomials on X are given respectively by:

$$E_n(f)_{p,\alpha_n} = \inf\{\|f - P_n\|_{p,\alpha_n} : P_n \in \mathbb{P}_n\}$$

(4)

$$\tilde{E}_n(f)_{p,\alpha_n} = \inf\{\|q_n - P_n\|_{p,\alpha_n} : P_n, q_n \in \mathbb{P}_n\}$$

(5)

Where:

$$P_n(x) \leq f(x) \leq q_n(x)$$

(6)

Definition 2.5

For $f \in L_{P,\alpha_n}(X), X = [0, \infty)$, the multiplier integral modulus of f of order $k, \delta \in \left[0, \frac{b-a}{k}\right]$ is defined by:

$$\omega_k(f,\delta)_{P,\alpha_n} = \sup_{0 \leqslant h \leqslant \delta} \left(\int_a^{1-kh} \left|\Delta_h^k f(x)\right|^P dx\right)^{1/P}$$

(7)

Definition 2.6

The multiplier local of smoothness of f of order k at point $x \in [a,b], \delta \in \left[0, \frac{b-a}{k}\right]$ is defined by:

$$\omega_k(f,x,\delta)_{p,\alpha_n} = \sup\left\{\|\Delta_h^k f(x)\|_{p,\alpha_n}\right.$$

(8)

Note that:

$t, t+kh \in \left\{\left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right] \cap [a,b]\right\}$, Additionally:

$$\Delta_h^k f(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{k\delta}{2} + ih\right)$$

(9)

Furthermore, $x \mp \frac{kh}{2} \in X$ is the difference of a function f of order k with step h at a point x.

Definition 2.7

The multiplier averaged modulus of smoothness of f of order k is defined by:

$$\tau_k(f,\delta)_{P,\alpha_n} = \|\omega_k(f,.,\delta)\|_{P,\alpha_n} = \left(\int_X |\omega_k(f\alpha_n,x,\delta)|^P dx\right)^{\frac{1}{P}}, k \in N, P \in [1,\infty)$$

(10)

Definition 2.8

For $f \in L_{P,\alpha_n}(X), X = [0,\infty), n \in N$, we define:

$\underset{\underline{h}}{H}(f,x) = f(x) - \frac{4}{h^2}\iint_0^{\frac{h}{2}} [2f(x+s+t) - f(x+2(s+t))]\alpha_n dsdt$
(11)

$\underset{\underset{+}{h}}{H}(f,x) = f(x) + \frac{4}{h^2}\int\int_0^{\frac{h}{2}} [2f(x+s+t) - f(x+2(s+t))]\alpha_n dsdt$ (12)

Where $x \geqslant 0, h \geqslant 0$. From the above inequalities, it is clear that:

$\underset{\underline{h}}{H}(f,x) \leq f(x) \leq \underset{\underset{+}{h}}{H}(f,x)$
(13)

3. Auxiliary Lemmas

Lemma 3.1

Assume that $f \in L_{p,\alpha_n}(X), X = [0,\infty)$, then $\omega_k(f,\delta)_{p,\alpha_n} \leq \tau_k(f,\delta)_{p,\alpha_n}$
(14)

Proof:

The proof of this lemma will be carried out via definition 2.2 (part 1) and definition 2.7 respectively.

$$\omega_k(f,\delta)_{p,\alpha_n} = \sup_{0\leq h\leq\delta}\left(\int_a^{b\_Kh} \left|\Delta_h^k f\alpha_n(x)\right|^p dx\right)^{\frac{1}{p}}$$

$$= \sup_{0\leqslant k\leqslant\delta}\left(\int_a^{b-kh} \left|\omega_k\left(f\alpha_n, x + \frac{x+kh}{a}, \delta\right)\right|^p dx\right)^{\frac{1}{p}}$$

$$= \sup_{0\leqslant k\leqslant\delta}\left(\int_{a+\frac{kh}{2}}^{b-\frac{kh}{2}} |\omega_k(f\alpha_n, x, \delta)|^p dx\right)^{\frac{1}{p}}$$

$$= \tau_k(f,\delta)_{p,\alpha_n}.$$

The proof is completed.

Lemma 3.2

Let $f \in L_{p,\alpha_n}(X), X = [0,\infty)$, then

$\| \underset{\underset{+}{h}}{H}(f,\cdot) - f(\cdot) \|_{p,\alpha_n} \leqslant c\omega_2\left(f, \frac{h}{2}\right)_{p,\alpha_n}$
(15)

Proof:
From definition 2.8, we have:

$$H_{\frac{h}{+}}(f,x) = f(x) + \frac{4}{h^2}\iint_0^{\frac{h}{2}} [2\, f(x+s+t) - f(x+2(s+t))]\alpha_n dsdt$$

$$H_{\frac{h}{+}}(f,x) - f(x) = \frac{4}{h^2}\iint_0^{\frac{h}{2}} [2f(x+s+t) - f(x+2(s+t))]\alpha_n dsdt$$

Taking norm to both sides, yields the below inequality:

$$\left\| H_{\frac{h}{+}}(f,x) - f(.)\right\|_{p,\alpha_n} \leq \left\| \frac{4}{h^2}\iint_0^{\frac{h}{2}} 2f(x+s+t) - f(x+2(s+t))]dsdt \right\|_{p,\alpha_n}$$

$$= \left\| \frac{4}{h^2}\iint_0^{\frac{h}{2}} \Delta^2_{s+t} f(x)dsdt \right\|_{p,\alpha_n}$$

$=c\omega_2(f, {}^{h}/_{2})_{p,\alpha_n}$.
The desired inequality is satisfied.
Lemma 3.3

Let $f \in L_{p,\alpha_n}(X), X = [0,\infty)$, then
$\| f(\cdot) - H_{\underline{h}}(f,x)\|_{p,\alpha_n} \leq C\omega_2(f,\cdot,h/2)_{p,\alpha_n}$
(16)
Where C is a constant.
Proof:
The proof will be carried out via definition 2.8 as shown below:

$$H_{\underline{h}}(f,x) = f(x) - \frac{4}{h^2}\iint_0^{\frac{h}{2}} [2\, f(x+s+t) - f(x+2(s+t))]\alpha_n dsdt$$

Then we get the below integral equation:

$$f(x) - H_{\underline{h}}(f,x) = \frac{4}{h^2}\iint_0^{h_{12}} [2f(x+s+t) - f(x+2(s+t))]\alpha_n dsdt$$

Taking the norm for both sides of the above integral equation would then yield:

$$\|f(\cdot) - H_{\underline{h}}(f,x)\|_{p,\alpha_n} \leqslant \left\| \frac{4}{h^2}\int\int_0^{h/2} \Delta^2 f(x+s+t) - f(x+2(s+t)]dsdt \right\|_{p,\alpha_n}$$

$= \left\| \frac{4}{h^2} \iint_0^{\frac{h}{2}} \Delta_{s+t}^2 f(x) dsdt \right\|_{P,\alpha_n}$

$\leqslant C\omega_2(f,\cdot,h/2)_{P,\alpha_n}$

The desired inequality is fulfilled.

4 Main Results

Several results will be proved and analyzed in this section.

Theorem 4.1

Let $f \in L_{P,\alpha_n}(X), X = [0,\infty)$, then

$\tilde{E}_n(f)_{P,\alpha_n} \leqslant c_1\tau_2\left(f,\frac{h}{2}\right)_{P,\alpha_n}$

(17)

C_1 is a constant.

Proof:

By taking the norm for both sides of the integral of the difference $\underset{\frac{h}{-}}{H}(f,x) - \underset{\frac{h}{+}}{H}(f,x)$ and by utilizing Lemmas 3.1, 3.2 and 3.3, respectively we get:

$$\left\|\underset{\frac{h}{-}}{H}(f,x) - \underset{\frac{h}{+}}{H}(f,x)\right\|_{P,\alpha_n} = \left\|\underset{\frac{h}{-}}{H}(f,x) - f(x) + f(x) - \underset{\frac{h}{+}}{H}(f,x)\right\|_{P,\alpha_n}$$

$$\leq \left\|f(x) - H_{\frac{h}{+}}(f,x)\right\|_{P,\alpha_n} + \left\|\underset{\frac{h}{-}}{H}(f,x) - f(x)\right\|_{P,\alpha_n}$$

$$= c\omega_2\left(f,\cdot,\frac{h}{2}\right)_{P,\alpha_n} + c\omega_2\left(f,\cdot,\frac{h}{2}\right)_{P,\alpha_n} = c_1\cdot\tau_2\left(f,\cdot,\frac{h}{2}\right)_{P,\alpha_n} + c_1\tau_2\left(f,\cdot,\frac{h}{2}\right)_{P,\alpha_n}$$

$\leqslant C\tau_2(f,h/2)_{P,\alpha_n}$.

The constant C is given by $C = c_1 + c_2$. This completes the proof.

Theorem 4.2

Let $\in L_{P,\alpha_n}(X), X = [0,\infty)$. Then:

$$\left\|H'_{\underline{h}}(f,x)-f'(x)\right\|_{p,\alpha_n}\leq \frac{c}{h}\omega_1\left(f,\frac{h}{2}\right)_{p,\alpha_n}+\frac{1}{h}\omega_1(f,h)_{p,\alpha_n}$$

(18) Proof:

From definition 2.8, it can be obtained:

$$H_{\underline{h}}(f,x)=f(x)-\frac{4}{h^2}\iint_0^{\frac{h}{2}}\left[2f(x+s+t)-f\big(x+2(s+t)\big)\right]dsdt$$

Then

$$H_{\underline{h}}(f,x)=f(x)-\frac{4}{h^2}\left[\iint_0^{\frac{h}{2}}(x+s+t)-\iint_0^{\frac{h}{2}}\big(x+2(s+t)\big)ds\right]dt$$

(19)

Let $u=x+s\rightarrow du=ds$, then $as\ \ s=0\ \rightarrow u=x\ and\ then\ s=\frac{h}{2}\rightarrow u=x+\frac{h}{2}$.

Suppose that $u=x+2s\rightarrow du=2ds$. This yields that:

$as\ \ s=0\ \rightarrow u=x\ and\ then\ s=\frac{h}{2}\rightarrow u=x+h.$ The function $H_{\underline{h}}(f,x)$ is then given by:

$$H_{\underline{h}}(f,x)=f(x)-\frac{4}{h^2}\left[\left[\iint_x^{x+\frac{h}{2}}2f(u+t)du-\iint_x^{x+h}\frac{1}{2}f(u+2t)du\right]dt\right] \quad (20)$$

Furthermore, $H'_{\underline{h}}(f,x)$ is also introduced by:

$$H'_{\underline{h}}(f,x)=f'(x)-\frac{4}{h^2}\int_0^{\frac{h}{2}}\left[2\left(f\left(x+\frac{h}{2}+t\right)-f(x+t)\right)-\frac{1}{2}(f(x+h+2t)-f(x+2t)\right]dt$$

Then

$$H'_{\underline{h}}(f,x)=f'(x)-\frac{4}{h^2}\int_0^{\frac{h}{2}}\left[2\Delta_{h/_2}f(x+t)-\frac{1}{2}\Delta_h f(x+2t)\right]dt$$

(21)

Then

$$\left\|H'_{\underline{h}}(f,x)-f'(.)\right\|_{p,\alpha_n}\leqslant\left\|\frac{4}{h^2}\int_0^{\frac{h}{2}}\Delta_{h/_2}f(x+t)dt\right\|_{p,\alpha_n}+\left\|\frac{4}{h^2}\int_0^{\frac{h}{2}}\frac{1}{2}\Delta_h f(x+2t)dt\right\|_{p,\alpha_n}$$

$$\leqslant\left[\frac{c}{h}\omega_1\left(f,\cdot,\frac{h}{2}\right)_{p,\alpha_n}+\frac{1}{h}\omega_1(f,.,h)_{p,\alpha_n}\right].$$

The final result is obtained from definition 2.5. The preferred inequality is verified.

Theorem 4.3

If $f\in L_{p,\alpha_n}(X), X=[0.\infty)$, then

$$\left\|\acute{H}_{\underset{+}{h}}(f,x)-f'(x)\right\|_{p,\alpha_n} \leqslant \frac{c}{h}\omega_1\left(f,{}^{h}/_{2}\right)_{p,\alpha_n}+\frac{1}{h}\omega_1(f,\cdot,h)_{p,\alpha_n} \quad (22)$$

Proof:

The proof can be carried out via definition 2.8 and by a similar way to the proof of theorem 4.2.

Theorem 4.4

Let $f \in L_{p,\alpha_n}(X),\ \ X=[0.\infty)$ then

$$\tilde{E}_n(f)_{p,\alpha_n} \leq \frac{c_1}{h}\tau_1(f,h/2)_{p,\alpha_n}+\frac{c_2}{h}\tau_1(f,h)_{p,\alpha_n}$$

(23)

Where c_1 and c_2 are constants.

Proof:

By utilizing lemma 3.1, theorem 4.2 and theorem 4.3, the desired inequality can be obtained as shown in the below analysis.

$$\left\|\acute{H}_{\underset{-}{h}}(f,.)-\acute{H}_{\underset{+}{h}}(f,.)\right\|_{p,\alpha_n} = \left\|\acute{H}_{\underset{+}{h}}(f,.)-\acute{H}_{\underset{-}{h}}(f,.)-f(x)+f(x)\right\|_{p,\alpha_n}$$

$$\leq \left\|\acute{H}_{\underset{+}{h}}(f,.)-\acute{f}(x)\right\|_{p,\alpha_n}+\left\|\acute{H}_{\underset{-}{h}}(f,.)-f(x)\right\|_{p,\alpha_n}$$

$$=\frac{c}{h}\omega_1\left(f,\cdot,\frac{h}{2}\right)_{p,\alpha_n}+\frac{1}{h}\omega_1(f,;,h)_{p,\alpha_n}+\frac{c}{h}\omega_1\left(f,\cdot,\frac{h}{2}\right)_{p,\alpha_n}+\frac{1}{h}\omega_1(f,\cdot,h)_{p,\alpha_n}$$

$$=\frac{c_1}{h}\omega_1\left(f,\frac{h}{2}\right)_{p,\alpha_n}+\frac{c_2}{h}\omega_1(f,\cdot,h)_{p,\alpha_n}$$

$$\leq \frac{c_1}{h}\tau_1\left(f,\frac{h}{2}\right)_{p,\alpha_n}+\frac{c_2}{h}\tau_1(f,h)_{p,\alpha_n}.$$

The proof is completed.

5 Conclusion

Through the operators $\underset{\underset{-}{h}}{H}(f,x)$ and $\underset{\underset{+}{h}}{H}(f,x)$, the Approximating of unbounded functions with the optimal multiplier in $L_{p,\alpha_n}(X)$-space $X=[0,\infty)$ is obtained. The relationship between this degree $\tilde{E}(f)_{p,\alpha_n}$ and $\omega_1\left(f,\frac{h}{2}\right)_{p,\alpha_n}$, $\omega_2(f,\frac{h}{2})_{p,\alpha_n}$ and $\tau_1(f,\frac{h}{2})_{p,\alpha_n}$.

References:

[1] Soleymani F, Stanimirović PS, Torregrosa JR, Saberi Nik H, Tohidi E., Recent theories and applications in approximation theory, The Scientific World Journal, 2015.
[2] Park JH., Chebyshev approximation of discrete polynomials and splines, PhD, Virginia Polytechnic Institute and State University, USA, 1999.
[3] AL-saidy, SK, Alaa, AM, The best one -sided approximation of unbounded functions in Lp,w spaces, PhD, Mustansiriyah University, Baghdad ,Iraq, 2014.
[4] Tachev GT., Refined Estimates for the Equivalence Between Ditzian-Totik Moduli of Smoothness and K-Functionals, Theory and Applications of Mathematics & Computer Science, 2(2), 48-54, 2012.
[5] Hristov, V.H. Ivanov, K.G., Operators for one sided approximation by algebraic in $L_p[-1,1]^d$, Mathematica Balkanica, 2, 374-390, 1988.
[6] Sendov, BI. Popov VA., The averaged modulus of smoothness, John Wiley & Sons, New York, USA, 1988.
[7] DeVore, RA, Lorentz, GG., Constructive approximation, Springer Science and Business Media, 1993.
[8] Ali HJ, Hassan RF., On Light Mapping and Certain Concepts by Using m x N-Open Sets, Baghdad Science Journal, 17(1), 371-377, 2020.
[9] Humadi NK, Ali HJ., On ω C-continuous functions, In Journal of Physics, Conference Series, 1294(3), 2019.
[10] Humadi NK, Ali HJ., New types of perfectly supra continuous functions, Iraqi Journal of Science, 26, 61(4), 811-819, 2020.
[11] Hussain, KA, Noman LM, Ali, HJ., On supra ωb_p-Lindelöf spaces, Italian Journal of Pure and Applied Mathematics, 47, 588–595, 2022.
[12] Chaouchi B, Kostic M, Velinov D., Metrical almost periodicity, metrical approximations of functions and applications, Turkish Journal of Mathematics, 47(2), 769-793, 2023, https://doi.org/10.55730/1300-0098.3393.
[13] Özkan S, Kula M, Kula S, Baran TM., Closure operators, irreducibility, Urysohn's lemma, and Tietze extension theorem for proximity spaces, Turkish Journal of Mathematics, 47(2),870-882, 2023, https://doi.org/10.55730/1300-0098.3398.