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The Modified Burr-III Distribution Properties, Estimation, Simulation, with Application on Real Data

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ABSTRACT

In this paper, a new modification of the Burr III distribution with a significantly improved functional form based on the Odd Lomax-G family is proposed, which is called the modified Odd Lomax Burr III distribution. This new modification has the ability to give the classical distribution high flexibility with the ability to model all Shapes of the hazard rate function including increasing, decreasing, bathtub, and inverted bathtub. Some of its primary properties are also presented, such as the moment function, the moment-generating function, the Quantile function, two-way skewness and Kurtosis, incomplete moments, ordered statistics, Rényi Entropy, in addition to the stress and strength functions in a clear and concise manner. In addition to estimating the model parameters using the maximum likelihood technique, least square, and Weighted Least Squares, also calculate the bias of the estimated parameters, Monte Carlo simulation is used to determine the bias. The effectiveness of the modified distribution was also confirmed by applying it to two types of real data consisting of complete and censored samples by making a comparison with some other distributions using a set of goodness criteria, which confirmed the superiority of the proposed model over other models through application on two types of real data.

1. Introduction

Burr devised a dynamic family of probability distributions such as: Burr They are widely used forms of Burr's distribution system. These distributions have received great attention from applied statisticians, and the main reason may be that these densities exist in simpler forms and can produce a range of shapes to model a variety of scenarios in various scientific fields. In [1] researchers argue that the most adaptable of these three is BIII, especially in reliability, survivability and environmental sciences. The BIII distribution is also called the Dagum distribution in studies of income, wages, and wealth distribution [2]. It is also known in [3] as the inverse Burr distribution. and kappa distribution in meteorological data [4]. Therefore, the cumulative distribution function (CDF) and probability density function (pdf) of the With shape parameters η, κ BIII distribution, respectively, are given below:

$$F(x) = \left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa}, \quad \eta, \kappa, x > 0 \tag{1}$$

$$f(x) = \frac{\eta\kappa}{x^{\eta+1}}\left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa-1}, \quad , \quad \eta, \kappa, x > 0 \tag{2}$$

When working with traditional statistical distributions we encounter some problems in specific applications. One of these problems is the inability to model it effectively, and the data may be abnormal, asymmetric, or contain extreme values. In order to solve such problems, many researchers have presented

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modeling solutions by introducing new families of distributions with the aim of giving flexibility to the basic distributions. Among those families that Eugene and others presented in 2002 were based on the Beta distribution function and called it Beta-G, and the limits of that family were between zero and one [5], this approach was developed by Alzaatreh and others and called it the T-X method, which was a breakthrough in the field of generating continuous distribution families [6]. Examples of these families include: shifted Gompertz-G by [7], shifted Gompertz-G by [8], OBP-G by [9], APMW-X by [10], And OLG, which is relied upon to build the new modified distribution, where OLG has CDF and PDF functions with two parameters θ and β, respectively, in the equations below [11]:

$$G(x) = 1 - \left(1 - \frac{F(x,\zeta).\,log\big(1 - F(x,\zeta)\big)}{\theta}\right)^{-\beta} \tag{3}$$

$$g(x) = \frac{\beta}{\theta} f(x,\zeta)\left(1 - \frac{F(x,\zeta).\log\big(1 - F(x,\zeta)\big)}{\theta}\right)^{-(\beta+1)} \left[\frac{F(x,\zeta)}{1 - F(x,\zeta)} - \log\big(1 - F(x,\zeta)\big)\right] \tag{4}$$

So that $\theta, \beta > 0$ are shape parameters of the OLG family and $F(x,\zeta)$ is baseline distribution.

The study aims to find a new statistical distribution according to the OLG family called the Modified Odd Lomax Burr III distribution (MODLBIII) with four parameters that are more flexible than the basic distribution and capable of modeling volatile and complex data, in addition to presenting some of the characteristics of the modified distribution, with a simulation of the parameters estimated by the maximum likelihood method, least square, and Weighted Least Squares. In addition to a practical application in the R language to confirm the flexibility of the MODLBIII distribution.

The paper consisted of a section on forming the MODLBIII with drawing the basic functions of the distribution, a second section that contained the purpose distribution, a third section that contained the expansion of the distribution functions with proof of the most important properties of the distribution, while the forth section contained an estimate of the parameters using the maximum likelihood method, while a Monte Carlo simulation was conducted in the fifth section, and the sixth and final section. A practical application was conducted on real data to demonstrate the flexibility of MODLBIII.

2. MODLBIII Distribution

To obtain the CDF, equation (1) can be substituted into equation (3) for the pdf by replacing equation (2) with equation (4), we can derive the CDF and pdf equations for the new distribution in the following format:

$$G(x) = 1 - \left(1 - \frac{1}{\theta}\left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1 - \left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta} \tag{5}$$

$$g(x) = \frac{\beta\eta\kappa}{\theta x^{\eta+1}}\left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa-1}\left(1 - \frac{1}{\theta}\left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa}.\log\left(1 - \left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-(\beta+1)} \times \left[\frac{\left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa}}{1 - \left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa}} - \log\left(1 - \left(1 + \frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right] \tag{6}$$

Below is a plot of the CDF and PDF functions of the MODLBIII with different values of the parameters

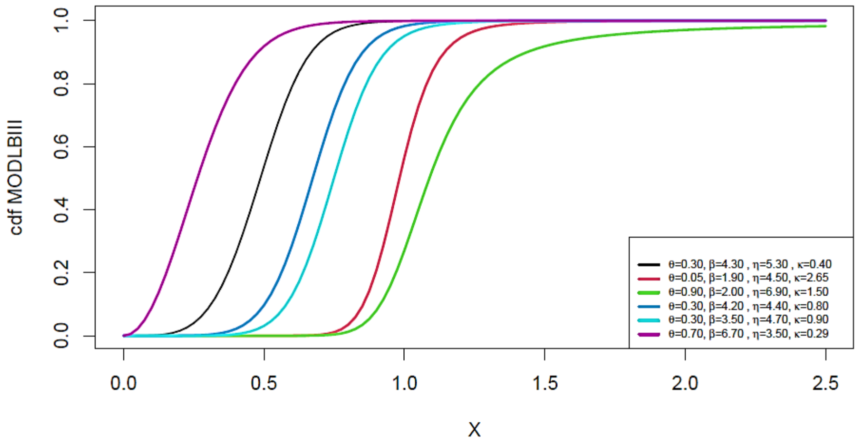


Figure 1. plot CDF function for MODLBIII dist. with different value of parameters

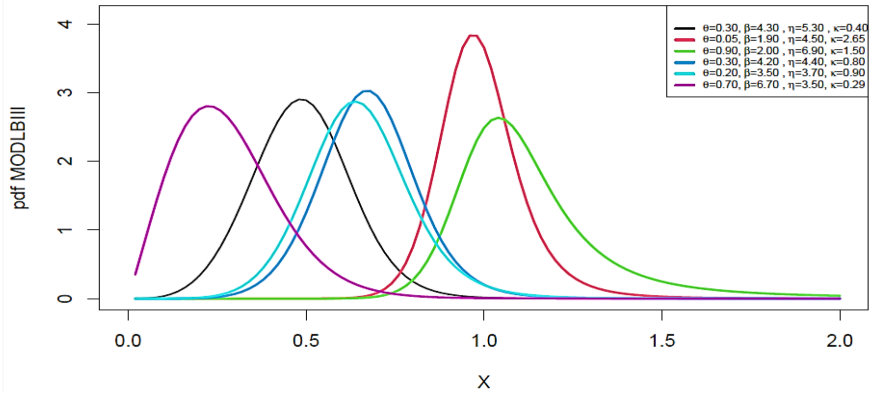


Figure 2. plot pdf function for MODLBIII dist. with different value of parameters

Some special cases of the distribution can be obtained by substituting values for its parameters in CDF, and pdf distribution, as follows:

Table 1. some special sub models of MODLBIII

model	θ	η	κ	β	x	$G(x)$
Odd F-(2,2) Burr III	**1**	-	-	**1**	-	$1-\frac{1}{1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)}$
Odd logistic Burr III	$\boldsymbol{Log(\lambda)}$	-	-	**1**	-	$\frac{-Log(\lambda)}{\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)}$
Odd lomax lomax	-	$\boldsymbol{-1}$	-	-	-	$1-\left(1-\frac{1}{\theta}(1+x)^{-\kappa}.log(1-(1+x)^{-\kappa})\right)^{-\beta}$
Odd lomax log-logistic (Odd lomax Fisk)	-	-	**1**	-	-	$1-\left(1-\frac{1}{\theta\left(1+\frac{1}{x^{\eta}}\right)}.log\left(\frac{1}{x^{\eta}\left(1+\frac{1}{x^{\eta}}\right)}\right)\right)^{-\beta}$
Odd lomax Dagum (Odd lomax Inverse Burr)	-	-	-	-	$\boldsymbol{\frac{1}{x}}$	$1-\left(1-\frac{1}{\theta}(1+x^{\eta})^{-\kappa}.log(1-(1+x^{\eta})^{-\kappa})\right)^{-\beta}$
Odd lomax logistic Type-I or Odd lomax Burr II or skew logistic	-	**1**	-	-	$\boldsymbol{e^{x}}$	$1-\left(1-\frac{1}{\theta}(1+e^{-x})^{-\kappa}.log(1-(1+e^{-x})^{-\kappa})\right)^{-\beta}$

The Survival function, which is the probability that the system will not fail after a period of time, is defined mathematically by the following relationship [12-13]:

$$S(x)_{MODLBIII}=1-G(x) \quad (7)$$

Therefore, when we substitute equation (5), we get in the above equation:

$$S(x)_{MODLBIII} = \left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta} \tag{8}$$

Below is a plot Survival function of MODLBIII with different values of the parameters

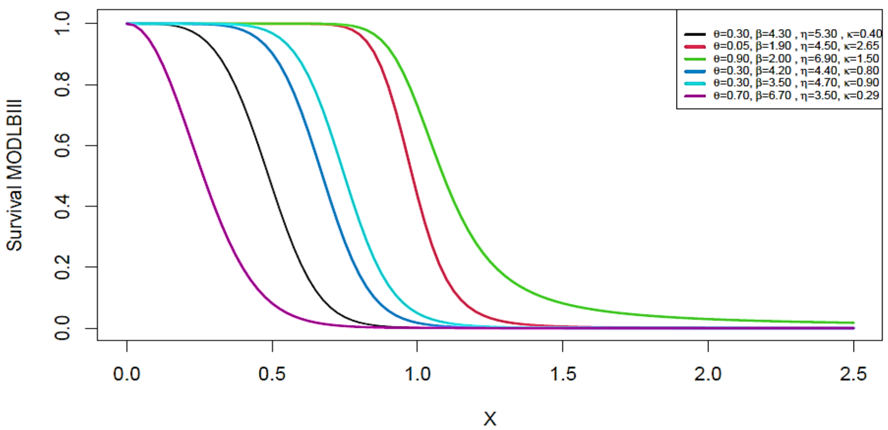


Figure 3. plot Survival function for MODLBIII dist. with different value of parameters

As for the Hazard function, which has great importance, especially with regard to life issues, many researchers have focused their attention on this function and finding statistical distributions of different forms for this function, and therefore it can be obtained from the following relationship [14-15]:

$$h(x)_{MODLBIII} = \frac{G(x)_{MODLBIII}}{S(x)_{MODLBIII}} \tag{9}$$

Therefore, when we substitute equation (5), and (8), we get in the above equation:

$$h(x)_{MODLBIII} = \frac{\frac{\beta\eta\kappa}{\theta x^{\eta+1}}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa-1}\left[\frac{\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}}{1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}} - \log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right]}{1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)} \tag{10}$$

Below is a plot Hazard function of MODLBIII with different values of the parameters

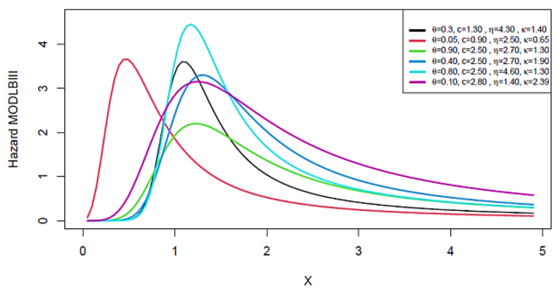


Figure 4. plot Hazard function for MODLBIII dist. with different value of parameters

3. Statistical Properties of MODLBIII

3.1 Expansion pdf and CDF of MODLBIII distribution

In order to find the characteristics of the MODLBIII distribution, the CDF and PDF functions are expanded using the binomial expansion and the logarithmic function. Where the CDF function is expanded as follows [16-18]:

$$\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta} = \sum_{j=0}^{\infty}\frac{\Gamma(\beta+j)\theta^{-j}}{j!\,\Gamma(\beta)}\left(\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)^{j}\left(log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{j}$$

Additionally, by utilizing the logarithm expansion of $\left(log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{j}$ in the following form:

$$\left(log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{j} = \sum_{i=0}^{\infty}(-1)^{i}\,d_{j,i}\left(\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)^{i+j}$$

where $d_{j,i} = i^{-1}\sum_{n=1}^{i}\frac{[n(j+1)-i]}{n+1}$ for $i \geq 0$ and $d_{j,0} = 1$

$$\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{\Gamma(\beta+j)\theta^{-j}}{j!\,\Gamma(\beta)}(-1)^{i}d_{j,i}\left(1+\frac{1}{x^{\eta}}\right)^{-(\kappa i+2j\kappa)}$$

where $d_{j,i} = i^{-1}\sum_{n=1}^{i}\frac{[n(j+1)-i]}{n+1}$ for $i \geq 0$ and $d_{j,0} = 1$

Then the term $\left(1+\frac{1}{x^{\eta}}\right)^{-(\kappa i+2j\kappa)}$ is expanded using the binomial expansion in the form:

$$\left(1+\frac{1}{x^{\eta}}\right)^{-(\kappa i+2j\kappa)} = \sum_{l=0}^{\infty}\frac{\Gamma(\kappa i+2j\kappa+l)}{l!\,\Gamma(\kappa i+2j\kappa)}x^{-\eta l}$$

Thus, the expansion of the CDF function is obtained in the form:

$$G(x) = 1-\Psi x^{-\eta l} \tag{11}$$

Where

$\Psi = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{i}d_{j,i}\frac{\Gamma(\beta+j)\theta^{-j}}{j!\Gamma(\beta)}\frac{\Gamma(\kappa i+2j\kappa+l)}{l!\Gamma(\kappa i+2j\kappa)}$

The CDF[W] has form:

$$G^{w}(x) = \left[1-\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta}\right]^{w} \tag{12}$$

To expand the above equation can using the binomial expansion by form:

$$\left[1-\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta}\right]^{w} = \sum_{r=0}^{\infty}(-1)^{r}\binom{w}{r}\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-r\beta}$$

And expansion by same way in expansion CDF we get:

$$\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-r\beta} = \sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty}\frac{\Gamma(\kappa s+2u\kappa+q)}{q!\,\Gamma(\kappa s+2u\kappa)}\frac{\Gamma(r\beta+u)\theta^{-u}}{u!\,\Gamma(r\beta)}(-1)^{s}d_{u,s}x^{-\eta q}$$

where $d_{u,s} = s^{-1}\sum_{n=1}^{s}\frac{[n(u+1)-s]}{n+1}$ for $s \geq 0$ and $d_{u,0} = 1$

Finally we get:

$$G^{w}(x) = Dx^{-\eta q} \tag{13}$$

where

$$D = \sum_{r=u=s=q=0}^{\infty}(-1)^{r+s}\binom{w}{r}d_{u,s}\frac{\Gamma(\kappa s+2u\kappa+q)}{q!\,\Gamma(\kappa s+2u\kappa)}\frac{\Gamma(r\beta+u)\theta^{-u}}{u!\,\Gamma(r\beta)}$$

To expand the pdf of MODLBIII distribution can using the binomial expansion by form:

$$\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-(\beta+1)} = \sum_{m=0}^{\infty}\frac{\Gamma(\beta+1+m)\theta^{-m}}{m!\,\Gamma(\beta+1)}\left(\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)^{m}\left(log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{m}$$

Thus, the expansion of the function pdf is obtained in the form:

$$g(x)=\sum_{m=0}^{\infty}\frac{\Gamma(\beta+1+m)\theta^{-m}\beta\eta\kappa}{m!\,\Gamma(\beta+1)\theta x^{\eta+1}}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa(m+1)-1}\left(log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{m} \times\left[\frac{\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}}{1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}}-\log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right]$$

Additionally, by utilizing the logarithm expansion of $\left(log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{m}$ in the following form:

$$\left(log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{m}=\sum_{n=0}^{\infty}(-1)^{n}\,d_{m,n}\left(\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)^{n+m}$$

where $d_{m,n}=n^{-1}\sum_{h=1}^{n}\frac{[h(m+1)-n]}{h+1}$ for $n\geq 0$ and $d_{m,0}=1$

$$g(x)=\sum_{m=n=0}^{\infty}(-1)^{n}d_{m,n}\frac{\Gamma(\beta+1+m)\theta^{-m}\beta\eta\kappa}{m!\,\Gamma(\beta+1)\theta x^{\eta+1}}\frac{\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa(2m+n+2)-1}}{1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}} -\sum_{m=n=0}^{\infty}(-1)^{n}d_{m,n}\frac{\Gamma(\beta+1+m)\theta^{-m}\beta\eta\kappa}{m!\,\Gamma(\beta+1)\theta x^{\eta+1}}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa(2m+n+1)-1}\log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)$$

Then the term $1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}$, and $\log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)$ are expanded using the binomial expansion in the form:

$$\frac{1}{1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}}=\sum_{z=0}^{\infty}(-1)^{z}\left(1+\frac{1}{x^{\eta}}\right)^{-z\kappa}$$

$$\log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)=\sum_{i=0}^{\infty}(-1)^{i}\,d_{1,i}\left(\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)^{i+1}$$

where $d_{1,i}=i^{-1}\sum_{p=1}^{i}\frac{[2p-i]}{p+1}$ for $i\geq 0$ and $d_{1,0}=1$

Then we get:

$$g(x)=\sum_{m=n=z=0}^{\infty}(-1)^{n+z}d_{m,n}\frac{\Gamma(\beta+1+m)\theta^{-m}\beta\eta\kappa}{m!\,\Gamma(\beta+1)\theta x^{\eta+1}}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa(2m+n+z+2)-1} -\sum_{m=n=i=0}^{\infty}(-1)^{n+i}d_{m,nd_{1,i}}\frac{\Gamma(\beta+1+m)\theta^{-m}\beta\eta\kappa}{m!\,\Gamma(\beta+1)\theta x^{\eta+1}}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa(2m+n+i+2)-1}$$

Finally, we get:

$$g(x)=\frac{\Phi\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa(2m+n+z+2)-1}-\Upsilon\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa(2m+n+i+2)-1}}{x^{\eta+1}} \quad (14)$$

where

$$\Phi=\sum_{m=n=z=0}^{\infty}(-1)^{n+z}d_{m,n}\frac{\Gamma(\beta+1+m)\theta^{-m}\beta\eta\kappa}{m!\,\Gamma(\beta+1)\theta}$$

$$\Upsilon=\sum_{m=n=i=0}^{\infty}(-1)^{n+i}d_{m,nd_{1,i}}\frac{\Gamma(\beta+1+m)\theta^{-m}\beta\eta\kappa}{m!\,\Gamma(\beta+1)\theta}$$

3.2 Quantile function of MODLBIII distribution

The Quantile function is defined as the inverse of the cumulative distribution function and is used to find the median, skewness, and kurtosis of distributions with large skewness values or that do not contain moments, and through it, random numbers of data can be generated for simulation study [19-20]:

$$Q(u) = F^{-1}(u)$$

where $Q(u)$ is the Quantity function $G(x)$ for each $u \in (0,1)$. Then the Quantile function of MODLBIII distribution by form:

$$Q_{G_{\text{MODLBIII}}} = Q_u\left(\left(\left(\frac{\theta-\frac{\theta}{(1-u)^{\frac{1}{\beta}}}+W_{-1}\left(\left(\theta-\frac{\theta}{(1-u)^{\frac{1}{\beta}}}\right)\exp\left[-\left(\theta-\frac{\theta}{(1-u)^{\frac{1}{\beta}}}\right)\right]\right)}{\theta-\frac{\theta}{(1-u)^{\frac{1}{\beta}}}}\right)^{\frac{1}{\kappa}}-1\right)^{-\frac{1}{\eta}}\right) \qquad (15)$$

Table 2: Explanation of the Quantile function for particular parameter values of the MODLBIII distribution

	$(\boldsymbol{\omega}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{a})$				
$\boldsymbol{u}$	(1.3,1.7,1.2,1.4)	(2.0,2.3,0.7,1.3)	(0.5,1.2,0.3,0.7)	(0.7,3.0,1.8,0.3)	(1.1,4.3,1.1,1.2)
0.1	0.6880	0.4913	0.0007	0.0306	0.3085
0.2	1.0054	0.9648	0.0055	0.0602	0.4449
0.3	1.3363	1.5928	0.0228	0.0922	0.5711
0.4	1.7332	2.5059	0.0773	0.1283	0.7029
0.5	2.2639	3.9656	0.2530	0.1707	0.8516
0.6	3.0671	6.6190	0.9096	0.2231	1.0324
0.7	4.5245	12.5365	4.3151	0.2929	1.2734
0.8	8.2192	32.0622	42.0734	0.3983	1.6451
0.9	31.4554	221.8755	6528.5998	0.6086	2.4284

The Median of the MODLBIII distribution can be determined by substituting $u = 0.5$ into Eq (15), resulting in the following form [21]:

$$Median = \left(\left(\frac{\theta-\frac{\theta}{(0.5)^{\frac{1}{\beta}}}+W_{-1}\left(\left(\theta-\frac{\theta}{(0.5)^{\frac{1}{\beta}}}\right)\exp\left[-\left(\theta-\frac{\theta}{(0.5)^{\frac{1}{\beta}}}\right)\right]\right)}{\theta-\frac{\theta}{(0.5)^{\frac{1}{\beta}}}}\right)^{\frac{1}{\kappa}}-1\right)^{-\frac{1}{\eta}} \qquad (16)$$

The measurements of skewness (S) and kurtosis (K) based on Quantile function were defined as follows [22-23]:

$$S = \frac{\mathcal{Q}\left(\frac{6}{8}\right) - 2\mathcal{Q}\left(\frac{4}{8}\right) + \mathcal{Q}\left(\frac{2}{8}\right)}{\mathcal{Q}\left(\frac{6}{8}\right) - \mathcal{Q}\left(\frac{2}{8}\right)}$$

$$K = \frac{\mathcal{Q}\left(\frac{7}{8}\right) - \mathcal{Q}\left(\frac{5}{8}\right) + \mathcal{Q}\left(\frac{3}{8}\right) - \mathcal{Q}\left(\frac{1}{8}\right)}{\mathcal{Q}\left(\frac{6}{8}\right) - \mathcal{Q}\left(\frac{2}{8}\right)} \qquad (17)$$

by substitute the substitute for $u = \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}$ in Equation (15) to get:

$$\mathcal{Q}\left(\tfrac{1}{8}\right) = \left(\left(\frac{\theta-\frac{\theta}{\left(\frac{7}{8}\right)^{\frac{1}{\beta}}}+W_{-1}\left(\left(\theta-\frac{\theta}{\left(\frac{7}{8}\right)^{\frac{1}{\beta}}}\right)\exp\left[-\left(\theta-\frac{\theta}{\left(\frac{7}{8}\right)^{\frac{1}{\beta}}}\right)\right]\right)}{\theta-\frac{\theta}{\left(\frac{7}{8}\right)^{\frac{1}{\beta}}}}\right)^{\frac{1}{\kappa}}-1\right)^{-\frac{1}{\eta}}$$

$$Q\left(\frac{2}{8}\right)=\left(\left(\frac{\theta-\frac{\theta}{\left(\frac{6}{8}\right)^{\frac{1}{\beta}}}+W_{-1}\left(\left(\theta-\frac{\theta}{\left(\frac{6}{8}\right)^{\frac{1}{\beta}}}\right)\exp\left[-\left(\theta-\frac{\theta}{\left(\frac{6}{8}\right)^{\frac{1}{\beta}}}\right)\right]\right)}{\theta-\frac{\theta}{\left(\frac{6}{8}\right)^{\frac{1}{\beta}}}}\right)^{\frac{1}{\kappa}}-1\right)^{-\frac{1}{\eta}}$$

$$Q\left(\frac{3}{8}\right)=\left(\left(\frac{\theta-\frac{\theta}{\left(\frac{5}{8}\right)^{\frac{1}{\beta}}}+W_{-1}\left(\left(\theta-\frac{\theta}{\left(\frac{5}{8}\right)^{\frac{1}{\beta}}}\right)\exp\left[-\left(\theta-\frac{\theta}{\left(\frac{5}{8}\right)^{\frac{1}{\beta}}}\right)\right]\right)}{\theta-\frac{\theta}{\left(\frac{5}{8}\right)^{\frac{1}{\beta}}}}\right)^{\frac{1}{\kappa}}-1\right)^{-\frac{1}{\eta}}$$

$$Q\left(\frac{4}{8}\right)=\left(\left(\frac{\theta-\frac{\theta}{\left(\frac{4}{8}\right)^{\frac{1}{\beta}}}+W_{-1}\left(\left(\theta-\frac{\theta}{\left(\frac{4}{8}\right)^{\frac{1}{\beta}}}\right)\exp\left[-\left(\theta-\frac{\theta}{\left(\frac{4}{8}\right)^{\frac{1}{\beta}}}\right)\right]\right)}{\theta-\frac{\theta}{\left(\frac{4}{8}\right)^{\frac{1}{\beta}}}}\right)^{\frac{1}{\kappa}}-1\right)^{-\frac{1}{\eta}}$$

$$Q\left(\frac{5}{8}\right)=\left(\left(\frac{\theta-\frac{\theta}{\left(\frac{3}{8}\right)^{\frac{1}{\beta}}}+W_{-1}\left(\left(\theta-\frac{\theta}{\left(\frac{3}{8}\right)^{\frac{1}{\beta}}}\right)\exp\left[-\left(\theta-\frac{\theta}{\left(\frac{3}{8}\right)^{\frac{1}{\beta}}}\right)\right]\right)}{\theta-\frac{\theta}{\left(\frac{3}{8}\right)^{\frac{1}{\beta}}}}\right)^{\frac{1}{\kappa}}-1\right)^{-\frac{1}{\eta}}$$

$$Q\left(\frac{6}{8}\right)=\left(\left(\frac{\theta-\frac{\theta}{\left(\frac{2}{8}\right)^{\frac{1}{\beta}}}+W_{-1}\left(\left(\theta-\frac{\theta}{\left(\frac{2}{8}\right)^{\frac{1}{\beta}}}\right)\exp\left[-\left(\theta-\frac{\theta}{\left(\frac{2}{8}\right)^{\frac{1}{\beta}}}\right)\right]\right)}{\theta-\frac{\theta}{\left(\frac{2}{8}\right)^{\frac{1}{\beta}}}}\right)^{\frac{1}{\kappa}}-1\right)^{-\frac{1}{\eta}}$$

$$Q\left(\frac{7}{8}\right)=\left(\left(\frac{\theta-\frac{\theta}{\left(\frac{1}{8}\right)^{\frac{1}{\beta}}}+W_{-1}\left(\left(\theta-\frac{\theta}{\left(\frac{1}{8}\right)^{\frac{1}{\beta}}}\right)\exp\left[-\left(\theta-\frac{\theta}{\left(\frac{1}{8}\right)^{\frac{1}{\beta}}}\right)\right]\right)}{\theta-\frac{\theta}{\left(\frac{1}{8}\right)^{\frac{1}{\beta}}}}\right)^{\frac{1}{\kappa}}-1\right)^{-\frac{1}{\eta}}$$

3.3 Moments

Moments play an important role in determining the mean, variance, skewness, and kurtosis of the probability distribution. The n^{th} moments of the MODLBIII distribution can be obtained by the equation μ_n [24-26]:

$$\mu_r=\int_0^{\infty} x^r g(x)dx \tag{19}$$

By substituting equation (12) into equation (19), we get:

$$\mu_r = \Phi \int_0^\infty \frac{1}{x^{\eta+r+1}} \left(1 + \frac{1}{x^\eta}\right)^{-\kappa(2m+n+z+2)-1} dx - \Upsilon \int_0^\infty \frac{1}{x^{\eta+r+1}} \left(1 + \frac{1}{x^\eta}\right)^{-\kappa(2m+n+i+2)-1} dx$$

By simplifying and integrating the above function, the moment function for the MODLBIII distribution is obtained in the form:

$$\mu_r = \Gamma\left(\frac{\eta + r}{\eta}\right) \left[-\frac{\Gamma\left(\frac{\eta\kappa(2m+n+z+2) - r}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+z+2)+1)} + \frac{\Gamma\left(\frac{\eta\kappa(2m+n+i+2) - r}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+i+2)+1)} \right] \tag{20}$$

The variance of the MODLBIII distribution is obtained by the following formula [27]:

$$\sigma^2 = \Gamma\left(\frac{\eta + 2}{\eta}\right) \left[-\frac{\Gamma\left(\frac{\eta\kappa(2m+n+z+2) - 2}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+z+2)+1)} + \frac{\Gamma\left(\frac{\eta\kappa(2m+n+i+2) - 2}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+i+2)+1)} \right] - \left(\Gamma\left(\frac{\eta + 1}{\eta}\right) \left[-\frac{\Gamma\left(\frac{\eta\kappa(2m+n+z+2) - 1}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+z+2)+1)} + \frac{\Gamma\left(\frac{\eta\kappa(2m+n+i+2) - 1}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+i+2)+1)} \right] \right)^2 \tag{21}$$

The skewness and kurtosis based on moments are defined by [28-29]:

$$S = \frac{\Gamma\left(\frac{\eta + 3}{\eta}\right) \left[-\frac{\Gamma\left(\frac{\eta\kappa(2m+n+z+2) - 3}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+z+2)+1)} + \frac{\Gamma\left(\frac{\eta\kappa(2m+n+i+2) - 3}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+i+2)+1)} \right]}{\left(\Gamma\left(\frac{\eta + 2}{\eta}\right) \left[-\frac{\Gamma\left(\frac{\eta\kappa(2m+n+z+2) - 2}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+z+2)+1)} + \frac{\Gamma\left(\frac{\eta\kappa(2m+n+i+2) - 2}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+i+2)+1)} \right] \right)^{\frac{3}{2}}} \tag{22}$$

$$K = \frac{\Gamma\left(\frac{\eta + 4}{\eta}\right) \left[-\frac{\Gamma\left(\frac{\eta\kappa(2m+n+z+2) - 4}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+z+2)+1)} + \frac{\Gamma\left(\frac{\eta\kappa(2m+n+i+2) - 4}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+i+2)+1)} \right]}{\left(\Gamma\left(\frac{\eta + 2}{\eta}\right) \left[-\frac{\Gamma\left(\frac{\eta\kappa(2m+n+z+2) - 2}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+z+2)+1)} + \frac{\Gamma\left(\frac{\eta\kappa(2m+n+i+2) - 2}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+i+2)+1)} \right] \right)^{2}} - 3 \tag{23}$$

Table 3: Numerical value of $\mu_1, \mu_2, \mu_3, \mu_4, \sigma^2, S,$ and K of the MODLBIII distribution

θ	β	η	κ	μ_1	μ_2	μ_3	μ_4	σ^2	S	K
0.03	6.3	5.3	0.4	0.261686	0.073903	0.022208	0.007037	0.005424	1.105386	1.288366
			0.8	0.509634	0.265674	0.141313	0.076545	0.005947	1.031946	1.084466
	6.7	2.3	0.6	0.129342	0.019706	0.003408	0.000656	0.002977	1.231893	1.688177
			0.9	0.255633	0.071224	0.021305	0.006778	0.005876	1.120832	1.336066
0.05	6.3	5.3	0.4	0.294506	0.093555	0.031608	0.011256	0.006821	1.10459	1.285994

		0.8	0.542061	0.30065	0.170196	0.098148	0.00682	1.032425	1.085825
6.7	2.3	0.6	0.155525	0.028506	0.005935	0.00138	0.004317	1.233101	1.697818
		0.9	0.290808	0.092344	0.031542	0.011487	0.007775	1.124026	1.346999

mgf is the moment generating function given by equation:

$$M_x(y)_{\text{MODLBIII}} = E(e^{yx}) = \int_{-\infty}^{\infty} e^{yx} g(x)dx$$

Then by used series expansion for e^{yx}, and from equation (20) we get the mgf of MODLBIII distribution [30]:

$$M_x(y)_{\text{MODLBIII}} = \sum_{s=0}^{\infty} \frac{y^s}{s!} \Gamma\left(\frac{\eta + r}{\eta}\right) \times \left[-\frac{\Gamma\left(\frac{\eta\kappa(2m+n+z+2)-r}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+z+2)+1)} + \frac{\Gamma\left(\frac{\eta\kappa(2m+n+i+2)-r}{\eta}\right)}{\eta\Gamma(\kappa(2m+n+i+2)+1)} \right] \quad (24)$$

2.4 Rényi Entropy

One may calculate the Rényi entropy for the MODLBIII distribution [31-32]:

$$I_R(t) = \frac{1}{1-t} log \int_0^{\infty} g(x)^t dx \quad (25)$$

By substituting equation (14) into the above equation, we get:

$$I_R(t) = \frac{1}{1-t} log \int_0^{\infty} \left[\frac{\Phi\left(1+\frac{1}{x^\eta}\right)^{-\kappa(2m+n+z+2)-1} - \Upsilon\left(1+\frac{1}{x^\eta}\right)^{-\kappa(2m+n+i+2)-1}}{x^{\eta+1}} \right]^t dx$$

$$I_R(t) = \frac{1}{1-t} log \left[\sum_{v=0}^{c} (-1)^m \binom{t}{v} \Phi\Upsilon \int_0^{\infty} \frac{\left(1+\frac{1}{x^\eta}\right)^{-\kappa(zv+2mt+nt+it+2t-iv)-t}}{x^{t\eta+t}} dx \right]$$

$$I_R(t) = \frac{1}{1-t} \times log \left[\sum_{v=0}^{t} - \frac{(-1)^m \binom{t}{v} \Phi\Upsilon\Gamma\left(\frac{\kappa\eta(zv+2mt+nt+it+2t-iv)+t+1}{\eta}\right)\Gamma\left(\frac{\eta t - t - 1}{\eta}\right)}{a\Gamma(\kappa(zv+2mt+nt+it+2t-iv)+t)} \right] \quad (26)$$

3.5 Order statistics

The probability density function (pdf) of the j^{th} order statistic, which represents the j^{th} smallest value in a random sample of size n from a distribution function $G(x)$ with an associated pdf $g(x)$, can be expressed as [33-34]:

$$g_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} [G(x)]^{j+r-1} g(x) \quad (27)$$

By inserting equation (3) and equation (4) into equation (20), we obtain the probability density function (pdf) of the j^{th} order statistics for a random sample of size n selected from the MODLBIII-distribution:

$$g_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} \left[1 - \left(1 - \frac{1}{\theta}\left(1+\frac{1}{x^\eta}\right)^{-\kappa} . log\left(1 - \left(1+\frac{1}{x^\eta}\right)^{-\kappa}\right)\right)^{-\beta}\right]^{j+r-1}$$
$$\frac{\beta\eta\kappa}{\theta x^{\eta+1}}\left(1+\frac{1}{x^\eta}\right)^{-\kappa-1}\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^\eta}\right)^{-\kappa}.\log\left(1-\left(1+\frac{1}{x^\eta}\right)^{-\kappa}\right)\right)^{-(\beta+1)} \times\left[\frac{\left(1+\frac{1}{x^\eta}\right)^{-\kappa}}{1-\left(1+\frac{1}{x^\eta}\right)^{-\kappa}} - \log\left(1-\left(1+\frac{1}{x^\eta}\right)^{-\kappa}\right)\right] \quad (28)$$

The $g_{j:n}(x)$ for minimal order statistics may be obtained by substituting $j = 1$ into equation (28), whereas the $g_{j:n}(x)$ for maximal order statistics can be obtained by substituting $j = n$ into equation (21).

4 Estimation

4.1 Maximum Likelihood Estimation

The method of maximum likelihood estimation is employed to estimate the parameters of the MODLBIII distribution. The log-likelihood function is computed using a random sample $x_1, x_2, \ldots, x_m$ that is distributed according to the probability density function (pdf) of the MODLBIII distribution [35]:

$$L(\Theta, x) = \prod_{i=1}^{m} g(x)$$

$$L(\Theta, x) = \prod_{i=1}^{m} \frac{\beta\eta\kappa}{\theta x^{\eta+1}}\left(1+\frac{1}{x^\eta}\right)^{-\kappa-1}\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^\eta}\right)^{-\kappa}.\log\left(1-\left(1+\frac{1}{x^\eta}\right)^{-\kappa}\right)\right)^{-(\beta+1)} \times\left[\frac{\left(1+\frac{1}{x^\eta}\right)^{-\kappa}}{1-\left(1+\frac{1}{x^\eta}\right)^{-\kappa}} - \log\left(1-\left(1+\frac{1}{x^\eta}\right)^{-\kappa}\right)\right]$$

The log-likelihood function L is derived using the following equation:

$$\begin{aligned} L &= mlog\beta + mlog\eta + mlog\kappa - mlog\theta - (\eta+1)\sum_{i=1}^{m} logx_i \\ &-(\kappa+1)\sum_{i=1}^{m}\log\left[1+\frac{1}{{x_i}^\eta}\right] \\ &-(\beta+1)\sum_{i=1}^{m}\log\left(1-\frac{1}{\theta}\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}.\log\left(1-\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}\right)\right) \\ &+\sum_{i=1}^{m}\log\left[\frac{\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}}{1-\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}} - \log\left(1-\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}\right)\right] \end{aligned} \quad (29)$$

The above equation is partially derived from the distribution parameters, so we obtain:

$$\frac{\partial L}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m}\log\left(1-\frac{1}{\theta}\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}.\log\left(1-\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}\right)\right) \quad (30)$$

$$\frac{\partial L}{\partial \theta} = -\frac{m}{\theta} - \sum_{i=1}^{m}\frac{(\beta+1)\frac{1}{\theta}\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}.\log\left(1-\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}\right)}{\theta^2\left(1-\frac{1}{\theta}\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}.\log\left(1-\left(1+\frac{1}{{x_i}^\eta}\right)^{-\kappa}\right)\right)} \quad (31)$$

$$\frac{\partial L}{\partial \kappa}=\frac{m}{\kappa}-\sum_{i=1}^{m}\log\left[1+\frac{1}{x_i{}^{\eta}}\right]-\sum_{i=1}^{m}\frac{\frac{2\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\log\left(1+\frac{1}{x_i{}^{\eta}}\right)}{1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}}+\frac{\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-2\kappa}\log\left(1+\frac{1}{x_i{}^{\eta}}\right)}{\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)^{2}}}{\frac{\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}}{1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}}-\log\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)}-(\beta+1)\times\sum_{i=1}^{m}\frac{\frac{\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\log\left(1+\frac{1}{x_i{}^{\eta}}\right)\log\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)}{\theta}-\frac{\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-2\kappa}\log\left(1+\frac{1}{x_i{}^{\eta}}\right)}{\theta\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)}}{1-\frac{1}{\theta}\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}.\log\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)} \tag{32}$$

$$\frac{\partial L}{\partial \eta}=\frac{m}{\eta}-\sum_{i=1}^{m}logx_i+(\kappa+1)\sum_{i=1}^{m}\frac{\log x_i}{1+x_i{}^{\eta}}+\sum_{i=1}^{m}\frac{\frac{2\kappa\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}logx_i}{(1+x_i{}^{\eta})\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)}+\frac{\kappa\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-2\kappa}logx_i}{(1+x_i{}^{\eta})\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)^{2}}}{\frac{\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}}{1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}}-\log\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)}+(\beta+1)\times\sum_{i=1}^{m}\frac{\frac{\kappa\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\log x_i\log\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)}{\kappa(1+x_i{}^{\eta})}-\frac{\kappa\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-2\kappa}\log x_i}{\theta(1+x_i{}^{\eta})\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)}}{1-\frac{1}{\theta}\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}.\log\left(1-\left(1+\frac{1}{x_i{}^{\eta}}\right)^{-\kappa}\right)} \tag{33}$$

The set of non-linear equations $\frac{\partial L}{\partial \eta}$=0, $\frac{\partial L}{\partial \kappa}$=0, $\frac{\partial L}{\partial \beta}$=0, and $\frac{\partial L}{\partial \theta}$=0 Their answers result in the maximum likelihood estimation (MLE) of the parameters $\eta,\theta,\kappa,$, and β. The sole method to obtain the solution was by numerical techniques using programs such as R, MAPLE, SAS, and others.

4.2 Least Squares Estimation

The parameters can be estimated using the Least Squares Estimation (LSE) method using the equation below [36]:

$$\varphi(\theta,\beta,\eta,\kappa)=\sum_{i=1}^{n}\left[G(x_i)-\frac{1}{n+1}\right]^{2}$$

$$\varphi(\theta,\beta,\eta,\kappa)=\sum_{i=1}^{n}\left[1-\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta}-\frac{1}{n+1}\right]^{2}$$

By partially deriving the above equation for the $\theta,\beta,\eta,$ and κ parameters, we obtain:

$$\frac{\partial\big(\varphi(\sigma,\beta,\alpha)\big)}{\partial\theta}=2\sum_{i=1}^{n}\left[1-\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta}-\frac{1}{n+1}\right]\times\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta} \tag{34}$$

$$\times \frac{\beta\ \left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)}{\theta^{2}\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)}$$

$$\frac{\partial(\varphi(\theta,\beta,\eta,\kappa))}{\partial\beta} = 2\sum_{i=1}^{n}\left[1-\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta}-\frac{1}{n+1}\right]$$
$$\times\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta} \tag{35}$$
$$\times\log\left[1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right]$$

$$\frac{\partial(\varphi(\theta,\beta,\eta,\kappa))}{\partial\eta} = 2\sum_{i=1}^{n}\left[1-\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta}-\frac{1}{n+1}\right]$$
$$\times\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta-1} \tag{36}$$
$$\times\left[\frac{\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\kappa\log x_i\, log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)}{\theta x_i^{\eta}\left(1+\frac{1}{x^{\eta}}\right)}+\frac{\left(1+\frac{1}{x^{\eta}}\right)^{-2\kappa}\kappa\log x_i}{\theta x_i^{\eta}\left(1+\frac{1}{x^{\eta}}\right)\left[1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right]}\right]$$

$$\frac{\partial(\varphi(\theta,\beta,\eta,\kappa))}{\partial\kappa} = 2\sum_{i=1}^{n}\left[1-\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta}-\frac{1}{n+1}\right]$$
$$\times\beta\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta-1} \tag{37}$$
$$\times\left[\frac{\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\log\left(1+\frac{1}{x^{\eta}}\right)log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)}{\theta}-\frac{\left(1+\frac{1}{x^{\eta}}\right)^{-2\kappa}\log\left(1+\frac{1}{x^{\eta}}\right)}{\theta\left[1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right]}\right]$$

By setting the previous equations equal to zero, we obtain the Least Squares estimation:

$$\frac{\partial\left(\varphi(\sigma,\beta,\alpha)\right)}{\partial\theta}=\frac{\partial(\varphi(\theta,\beta,\eta,\kappa))}{\partial\beta}=\frac{\partial(\varphi(\theta,\beta,\eta,\kappa))}{\partial\eta}=\frac{\partial(\varphi(\theta,\beta,\eta,\kappa))}{\partial\kappa}=0$$

It is seen that the equations are equivalent to zero. Evidently, deriving the closed form of the aforementioned equations is unattainable, and solving them manually poses a challenge. Thus, it is imperative to utilize computer programs or numerical procedures to ascertain an approximation of these parameters.

4.3 Weighted Least Squares Estimators (WLSE)

The weighted least squares estimators can be obtained by the equation (5) [37-38]:

$$\mathrm{W}\,(\theta,\beta,\eta,\kappa)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\left(1-\frac{1}{\theta}\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}.\,log\left(1-\left(1+\frac{1}{x^{\eta}}\right)^{-\kappa}\right)\right)^{-\beta}-\frac{i}{n+1}\right]^{2} \tag{38}$$

By partially deriving the above equation for the θ,β,η, and κ parameters, in order to obtain estimates for the parameters.

5 Simulation

The MODLBIII distribution's MLEs, LSE, and WLSE are evaluated using a Monte Carlo simulation in R. The study comprised 50, 100, 150 and 200 samples. In order to obtain the exact Table 4 parameters, we collect a total of 1000 samples. The mean values are determined by averaging the greatest likelihood estimators of the model parameters. Subsequently, bias and root mean square errors (RMSEs) are determined using [20] and [24].

Table 4 : Monte Carlo simulations conducted for the MODLBIII distribution

$\theta = 1.5$, $\beta = 4.3$, $\eta = 5.3$, $\kappa = 2.4$					
n	Est.	Est.par.	MLE	LSE	WLSE
50	MEAN	$\hat{\theta}$	5.345371e+03	1.51751479	1.3582008
		$\hat{\beta}$	1.736813e+04	5.470187	6.002160
		$\hat{\eta}$	6.2144646	5.303065735	4.775659
		$\hat{\kappa}$	2.5472295	2.6801830	2.7905173
	MSE	$\hat{\theta}$	1.261281e+09	1.80245214	1.0017079
		$\hat{\beta}$	1.144928e+10	13.032840	19.203354
		$\hat{\eta}$	6.5936608	7.720787734	3.817021
		$\hat{\kappa}$	0.7687015	0.6338495	0.6948649
	RMSE	$\hat{\theta}$	3.551452e+04	1.34255433	1.0008536
		$\hat{\beta}$	1.070013e+05	3.610102	4.382163
		$\hat{\eta}$	2.5678125	2.778630550	1.953720
		$\hat{\kappa}$	0.8767563	0.7961467	0.8335856
	BAIS	$\hat{\theta}$	5.343871e+03	0.01751479	0.1417992
		$\hat{\beta}$	1.736383e+04	1.170187	1.702160
		$\hat{\eta}$	0.9144646	0.003065735	0.524341
		$\hat{\kappa}$	0.1472295	0.2801830	0.3905173
100	MEAN	$\hat{\theta}$	5.834706e+02	1.3720389	3.688289
		$\hat{\beta}$	7.778956e+02	5.451965	10.67046
		$\hat{\eta}$	6.1260485	4.9382109	4.692966
		$\hat{\kappa}$	2.34212509	2.7141464	2.7985915
	MSE	$\hat{\theta}$	6.983158e+07	1.6977030	470.284513
		$\hat{\beta}$	1.240438e+08	14.190658	2161.08633
		$\hat{\eta}$	3.2520515	3.1139042	2.574678
		$\hat{\kappa}$	0.26073566	0.4598445	0.6331836
	RMSE	$\hat{\theta}$	8.356529e+03	1.3029593	21.686044
		$\hat{\beta}$	1.113750e+04	3.767049	46.48749
		$\hat{\eta}$	1.8033445	1.7646258	1.604580
		$\hat{\kappa}$	0.51062282	0.6781183	0.7957284
	BAIS	$\hat{\theta}$	5.819706e+02	0.1279611	2.188289
		$\hat{\beta}$	7.735956e+02	1.151965	6.37046
		$\hat{\eta}$	0.8260485	0.3617891	0.607034
		$\hat{\kappa}$	0.05787491	0.3141464	0.3985915
150	MEAN	$\hat{\theta}$	5.549643e+03	1.831566	6.871573e+03
		$\hat{\beta}$	9.834520e+03	6.084883	2.399027e+04
		$\hat{\eta}$	5.8658943	4.896935	4.9412047
		$\hat{\kappa}$	2.34580322	2.6235091	2.5432614
	MSE	$\hat{\theta}$	5.526486e+09	22.137230	5.333301e+09
		$\hat{\beta}$	1.849041e+10	138.109362	6.500868e+10
		$\hat{\eta}$	1.7901399	1.689651	1.3137615
		$\hat{\kappa}$	0.18461147	0.2830941	0.2064004
	RMSE	$\hat{\theta}$	7.434034e+04	4.705022	7.302946e+04
		$\hat{\beta}$	1.359794e+05	11.751994	2.549680e+05
		$\hat{\eta}$	1.3379611	1.299866	1.1461944
		$\hat{\kappa}$	0.42966437	0.5320659	0.4543131
	BAIS	$\hat{\theta}$	5.548143e+03	0.331566	6.870073e+03
		$\hat{\beta}$	9.830220e+03	1.784883	2.398597e+04
		$\hat{\eta}$	0.5658943	0.403065	0.3587953

		$\hat{\kappa}$	0.05419678	0.2235091	0.1432614
200	MEAN	$\hat{\theta}$	2.742094	3.491361	1.6155496
		$\hat{\beta}$	5.2614663	10.089273	4.793763
		$\hat{\eta}$	5.6427914	5.1100473	5.27664803
		$\hat{\kappa}$	2.401198667	2.5337544	2.45546604
	MSE	$\hat{\theta}$	13.387749	695.787118	0.5176883
		$\hat{\beta}$	10.5328128	4432.309385	4.509236
		$\hat{\eta}$	1.3700284	1.3672749	0.79680766
		$\hat{\kappa}$	0.162938299	0.1808936	0.12947818
	RMSE	$\hat{\theta}$	3.658927	26.377777	0.7195056
		$\hat{\beta}$	3.2454295	66.575592	2.123496
		$\hat{\eta}$	1.1704821	1.1693053	0.89264083
		$\hat{\kappa}$	0.403656164	0.4253159	0.35983077
	BAIS	$\hat{\theta}$	1.242094	1.991361	0.1155496
		$\hat{\beta}$	0.9614663	5.789273	0.493763
		$\hat{\eta}$	0.3427914	0.1899527	0.02335197
		$\hat{\kappa}$	0.001198667	0.1337544	0.05546604

$\theta = 2.8, \quad \beta = 6.3, \quad \eta = 1.7, \quad \kappa = 3.3$

n	Est.	Est.par.	MLE	LSE	WLSE
50	MEAN	$\hat{\theta}$	7.809163e+04	2.86416937	4.966504e+04
		$\hat{\beta}$	7.689101e+04	8.064287	1.678635e+05
		$\hat{\eta}$	1.8366607	1.60183522	1.5944730
		$\hat{\kappa}$	3.8400674	3.5842128	3.6517170
	MSE	$\hat{\theta}$	1.242802e+11	11.24134795	4.069492e+11
		$\hat{\beta}$	7.393434e+10	20.111660	4.648907e+12
		$\hat{\eta}$	0.7324774	0.43958901	0.3665253
		$\hat{\kappa}$	4.2193062	0.7134215	0.6568189
	RMSE	$\hat{\theta}$	3.525339e+05	3.35281195	6.379257e+05
		$\hat{\beta}$	2.719087e+05	4.484603	2.156132e+06
		$\hat{\eta}$	0.8558489	0.66301509	0.6054134
		$\hat{\kappa}$	2.0540950	0.8446428	0.8104436
	BAIS	$\hat{\theta}$	7.808883e+04	0.06416937	4.966224e+04
		$\hat{\beta}$	7.688471e+04	1.764287	1.678572e+05
		$\hat{\eta}$	0.1366607	0.09816478	0.1055270
		$\hat{\kappa}$	0.5400674	0.2842128	0.3517170
100	MEAN	$\hat{\theta}$	2.559759e+04	2.025829e+04	5.928752e+04
		$\hat{\beta}$	2.834725e+04	4.202144e+04	1.495324e+05
		$\hat{\eta}$	1.79880223	1.5615797	1.5656145
		$\hat{\kappa}$	3.4637002	3.6215560	3.5502009
	MSE	$\hat{\theta}$	2.401295e+10	8.821506e+10	4.227686e+11
		$\hat{\beta}$	2.518494e+10	3.795110e+11	2.697523e+12
		$\hat{\eta}$	0.21431648	0.2897513	0.1890837
		$\hat{\kappa}$	0.3112884	0.4498621	0.3595743
	RMSE	$\hat{\theta}$	1.549611e+05	2.970102e+05	6.502066e+05
		$\hat{\beta}$	1.586976e+05	6.160447e+05	1.642414e+06
		$\hat{\eta}$	0.46294327	0.5382855	0.4348376
		$\hat{\kappa}$	0.5579323	0.6707176	0.5996451
	BAIS	$\hat{\theta}$	2.559479e+04	2.025549e+04	5.928472e+04
		$\hat{\beta}$	2.834095e+04	4.201514e+04	1.495261e+05
		$\hat{\eta}$	0.09880223	0.1384203	0.1343855
		$\hat{\kappa}$	0.1637002	0.3215560	0.2502009
150	MEAN	$\hat{\theta}$	1.408614e+04	2.70006588	2.78474422
		$\hat{\beta}$	1.981617e+04	8.051607	7.639880
		$\hat{\eta}$	1.7549513	1.5867696	1.63182323
		$\hat{\kappa}$	3.39874827	3.5069844	3.4625574
	MSE	$\hat{\theta}$	6.948594e+09	2.33231724	2.18934405
		$\hat{\beta}$	1.793435e+10	31.663544	25.566764
		$\hat{\eta}$	0.1262399	0.1998382	0.14099851

		$\hat{\kappa}$	0.17584021	0.2507470	0.2107915
	RMSE	$\hat{\theta}$	8.335823e+04	1.52719260	1.47964322
		$\hat{\beta}$	1.339192e+05	5.627037	5.056359
		$\hat{\eta}$	0.3553025	0.4470326	0.37549769
		$\hat{\kappa}$	0.41933305	0.5007465	0.4591204
	BAIS	$\hat{\theta}$	1.408334e+04	0.09993412	0.01525578
		$\hat{\beta}$	1.980987e+04	1.751607	1.339880
		$\hat{\eta}$	0.0549513	0.1132304	0.06817677
		$\hat{\kappa}$	0.09874827	0.2069844	0.1625574
200	MEAN	$\hat{\theta}$	7.739908e+03	2.9849108	5.141126e+04
		$\hat{\beta}$	9.289976e+03	7.496490	1.027360e+05
		$\hat{\eta}$	1.79023834	1.6390257	1.68652276
		$\hat{\kappa}$	3.32109492	3.38407146	3.32218907
	MSE	$\hat{\theta}$	4.731574e+09	2.9213461	2.880687e+11
		$\hat{\beta}$	6.755221e+09	9.020148	1.150317e+12
		$\hat{\eta}$	0.09565266	0.1517925	0.07823345
		$\hat{\kappa}$	0.13153767	0.12973588	0.09748519
	RMSE	$\hat{\theta}$	6.878644e+04	1.7091946	5.367203e+05
		$\hat{\beta}$	8.219015e+04	3.003356	1.072528e+06
		$\hat{\eta}$	0.30927764	0.3896055	0.27970244
		$\hat{\kappa}$	0.36268122	0.36018868	0.31222619
	BAIS	$\hat{\theta}$	7.737108e+03	0.1849108	5.140846e+04
		$\hat{\beta}$	9.283676e+03	1.196490	1.027297e+05
		$\hat{\eta}$	0.09023834	0.0609743	0.01347724
		$\hat{\kappa}$	0.02109492	0.08407146	0.02218907

Tables 3 show the consistency of all estimators. As the sample size increases, the average parameter estimates get closer to the true values of the coefficients. Moreover, the best method is the MLE method for small samples and the WLSE method for large samples because the mean square errors (MSEs) show a decrease in magnitude as the sample size grows.

5. Application

This section demonstrates the practical implementation using actual data. The data consists of Transactions on Electronic Devices ED-34 failure Specimens [35] and the monthly price share of Baghdad Bank from 2007 to 2012. The efficacy of MODLBIII distribution is precisely demonstrated in this data. The app demonstrates the advantages of MODLBIII and its exceptional data interoperability.

This comparison utilizes six distinct distributions, which are enumerated as follows:

- ❖ Beta Exponential Burr III (BeBIII) (new)
- ❖ Kumaraswamy Burr III (KuBIII) (new)
- ❖ Exponential Generalized Burr III (EGBIII) (new)
- ❖ Log Gamma Burr III (LGBIII) (new)
- ❖ [0,1]Truncated Exponentiated Exponential Burr III ([0,1] TEEBIII) (new)
- ❖ Burr III (BIII)

To perform this comparison, we utilize eight metrics. Two statistical measures employed are the Kolmogorov-Smirnov statistic (KS) and the Anderson-Darling statistic (A). The analysis also takes into account the Cramér-von Mises (W) statistic and the HQIC, BIC, AIC by [39-40-41], and CAIC information criteria. Moreover, it employs the p-value obtained from the Kolmogorov-Smirnov test. These criteria are frequently employed to evaluate the quality of the match.

Table 5: Estimates of models for data1

Dist.	-2L	AIC	CAIC	BIC	HQIC
MODLBIII	111.3699	230.7398	231.4806	239.05	233.9838
BeBIII	111.714	231.4281	232.1688	239.7382	234.672
KuBIII	111.5382	231.0764	231.8172	239.3866	234.320

EGBIII	111.5107	231.0215	231.7622	239.3316	234.2654
LGBIII	111.5666	231.1332	231.8739	239.4433	234.3771
[0,1] TEEBIII	111.535	231.07	231.8107	239.3801	234.3139
BIII	126.6927	257.3854	257.5997	261.5405	259.0074

Table 6: Evaluate statistical metrics for the data1

Dist.	W	A	K-S	p-value
MODLBIII	0.03707751	0.2101993	0.06424404	0.9549713
BeBIII	0.03890086	0.2309838	0.06980883	0.9166754
KuBIII	0.03728512	0.2171501	0.06319673	0.9606704
EGBIII	0.03916518	0.2248682	0.06170907	0.9679512
LGBIII	19.16381	117.8715	0.9982075	7.771561e-16
[0,1] TEEBIII	0.03904305	0.2248934	0.06294089	0.9619902
BIII	0.2431589	1.460834	0.1790567	0.03982132

Table 7: parameter estimators by MLE for the data1

Dist.	θ	β	η	κ
MODLBIII	0.03794086	21.4981489	0.81570094	17.8876532
BeBIII	1.050644	51.854301	1.155388	41.452487
KuBIII	1.570387	85.562028	1.072134	25.887032
EGBIII	84.159885	0.771039	1.188867	53.105248
LGBIII	0.7545401	65.2492090	1.2190964	54.3840325
[0,1] TEEBIII	81.0385180	0.7270798	1.2284767	57.5144360
BIII	-	-	2.888419	177.334276

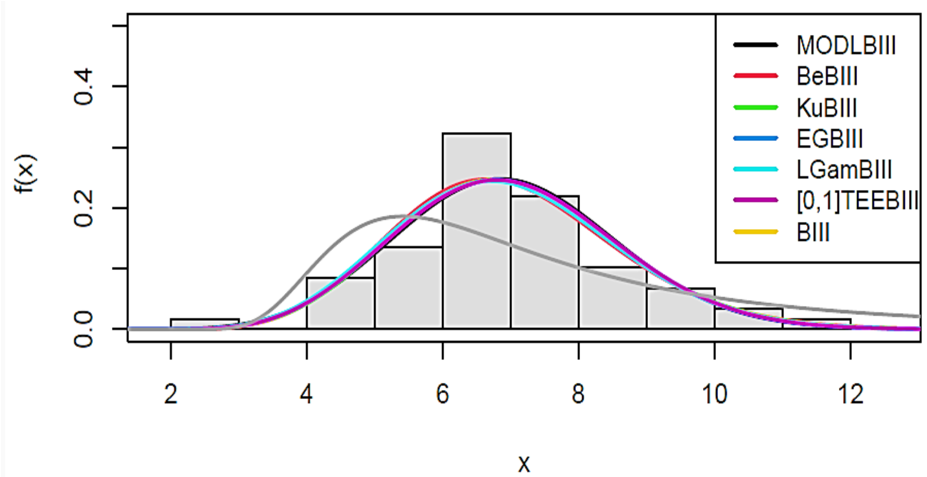
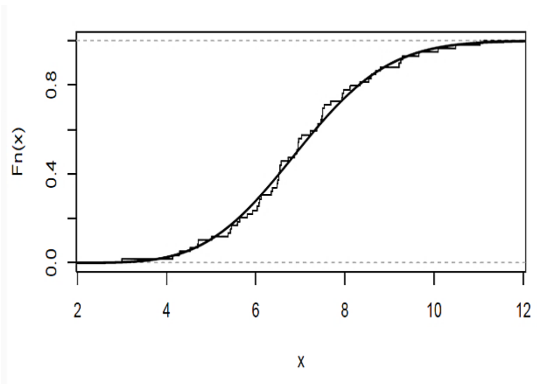
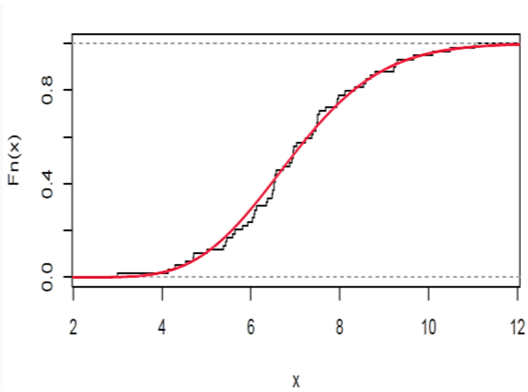


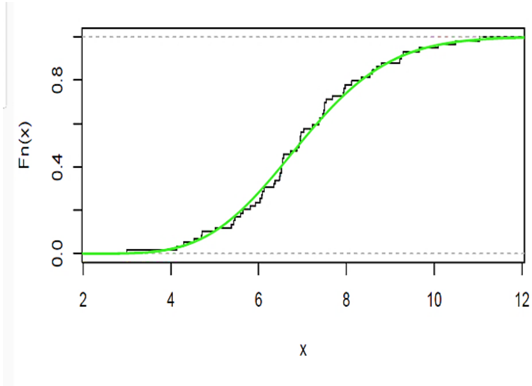
Figure 5: Fitted densities for Data1



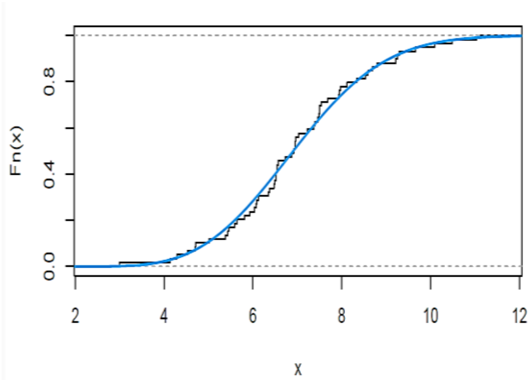
Empirical CDF for MODLBIII



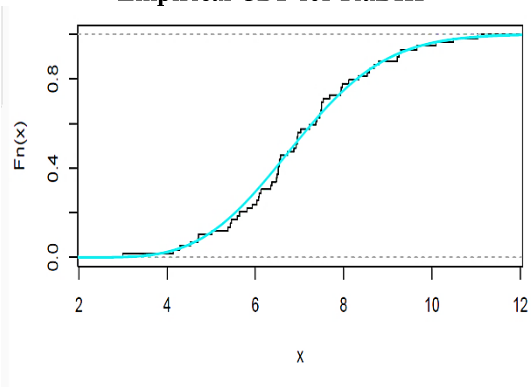
Empirical CDF for BeBIII



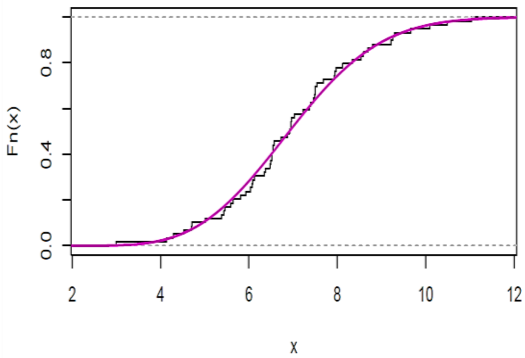
Empirical CDF for KuBIII



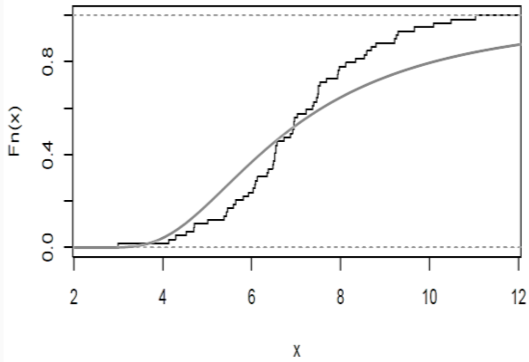
Empirical CDF for EGBIII



Empirical CDF for LGBIII



Empirical CDF for [0,1] TEEBIII



Empirical CDF for BIII

Figure.6 Empirical CDF for Data**1**

Table 8: Estimates of models for data2

Dist.	-2L	AIC	CAIC	BIC	HQIC
MODLBIII	66.90305	141.8088	142.4059	150.9155	145.4342
BeBIII	68.51612	145.047	145.6441	154.1537	148.6725
KuBIII	67.51492	143.0522	143.6492	152.1589	146.6776
EGBIII	69.25097	146.5301	147.1271	155.6367	150.1555
LGBIII	68.42045	144.8693	145.4664	153.976	148.4947
[0,1] TEEBIII	69.13167	146.2814	146.8784	155.3881	149.9068
BIII	74.60402	153.2514	153.4253	157.8048	155.0641

Table 9: Evaluate statistical metrics for the data2

Dist.	W	A	K-S	p-value
MODLBIII	0.04766311	0.3899924	0.07082137	0.8630367
BeBIII	0.104646	0.7980882	0.0988238	0.4828802
KuBIII	0.07334565	0.5815222	0.08876232	0.6218274
EGBIII	0.1228968	0.9220773	0.1007446	0.4579877
LGBIII	24.16727	143.8307	0.985822	0

[0,1] TEEBIII	0.1085448	0.8296792	0.09764551	0.4984664
BIII	0.2252649	1.577803	0.1217605	0.236124

Table 10: parameter estimators by MLE for the data2

Dist.	θ	β	η	κ
MODLBIII	0.1516091	7.3544117	1.2405778	7.3134921
BeBIII	4.141538	5.310211	1.879273	4.559861
KuBIII	3.733340	6.530093	1.996238	3.731251
EGBIII	4.797445	3.969605	1.695221	5.415823
LGBIII	4.268642	6.445361	1.907220	4.355687
[0,1] TEEBIII	11.003107	2.217134	1.487586	8.588491
BIII	-	-	3.58801	13.81385

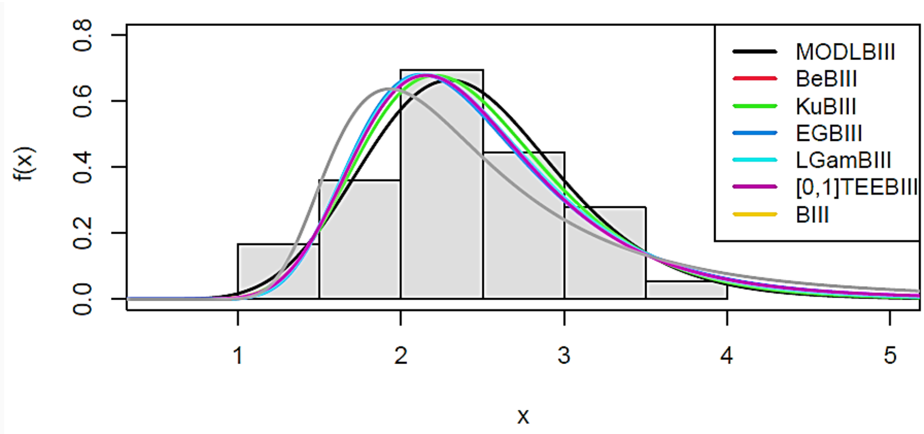
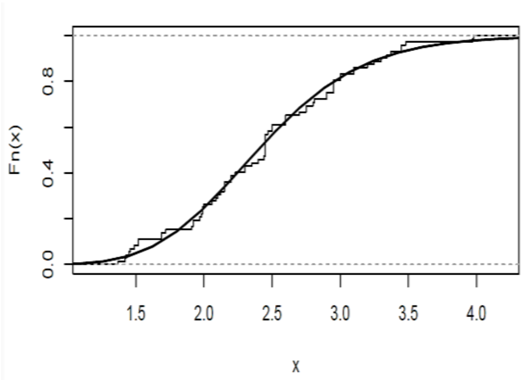
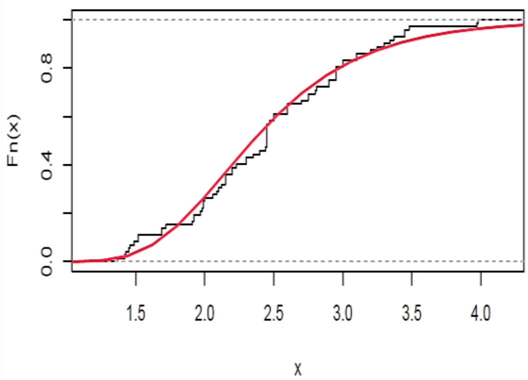


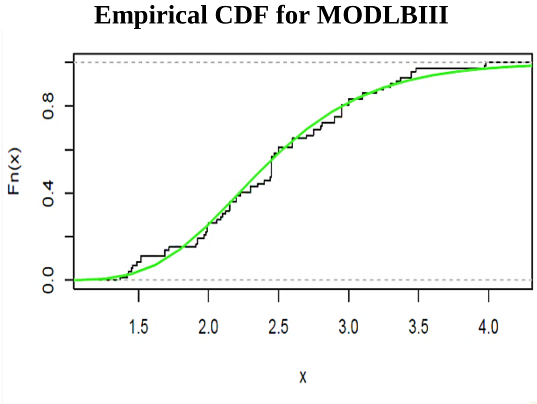
Figure 7: Fitted densities for Data2



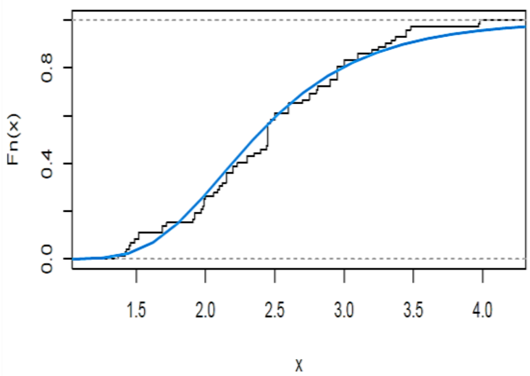
Empirical CDF for MODLBIII



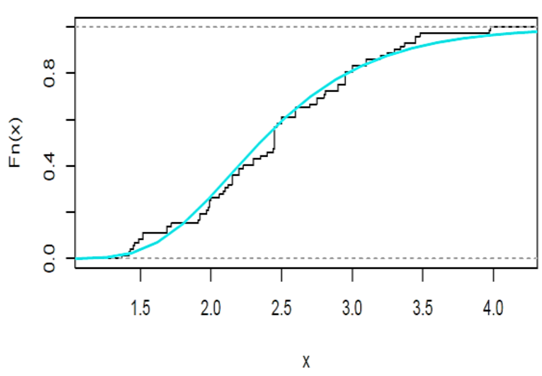
Empirical CDF for BeBIII



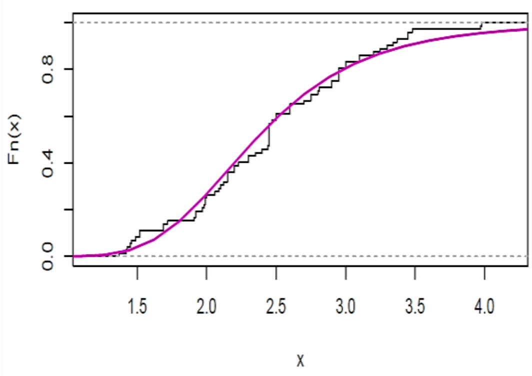
Empirical CDF for KuBIII



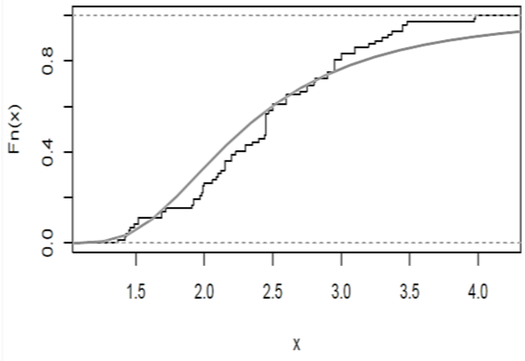
Empirical CDF for EGBIII



Empirical CDF for LGBIII



Empirical CDF for [0,1] TEEBIII



Empirical CDF for BIII

Figure.8 Empirical CDF for Data**2**

Conclusions

In this study, a new model called the MODLBIII distribution is proposed which extends the BIII distribution according to the Odd Lomax-G family. One of the main reasons for this generalization of the standard distribution is that the modified model provides greater flexibility in modelling real data. Several statistical properties of the modified distribution were also found. Estimation The parameters are treated using the maximum likelihood method. In addition to conducting a practical application of the MODLBIII distribution on two types of real data, which showed that the modified distribution can be used effectively to provide a better fit than the BIII distribution as well as a number of other distributions.

Conflicts of interest

Non

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