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Comparison of Some Methods for Estimating the Parameters of the Type – II Generalized Log Logistic Distribution Using Simulation

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ABSTRACT

In this research, the type – II generalized log logistic distribution with three parameters was used, as this distribution is considered a continuous distribution. Three methods were taken to estimate the parameters of this distribution (the maximum likelihood method, the method of moments, and the ordinary least squares method) and to compare them with each other using the simulation method and with different sample sizes (10, 30, 60, 100) and with scale integral mean square error and writing the program in a programming language MATLAB, concluding that the maximum likelihood method is the best.

1. Introduction

Many probabilistic models have been studied over different years and applied in different fields, such as in biological, actuarial, clinical, and other sciences [6]. Logistic distributions [3] are probabilistic models, as the generalized log logistic distribution is of the second type (Type – II Generalized Log Logistic Distribution). Of the continuous distributions, the scientist Rosaiah and others presented this distribution in (2008) [7]. It is symbolized by (TGLD), and the formula of the probability density function (pdf) for this distribution is:

$$f(t)=\frac{\lambda\theta}{\sigma}\frac{\left(t/\sigma\right)^{\lambda-1}}{\left[1+\left(t/\sigma\right)^{\lambda}\right]^{\theta+1}}\quad ,\ t>0\ ,\ \sigma>0\ ,\ \theta>0\ ,\ \lambda>1 \qquad \text{.......... (1)}$$

σ : Scale Parameter, θ, λ : Shape Parameters

And the cdf:

$$F(t)=1-\left[1+\left(t/\sigma\right)^{\lambda}\right]^{-\theta} \qquad \text{.......... (2)}$$

And the Reliability Function

$$R(t)=1-F(t)=\left[1+\left(t/\sigma\right)^{\lambda}\right]^{-\theta} \qquad \text{........... (3)}$$

The formula for the momentum around the origin is [6]:

$$\mu_r = E(t^r)=\frac{r\sigma^r}{\lambda}\frac{\Gamma\left(\frac{r}{\lambda}\right)\Gamma\left(\theta-\frac{r}{\lambda}\right)}{\Gamma(\theta)} \qquad \text{........... (4)}$$

Three estimation methods will be discussed: the maximum likelihood method, the moments method, and the ordinary least squares method, as follows:

2. **Methodology**

2. 1. Maximum Likelihood Method

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This method is used to estimate the parameters of the Type II Generalized Log Logistic Distribution (TGLLD), and the formula of the Probability Density Function (PDF) for this distribution is [7]:

$$f(t)=\frac{\lambda\theta}{\sigma}\frac{\left(t/\sigma\right)^{\lambda-1}}{\left[1+\left(t/\sigma\right)^{\lambda}\right]^{\theta+1}}\ ,\ t\succ 0\ ,\ \sigma\succ 0\ ,\theta\succ 0\ ,\lambda\succ 1 \quad \text{......... (5)}$$

And that:

σ : They represent the Scale Parameter, θ,λ: They represent the Shape Parameters, and the formula for the Cumulative Distribution Function (CDF) is [7]:

$$F(t)=1-\left[1+\left(t/\sigma\right)^{\lambda}\right]^{-\theta} \quad \text{........... (6)}$$

The formula of the reliability function [6] for this distribution is:

$$R(t)=1-F(t)=\left[1+\left(t/\sigma\right)^{\lambda}\right]^{-\theta} \quad \text{............. (7)}$$

If we have a random sample of size (n) drawn from the distribution defined by formula (1), then the likelihood function (L) will be as follows [1]:

$$L=\prod_{i=1}^{n}\frac{\lambda\theta}{\sigma}\frac{\left(t/\sigma\right)^{\lambda-1}}{\left[1+\left(t/\sigma\right)^{\lambda}\right]^{\theta+1}} \quad \text{.............. (8)}$$

Taking the logarithm of both sides of the above formula [2]:

$$LogL=nLog\lambda+nLog\theta-nLog\sigma+(\lambda-1)\sum_{i=1}^{n}Log\left(t_i/\sigma\right)-(\theta+1)\sum_{i=1}^{n}Log\left[1+\left(t_i/\sigma\right)^{\lambda}\right] \quad \text{...(9)}$$

By taking the partial derivative of the above formula with respect to the distribution parameters and setting it equal to zero, we obtain the following:

For the parameter (σ)

$$\frac{\partial LogL}{\partial\hat{\sigma}}=\frac{-n}{\hat{\sigma}}-(\hat{\lambda}-1)\sum_{i=1}^{n}\frac{\left(t_i/\hat{\sigma}^2\right)}{\left(t_i/\hat{\sigma}\right)}+\hat{\lambda}(\hat{\theta}+1)\sum_{i=1}^{n}\frac{\left[t_i^{\hat{\lambda}}/\hat{\sigma}^{\hat{\lambda}+1}\right]}{\left[1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right]}=0 \quad \text{............ (10)}$$

By simplifying the above formula we get:

$$\frac{-n\hat{\lambda}}{\hat{\sigma}}+\frac{\hat{\lambda}(\hat{\theta}+1)}{\hat{\sigma}}\sum_{i=1}^{n}\frac{\left[t_i/\hat{\sigma}\right]^{\hat{\lambda}}}{\left[1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right]}=0 \quad \text{........ (11)}$$

And For the parameter (λ)

$$\frac{\partial LogL}{\partial\hat{\lambda}}=\frac{n}{\hat{\lambda}}+\sum_{i=1}^{n}Log\left(t_i/\hat{\sigma}\right)-(\hat{\theta}+1)\sum_{i=1}^{n}\frac{\left[t_i/\hat{\sigma}\right]^{\hat{\lambda}}Log\left(t_i/\hat{\sigma}\right)}{\left[1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right]}=0 \quad \text{.. (12)}$$

And For the parameter (θ)

$$\frac{\partial LogL}{\partial\hat{\theta}}=\frac{n}{\hat{\theta}}-\sum_{i=1}^{n}Log\left[1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right]=0 \quad \text{.... (13)}$$

2. 2. Moment Method

This method relies on equating the population moment with the sample moment and my agencies [6]:

$$\frac{\sum_{i=1}^{n}t_i}{n}=\frac{\hat{\sigma}}{\hat{\lambda}}\frac{\Gamma\left(\frac{1}{\hat{\lambda}}\right)\Gamma\left(\hat{\theta}-\frac{1}{\hat{\lambda}}\right)}{\Gamma(\hat{\theta})} \quad \text{............. (14)}$$

$$\frac{\sum_{i=1}^{n}t_i^2}{n}=\frac{2\hat{\sigma}}{\hat{\lambda}}\frac{\Gamma\left(\frac{2}{\hat{\lambda}}\right)\Gamma\left(\hat{\theta}-\frac{2}{\hat{\lambda}}\right)}{\Gamma(\hat{\theta})} \quad \text{.............. (15)}$$

$$\frac{\sum_{i=1}^{n}t_i^3}{n}=\frac{3\hat{\sigma}}{\hat{\lambda}}\frac{\Gamma\left(\frac{3}{\hat{\lambda}}\right)\Gamma\left(\hat{\theta}-\frac{3}{\hat{\lambda}}\right)}{\Gamma(\hat{\theta})} \quad \text{............... (16)}$$

2. 3. Ordinary Least Squares Method

This method is used to estimate parameters based on minimizing the sum of squared errors as follows:

$$L.S=\sum_{i=1}^{n}\left(F(t_i)-\frac{i}{n}\right)^2 \quad \text{........... (17)}$$

$$L.S=\sum_{i=1}^{n}\left(\left[1-\left(1+\left(t_i/\sigma\right)^{\lambda}\right)^{-\theta}\right]-\frac{i}{n}\right)^2 \quad \text{........... (18)}$$

Taking the partial derivative of the above formula with respect to the distribution parameters and setting it equal to zero, we obtain the following:

For the parameter (σ)

$$\frac{\partial L.S}{\partial\hat{\sigma}}=2\sum_{i=1}^{n}\left[\begin{array}{l}\left(\left[1-\left(1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)^{-\hat{\theta}}\right]-\frac{i}{n}\right)\\ *\left(-\hat{\theta}\hat{\lambda}\left(1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)^{-\hat{\theta}-1}\left(t_i^{\lambda}/\hat{\sigma}^{\hat{\lambda}+1}\right)\right)\end{array}\right]=0 \quad \text{... (19)}$$

$$\sum_{i=1}^{n}\left[\begin{array}{l}\left(\frac{\hat{\theta}\hat{\lambda}}{\hat{\sigma}}\left(1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)^{-\hat{\theta}-1}\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)\\ *\left(\left(1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)^{-\hat{\theta}}-1+\left(\frac{i}{n}\right)\right)\end{array}\right]=0 \quad \text{....... (20)}$$

And For the parameter (λ)

$$\sum_{i=1}^{n}\left[\begin{array}{l}\left(\hat{\theta}\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}} Log\left(t_i/\hat{\sigma}\right)\left(1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)^{-\hat{\theta}-1}\right)\\ *\left[\left(1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)^{-\hat{\theta}}-1+\left(\frac{i}{n}\right)\right]\end{array}\right]=0 \quad \dots (21)$$

And For the parameter (θ)

$$\sum_{i=1}^{n}\left[\begin{array}{l}Log\left(1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)\\ *\left[\left(1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)^{\hat{\theta}}-1-\left(1+\left(t_i/\hat{\sigma}\right)^{\hat{\lambda}}\right)^{\hat{\theta}}\left(\frac{i}{n}\right)\right]\end{array}\right]=0 \quad \dots (22)$$

Since the estimated formulas for the three methods above are non-linear functions, they can only be solved using numerical methods. Therefore, the Newton-Raphson method will be used to obtain the parameter estimates.

3. Simulation

Default values were chosen for the measurement parameter (3, 3.5) and for the two shape parameters λ with values (3, 4) and θ with values (3.5, 5). Four different sample sizes were also specified (10, 30, 60, 100) and the simulation experiments were repeated (1000).) a thousand times and generate data according to the formula:

$$t=\left[\left(\frac{1}{1-u}\right)^{\frac{1}{\theta}}-1\right]^{\frac{1}{\lambda}}\sigma$$

To compare the methods, the mean square integral error was used using the MATLAB programming language. The results shown in the tables below were obtained:

Table (1): shows the mean square of the integral error when (σ=3 ،λ=3 ،θ=3.5)

n	Parameters	LS	MO	ML
10	σ	1.95218316450792	1.28048856567155	0.169046721866298
	λ	3.10368899345219	0.793737194998602	0.0203228093578873
	θ	10248.1151116509	52.5318697297061	1.00396154713445
30	σ	0.647770929364961	0.230669648619980	0.0170376201374454
	λ	1.02591040751000	0.0526765089467297	0.00361937060579106
	θ	2889.03101929545	2.55588516157478	0.178973538365688
60	σ	0.321519492316845	0.107708264332858	0.00607665266624812
	λ	0.500202945896168	0.00852813775835625	0.00186616302564994
	θ	1291.15493127535	0.144607584844418	0.0943585273770516
100	σ	0.195271093009948	0.0648029080682909	0.00312352043506342
	λ	0.305415459730375	0.00245504722349148	0.00112928073990521
	θ	785.983295781449	0.0372650703033875	0.0584889699388171

Table (2): shows the mean square of the integral error when (σ=3 ،λ=3 ،θ=5)

n	Parameters	LS	MO	ML
10	σ	0.780236791247946	1.69176735738062	0.316075755501129
	λ	1.96618376579135	0.639428994121007	0.0379206782976298

	θ	178187.052255192	135.057470201331	1.79292258346249
30	σ	0.257279042669274	0.327451559875142	0.0404913421016737
	λ	0.660777439035036	0.0514461121938182	0.00118266880886553
	θ	56370.8916085429	11.6502595038730	0.244654502015787
60	σ	0.128973165896228	0.151612359935814	0.0113953820529719
	λ	0.325496632231118	0.00588772527904120	0.000374732778052942
	θ	28327.7037299138	0.736211430401806	0.0939891559726574
100	σ	0.0766805183451086	0.0898244532412642	0.00487349647717109
	λ	0.196260480058373	0.00118806428487675	0.000210515115483357
	θ	17446.9531620069	0.143875242319828	0.0536554495367078

Table (3): shows the mean square of the integral error when (σ=3 ،λ=4 ،θ=3.5)

n	Parameters	LS	MO	ML
10	σ	2.10294749341023	2.99720419945295	0.195698755906669
	λ	2.30720176120262	0.375769103508912	15.6291956647588
	θ	0.703923527122397	47.2706232596797	1.24198805597356
30	σ	0.704189443337824	0.569022908646238	0.0252084650727771
	λ	0.773533520496604	0.0405474463676142	0.0155029724882326
	θ	0.231839834954435	6.21209715000018	0.210245702256732
60	σ	0.349140250897970	0.261212257226109	0.00941718487582653
	λ	0.381233634252459	0.00933737507649270	0.00729430168561492
	θ	0.115471673499169	0.931176771918787	0.112139075088747
100	σ	0.210221967364965	0.154235388078801	0.00494650379233530
	λ	0.230115887184207	0.00370845185534940	0.00430310044064445
	θ	0.0689557694928188	0.164248968708998	0.0692979578952431

Table (4): shows the mean square of the integral error when (σ=3 ،λ=4 ،θ=5)

n	Parameters	LS	MO	ML
10	σ	0.888589455801980	3.52650943543442	0.375324829548308
	λ	1.52171642676713	0.390735774955770	158.427331747371
	θ	145345.638178059	125.050071212455	3.34708712606159
30	σ	0.302034319111518	0.692050167002009	0.0474025087904249
	λ	0.510852749296744	0.0451007397452965	1.52502408037690
	θ	48738.4081311104	12.5360480860850	0.268530395100448
60	σ	0.148846168504546	0.313878094254240	0.0140693646688930
	λ	0.249110842099732	0.0126534492604782	0.0102016413691292
	θ	24623.9652274798	1.75325580746470	0.109304433463959
100	σ	0.0892733982761058	0.183690375779514	0.00637184596651452
	λ	0.151800634527954	0.00568206155890847	0.00607467049497707
	θ	15173.6038950927	0.328870202367386	0.0640946106615050

Table (5): shows the mean square of the integral error when (σ=3.5 ،λ=3 ،θ=3.5)

n	Parameters	LS	MO	ML
10	σ	1.81120356893793	0.798002800767702	0.323092011178583
	λ	3.69567754328151	1.41056760431065	0.110714737566791
	θ	4154.81612093461	80.5385721413216	1.45261159286758
30	σ	0.607154732640139	0.110595826933886	0.0446456373381499

	λ	1.22962241863451	0.158973788488482	0.0213462092570317
	θ	1167.25889869962	14.0995333122493	0.211600685047895
60	σ	0.298693514099940	0.0482442563652094	0.0142353290459698
	λ	0.598349666131562	0.0274146322294678	0.0107165075278280
	θ	810.635524565859	0.751235031525340	0.104479748965494
100	σ	0.180366684266244	0.0286625311197082	0.00684269501414839
	λ	0.364802995381009	0.00916471851885297	0.00650756221167202
	θ	388.181050069552	0.0571209511212876	0.0636219185157907

Table (6): shows the mean square of the integral error when (σ=3.5, λ=3, θ=5)

n	Parameters	LS	MO	ML
10	σ	0.736981013391033	1.11250539148012	0.495582759962156
	λ	2.44552178299788	1.06803945262581	0.0998007427856711
	θ	129295.697599681	161.835727552652	7.89211569792721
30	σ	0.245999882372392	0.168816510757076	0.0565019721490784
	λ	0.826588371907216	0.108702486355174	0.0119552005568438
	θ	42512.9469348793	17.4351717246209	0.392278668179907
60	σ	0.118776645067594	0.0740880024419504	0.0128693052422621
	λ	0.400530989919525	0.0188158468891001	0.00533704059123560
	θ	21602.7195069873	1.71789232150187	0.118857762190860
100	σ	0.0713733639380119	0.0432036146494664	0.00405164670721602
	λ	0.242225839311240	0.00459312354148153	0.00316156631785646
	θ	12743.7095265747	0.141476963822458	0.0630232918088573

Table (7): shows the mean square of the integral error when (σ=3.5 ،λ=4 ،θ=3.5)

n	Parameters	LS	MO	ML
10	σ	1.99535059026374	2.13402663287297	0.208197257443523
	λ	2.86765278593346	0.551880758781549	0.0440961292166181
	θ	0.674996318630147	48.4001294111776	1.07457303916896
30	σ	0.662584445099103	0.352781877162041	0.0262792617380084
	λ	0.957733839974807	0.0522441983249116	0.00140854469060976
	θ	0.222888708881553	3.18869291723540	0.201270417893898
60	σ	0.326843147690726	0.156194478976220	0.0100047323853551
	λ	0.470278804754791	0.0110019157421325	0.000538503330977550
	θ	12.4365724214481	0.517943060580863	0.105543571981417
100	σ	0.197187273035939	0.0913824603188180	0.00527440288984957
	λ	0.281319037774885	0.00287050028028847	0.000295536805680183
	θ	7.46188332976410	0.117045494578903	0.0654435914784053

Table (8): shows the mean square of the integral error when (σ=3.5 ،λ=4 ،θ=5)

n	Parameters	LS	MO	ML
10	σ	0.856342508695542	2.56384845863489	0.529702045536906
	λ	2.09074043982753	0.622938559173498	10.4220070751583
	θ	81257.0983463537	145.244362851399	43.1491976545850
30	σ	0.286055177835668	0.438468325101922	0.0583773875128647
	λ	0.697788803553710	0.0649265035490974	0.00979675887849211
	θ	26094.4301462264	15.6555817928138	0.329965438967870

60	σ	0.142645260810766	0.192120890294871	0.0159581717399215
	λ	0.340441617275571	0.0127599776804449	0.00235723143275822
	θ	12903.4025935697	1.84442189580061	0.102856228252894
100	σ	0.0858265744497718	0.111180201824839	0.00642042945499657
	λ	0.205756712015645	0.00344949767718523	0.00111814920618280
	θ	7719.48993213508	0.514466308588860	0.0545050925793897

As for the values of the reliability function based on the maximum likelihood method being the best method, they are as in Table (9) below:

Table (9) shows the values of the reliability function

t	1	2	3	4	5	6	7	8	9	10
R	1	0.99999	0.99965	0.99549	0.96767	0.8513	0.57031	0.24746	0.075471	0.020242

3. Results and discussion

From the table above it is clear that the maximum places method (ML) is better than the two moments methods (MO) and the ordinary least squares method (LS).

4. Conclusions

1- The Maximum Likelihood (ML) method was the best, followed by the Moments (MO) method, and finally the Ordinary Least Squares (LS) method.
2- The mean square integral error decreases as the sample size increases.
3- We notice that the values of the reliability function decrease as the values of (t) increase.

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