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Modelling the Fractional Diffusion Equation for a Truncated Levy Process with Financial Applications

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ABSTRACT

The truncated Levy process (TLP) modifies the heavy-tailed Levy distribution by transitioning to a fast-decaying probability distribution, solving the second-moment divergence problem. We present an extension of the fractional diffusion equation that simulates a process with a truncated Levy power-law behavior with an exponent of 5-α. This results in a closed-form discriminant function, where the displacement probability density function transitions to a Gaussian in essence while preserving the power-law tail. Truncated Levy processes are promising for financial modelling, as they provide finite moments and capture short-term divergence and long-term Gaussian convergence. We validate the truncated Levy process model using simulated data. This paper is part of a doctoral thesis, where we relied on simulation in the applied aspect, exploring option hedging in a Levy-dominated context, comparing optimal strategies with delta hedging, and revealing key differences. In addition, we derive a generalized option pricing formula for assets under the truncated Levy process model.

1. Introduction

Diffusion is a fundamental physical process that describes the movement of particles from regions of higher concentration to regions of lower concentration, driven by random thermal motion. Classical diffusion, as articulated by Fick's laws, assumes that the mean squared displacement (MSD) of particles grows linearly with time, a behavior typically observed in homogeneous and isotropic media. However, numerous experimental and theoretical studies have revealed that many systems exhibit non-linear diffusion behavior, which diverges from classical predictions. This phenomenon is termed anomalous diffusion.

Anomalous diffusion is characterized by a power-law relationship between the MSD and time:

$$\langle x^2(t)\rangle \sim t^\alpha \qquad (1)$$

where $0 < \alpha \neq 1$ (Metzler & Klafter, 2000). The parameter α serves as a crucial descriptor of the diffusion process, categorizing it into three distinct regimes: normal diffusion $(\alpha = 1)$, subdiffusion $(0 < \alpha < 1)$, and superdiffusion $(\alpha > 1)$.

The significance of anomalous diffusion extends beyond theoretical interest; it has profound implications across various scientific disciplines. In biological systems, for instance, the movement of proteins and other biomolecules often exhibits subdiffusive characteristics due to crowding effects and the complex internal structures of cells (Weiss et al., 2004). In materials science, anomalous diffusion plays a vital role in understanding transport phenomena in porous materials, batteries, and catalysts, where diffusion

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pathways are often obstructed or altered by the medium's heterogeneity (Deng et al., 2018).

The mechanisms driving anomalous diffusion are diverse and can include factors such as heterogeneous environments, trapping effects, and long-range correlations. For example, in crowded cellular environments, the presence of organelles and macromolecules can significantly hinder the movement of diffusing particles, leading to subdiffusive behavior (Ghosh et al., 2020). Conversely, in systems exhibiting superdiffusion, particles may exhibit correlated movements that result in faster-than-expected spreading (Ben-Avraham & Havlin, 2000).

Mathematical modeling of anomalous diffusion has evolved significantly, with approaches such as fractional calculus, Levy flights, and continuous time random walks (CTRW) providing frameworks to capture the complexities of these processes. These models not only enhance our understanding of the underlying physics but also facilitate the prediction of diffusion behaviors in various applications. Recently, attempts have been made to use anomalous diffusion in finance and economics. This paper uses the fractional diffusion equation for the first time to calculate a European call option with an application.

Given the multifaceted nature of anomalous diffusion and its relevance across disciplines, this review aims to provide a comprehensive synthesis of the current literature. By exploring the definitions, mechanisms, mathematical models, experimental observations, and practical applications of anomalous diffusion, we aim to highlight its importance and the ongoing challenges in the field.

2. Methodology

Truncated levy flight are stochastic processes which display a crossover from a heavy-tailed levy behavior to a faster decaying probability distribution function (pdf). In this section, we introduce a fractional diffusion equation, whose solution defines a process in which a levy flight of exponent α is truncated by a power-law of exponent $5-\alpha$. A closed form for the characteristic function of the process is derived. In financial applications are discussed (Ghosh et al., 2020). The equation we propose for truncated levy flights with the power-law cutoff has following form (Ben-Avraham & Havlin, 2000).

$$\left(1-K_\alpha \frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}}\right)\frac{\partial P(x,t)}{\partial t} = C\,\frac{\partial^2 P(x,t)}{\partial x^2} \qquad (2)$$

Where C is the diffusion coefficient governing the long-time asymptotic behavior, and the scale factor $K_\alpha = {C}/{k}$ is a coefficient governing the intermediate time levy- like one. The fractional diffusion equation is particularly useful in modeling processes that exhibit anomalous diffusion, such as those described by levy process. In the context of financial modeling, the probability density function $P(x,t)$ for a power-law- truncated levy process.

The equation is given as

$$p(x,t) = \frac{K_\alpha^2\, k\, t\, \Gamma(5-\alpha)\sin(\pi\alpha/2)}{\pi |x|^{5-\alpha}} \qquad (3)$$

Where

K_α is a normalization constant that depends on the parameter α is parameter that typically lies in the range (0,2) and indicates the degree of anomalous diffusion. Γ is the gamma function, which generalizes the factorial function. k is a constant related to the process, and $|x|^{5-\alpha}$indicates the power-law behavior of the process

To analyze the distribution further, we can compute its moments. The first moment (mean) given the symmetry of the PDF around zero for $\alpha < 5$, the first moment will be zero $\mu_1(t)=0$ (Scher & Lax, 1973), the second moment $\mu_2(t)$ is given by

$$\mu_2 = \int_{-\infty}^{\infty} x^2\; P(x,t)dx$$

$$= 2\int_0^\infty x^2\, \frac{K_\alpha^2\, k\, t\, \Gamma(5-\alpha)\, sin(\pi\alpha/2)}{\pi |x|^{5-\alpha}}\; dx$$

$$= 2\,\frac{K_\alpha^2\, k\, t\, \Gamma(5-\alpha)\, sin(\pi\alpha/2)}{\pi}\int_0^\infty x^2\,\frac{1}{|x|^{5-\alpha}}\; dx$$

$$= \; 2\,\frac{K_\alpha^2\, k\, t\, \Gamma(5-\alpha)\, sin(\pi\alpha/2)}{\pi}\int_0^\infty x^{\alpha-3}\;\; dx$$

The integral $\int_0^\infty x^{\alpha-3}\;\; dx$ converges for $\alpha < 2$ and diverges otherwise. If it converges, we can evaluate it using the gamma function. $\int_0^\infty x^{\alpha-3}\ \ dx = \Gamma(\alpha-2)$ for $\alpha < 2$. Thus, the second moment becomes

$$\mu_2 = 2\ \frac{K_\alpha^2\ k\ t\ \Gamma(5-\alpha)\ sin(\pi\alpha/2)}{\pi}\Gamma(\alpha-2)$$

The variance is the second central moment, defined as: $\sigma^2(t)\ =\ E[(X\ -\ \mu)^2]$. Since we stated that the mean μ_1 is zero for $\alpha\ <\ 5$, the variance simplifies to: $\sigma^2(t) = \mu_2$

These moments provide insights into the behavior of the fractional diffusion equation driven by a power-law truncated levy process.

The fractional diffusion equation can effectively model the distribution of asset returns with jumps by incorporating the characteristics of both fractional calculus and levy processes, the fractional diffusion equation is an extension of the classical diffusion equation. It accounts for anomalous diffusion, which is characterized by non-Gaussian behavior and heavy tails in the distribution of returns. This is particularly relevant in finance, where asset returns often exhibit.

- Leverage effects: Negative returns may lead to increased volatility
- Fat tails: Extreme events (jumps) are more common than predicted by normal distributions.

we need estimate the diffusion coefficient C from the given fractional diffusion equation, we can derive relationships on the characteristic of the probability density function P(x,t) and its implication for diffusion processes.

Starting with the equation (3) this form suggests that as $|x|$ increases, the density function $P(x,t)$ behaves like:

$$P(x,t){\sim}\ \frac{1}{|x|^{5-\alpha}}$$

This indicates that the distribution has a heavy tail, which is characteristic of anomalous diffusion processes, to relate this diffusion, we can calculate the moments of the distribution. The second moment is particularly important for estimating the diffusion coefficient.

For a diffusion process, the second moment grows linearly with time(Mandelbrot, B. B., & Van Ness, J. W. (1968)) :

$$E[\,X^2]{\sim}\ 2Ct \qquad (4)$$

equation two expressions give:

$$2Ct = 2\ \frac{K_\alpha^2\ k\ t\ \Gamma(5-\alpha)\ sin(\pi\alpha/2)}{\pi\ \ (2-\ \alpha)}$$

We obtain

$$C = \frac{K_\alpha^2\ k\ \ \Gamma(5-\alpha)\ sin(\pi\alpha/2)}{\pi\ \ (2-\ \alpha)}$$

Now, we will determine the value of K_α^2, to find K_α we need to satisfy the normalization condition $\int_{-\infty}^{\infty} P(x,t)\,dx = 1$

Assuming $k = 1$ and $t = 1$ for simplicity, we have.

$$\int_{-\infty}^{\infty} P(x,t)\,dx = \frac{K_\alpha^2\ k\ t\ \Gamma(5-\alpha)\ sin(\pi\alpha/2)}{\pi}\int_{-\infty}^{\infty}\frac{1}{|x|^{5-\alpha}}\ \ dx$$

the integral $\int_{-\infty}^{\infty}\frac{1}{|x|^{5-\alpha}}\ \ dx$ can be evaluated as follows:

1- For $5-\alpha < 1\ (1.e., \alpha > 4)$; the integral diverges

2- For $5-\alpha = 1\ (1.e., \alpha = 4)$; the integral diverges logarithmically

3- For $5-\alpha > 1\ (1.e., \alpha < 4)$;the integral converges.

for convergence, we have:

$$\int_{-\infty}^{\infty}|x|^{-(5-\alpha)}\,dx = 2\int_0^{\infty} x^{-(5-\alpha)}\,dx$$

$$= 2 \left. \frac{x^{-(4-\alpha)}}{-(4-\alpha)} \right|_0^\infty$$

This integral converges if $4 - \alpha > 0$ $(1. e., \alpha < 4)$ and the result is

$$\int_0^\infty x^{-(5-\alpha)} \, dx = \frac{1}{4-\alpha}$$

Thus

$$\int_{-\infty}^\infty |x|^{-(5-\alpha)} \, dx = \frac{2}{4-\alpha}$$

Now, substituting back into the normalization condition

$$\frac{K_\alpha^2 \ \Gamma(5-\alpha) \, sin(\pi\alpha/2)}{\pi} \frac{2}{4-\alpha} = 1$$

$$K_\alpha^2 = \frac{\pi \, (4-\alpha)}{2 \, \Gamma(5-\alpha) \sin\left(\frac{\pi\alpha}{2}\right)}$$

Taking the square root

$$K_\alpha = \sqrt{\frac{\pi \, (4-\alpha)}{2 \, \Gamma(5-\alpha) \, sin\left(\frac{\pi\alpha}{2}\right)}} \qquad (5)$$

This formula is valid for $\alpha < 4$ to ensure the integral converges.

3. Results and discussion

The result (44.35749) represents the estimated price of a call option based on the power-law truncated Levy process simulation. Here we will interpret this value :

This value indicates that, under the specified parameters (initial asset price, strike price, time to maturity, number of simulations, and the characteristics of the Levy process), the average price you would pay for the call option is approximately (44.36). In a financial context, a call option gives the holder the right (but not the obligation) to buy an underlying asset at a predetermined strike price (in this case, $K = 105$) before or at expiration (in this case, $T = 1$ year. The calculated option price reflects the market's expectation of the potential profit from holding this option. A higher option price suggests a greater probability that the asset price will exceed the strike price at maturity, thus making the option valuable. This result can also be used to assess risk. If the option price is significantly lower than expected future asset prices, it may indicate a potentially undervalued opportunity or a market expectation of less volatility. Investors can compare this option price to other options with different parameters or to historical data to gauge whether it is a good investment. In summary, (44.35749) is the estimated fair market price for the call option under the modeled conditions, providing insight into the potential profitability and risk associated with this financial instrument.

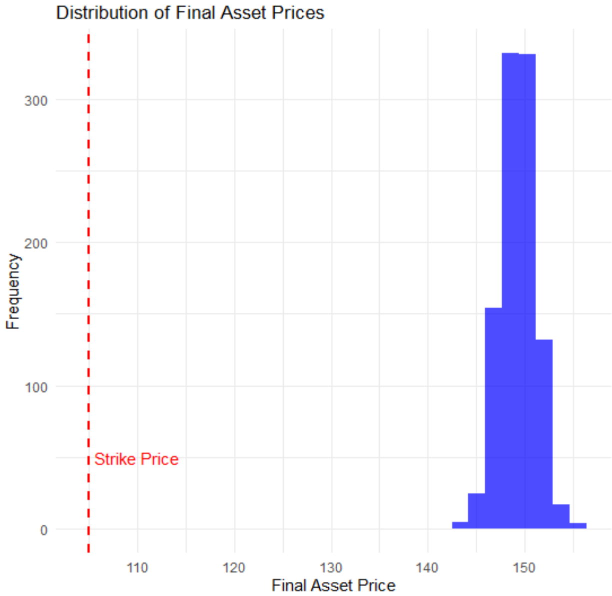

Figure 1. : distribution of Final Asset Prices by using the R program

4. Conclusions

This brief paper deals with a fundamental analysis of the abnormal diffusion process by modeling the fractional diffusion equation of the truncated Levy process and its use in financial applications, which is one of the first attempts in this field. Using simulation and the R programming language to calculate the European call option as a model, the results showed that the use of the fractional diffusion equation in financial modeling allows for a more accurate representation of market behavior, especially in capturing the complexities of asset price movements affected by rare but important events. This approach can be particularly useful in developing robust trading strategies and risk management frameworks. We view the proposed stochastic technique as a promising model for the complex phenomenon of abnormal diffusion. Future research will focus on studying diffusion processes systematically in finance.

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