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A Survival Function of Proposed Versions of Three-Parameter Lindley Distribution (ThLD)

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ABSTRACT

The research aims to study one of the most important distributions used in estimating the survival function, the Lindley distribution, which is considered a mix of two continuous distributions. The Lindley distribution has a great ability to represent life and survival systems. We will study the Lindley distribution with three parameters because of its high flexibility in modelling life data. We proposed three versions of the three parameters of the Lindley distribution by making changes to the mixing weights and mixing parameters that are used in building the Lindley distribution function; we will denote these three proposed versions, respectively, as ThLD1st, ThLD2nd, and ThLD3rd. We will compute and prove some statistical properties for these three versions. We have used two methods for estimating the survival functions: the moments method and maximum likelihood estimation. We have compared the estimates of the survival functions using some criteria of accuracy (IMSE, -2 lnL, AIC, CAIC, and BIC). The simulation results have shown that (ThLD3rd) was superior at all sample sizes, followed by (ThLD2nd), (ThLD1st), and finally (ThLD). The moments method was superior to the sample size (10), and the maximum likelihood method was superior to the sample size (50, 100). The real data in the application also showed that the survival function estimate for ThLD3rd is the best one.

1. Introduction

Survival analysis is a statistical field that focuses on evaluating data in which the variable of interest is the time it takes for a biological organism to experience an event or for electrical equipment to fail. Units such as days, weeks, years, and so on can quantify the duration. Various continuous distributions, including exponential, gamma, Weibull, and log-normal distributions, are often used in statistical literature to represent lifespan data.

Recently, the one-parameter Lindley distribution has received great attention from academics in the fields of life-time function distribution and survival systems. Scholars have noted the significant performance of this distribution in numerous applications. Lindley introduced this distribution in 1958 (Lindley, 1958), when he used the distribution in the framework of Bayesian estimations in credit statistics. (Zakerzadeh& Dolati, 2009) proposed the generalized Lindley distribution with three parameters, which is a mixture of the gamma distribution (σ,δ) and the second gamma distribution $(\sigma+1,\delta)$, and estimated the model parameters and the survival function using the maximum likelihood method. (Ghitany et al., 2013) proposed a power Lindley distribution and conducted relevant statistical analyses. (Abd El-Monsef et al., 2014) propose a weighted Lindley distribution (3-WLD) with three parameters and use the

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MLH and interval estimation to estimate the parameters. (Shanker et al., 2017) proposed a weighted Lindley distribution with three parameters and discussed its applications in modeling survival time. The adequacy of TPWLD has been examined using survival time data from a group of patients with head and neck cancer. (Ekhosuehi & Opone, 2018) suggested a three parameter generalized Lindley distribution: properties with applications. (Peter et al., 2019) introduced the A Gompertz-Lindley Distribution (Gompertz-G) with three- parameters. (Karma, 2021) conducted a study on estimating the Bayesian survival function of the three-parameter Lindley distribution and its practical application. The study found that using the Lindley distribution function with three parameters yielded more accurate findings compared to using the Lindley function with two parameters, as well as the estimating techniques employed in practical applications. (Mathil & Raoudha, 2023) studied the three-parameter characteristics of the Lindley distribution. They used a simulation experiment and showed the consistency of the three estimators and the reduction in mean squared errors (MSEs) when the sample size increased.

In this paper, we propose three versions of the three-parameter Lindley distribution (ThLD) by changing the mixing weights in the mixing equation as well as the mixing parameters that were used in building the distribution function. We discuss many essential mathematical properties of the three versions and estimate the parameters and survival functions using two methods of estimation, the moments and MLE, and compare the survival functions using some criteria of accuracy (IMSE, -2 Lnl, AIC, CAIC, and BIC). We used actual data from AL-Hussain Teaching Hospital in Nasiriya City to estimate the survival time of patients diagnosed with COVID-19 using ThLD, ThLD1st, ThLD2nd, and ThLD3rd. We ran simulations to estimate the distributions' parameters and compare them, as well as the estimation methods used. Finally, the conclusions we reached and the sources adopted in the study.

2. The Three Parameters Lindley Distribution (ThLD)

The probability density function of the three-parameter Lindley distribution (ThLD), as proposed by Shanker et al. (2017)[9], may be represented as a combination of two distributions: an exponential distribution with a parameter (σ) and a gamma distribution with two parameters (2,σ). The mixture is determined by a certain mixing percentage:

$$P = \frac{\sigma\delta}{\sigma\delta+\theta} \quad \ldots\ldots (1)$$

By using the mixing formula: $f(t;\sigma,\delta,\theta) = Pf1(t) + (1-P)\, f2(t)$ ……. (2)

$$f(t;\sigma,\delta,\theta) = \left(\frac{\sigma^2}{\sigma\delta+\theta}\right)(\delta+\theta t\,)e^{-\,\sigma t}\ ;\ t,\sigma,\delta > 0, \sigma\delta+\theta\ > 0 \quad \ldots\ldots (3)$$

The associated cumulative distribution function (C.D.F.) is:

$$\mathrm{F}(\mathrm{t};\ \sigma,\delta,\theta) = 1 - \left[\frac{\sigma\delta+\theta+\eta\theta t}{(\sigma\delta+\theta)}\right]e^{-\eta t}\ ;\ \mathrm{t}, \sigma,\delta > 0\ ,\ \ \sigma\delta+\theta > 0 \quad \ldots\ldots (4)$$

The survival (Reliability) function S(t) is:

$$\mathrm{S}(\mathrm{t};\ \sigma,\delta,\theta) = \left[\frac{\sigma\delta+\theta+\eta\theta t}{(\sigma\delta+\theta)}\right]e^{-\eta t}\ ;\ \mathrm{t}, \sigma,\delta > 0\ ,\ \ \sigma\delta+\theta > 0 \quad \ldots\ldots (5)$$

The hazard function is as follows:

$$\mathrm{h}(\mathrm{t};\ \sigma,\delta,\theta)\ = \frac{\sigma^2(\delta+\theta t)}{\sigma\delta+\theta+\sigma\theta t}\ \ ;\ \mathrm{t},\sigma,\theta\ > 0\ ,\ \ \sigma\delta+\theta > 0 \quad \ldots (6)$$

The MTTF for (ThLD) is: MTTF =

$$\int_0^\infty \mathrm{S}(\mathrm{t};\ \sigma,\delta,\theta)\,dt = \frac{\sigma\delta+2\theta}{\sigma(\sigma\delta+\theta)} \quad \ldots\ldots (7)$$

$$The\ variance:\ var(t) = \frac{\sigma^2\delta^2+4\,\sigma\delta\theta+2\theta^2}{\sigma^2(\,\sigma\delta+\theta)^2} \quad (8)$$

3. Propose Three Parameters Lindley Distribution 1st version (ThLD1st)

We propose the 1st version three parameters Lindley distribution **(ThLD1st)** throws using the following assumptions:

i. Assuming the exponential distribution as follows: f_1 (t, σ) =$\sigma\ e^{-\sigma t}$, t > 0 ……. (9)

And the gamma distribution as follows: $f_2(t,2,\sigma) = \sigma^2 t\ e^{-\sigma t}$, t > 0 ……. (10)

ii. Define the mixing proportion as follows:

$$P = \frac{\sigma\delta\theta}{\sigma\delta\theta+1} \quad \ldots\ldots. (11)$$

iii. By using the mixing equation in (2)Then we can get the **(ThLD1st)** as follows:

$$f(t,\sigma,\delta,\theta) = \frac{\sigma^2}{\sigma\delta\theta+1}(\delta\theta+t)\, e^{-\sigma t} \quad ; \quad t,\sigma,\delta,\theta > 0 \quad \ldots\ldots. (12)$$

δ,θ : Shape parameters, : Rate parameter, where, $\delta,\theta > 0$.

We can prove that:

$$\int_0^\infty \frac{\sigma^2}{\sigma\delta\theta+1}(\delta\theta+t)\, e^{-\sigma t}\, dt = 1$$

so the **(ThLD1st)** is a p.d.f

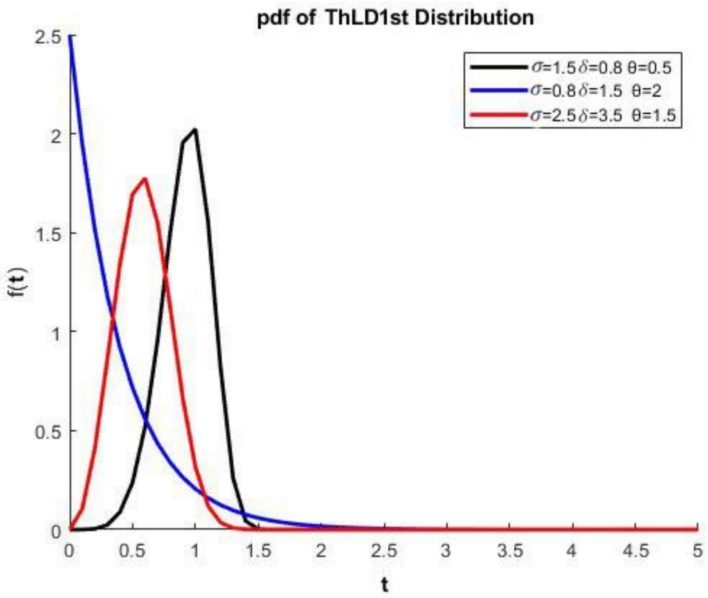


Figure 1. Plot for probability density function of (ThLD1st)

The corresponding cumulative distribution function (c.d.f.) is defined as follows:

$$F_{(t)} = 1 - \left[\frac{\sigma\delta\theta+\sigma t+1}{\sigma\delta\theta+1}\right] e^{-\sigma t} \quad \ldots\ldots. (13)$$

The Survival function S(t) is given by:

$$S(t,\sigma,\delta,\theta) = P(T > t) = \left[\frac{\sigma\delta\theta+\sigma t+1}{\sigma\delta\theta+1}\right] e^{-\sigma t};\ t,\sigma,\theta > 0,\ \sigma\delta+\theta > 0 \quad \ldots\ldots. (14)$$

And the hazard function can be shown as follows:

$$h(t,\sigma,\delta,\theta) = \frac{\sigma^2(\delta\theta+t)}{\sigma\delta\theta+\sigma t+1} \quad ; t,\sigma,\delta,\theta > 0 \quad \ldots\ldots. (15)$$

The r^{th} non-central moments about origin $\mu_r' = E(t)^r$ are:

$$\mu_r' = \int_0^\infty t^r \frac{\sigma^2}{\sigma\delta\theta+1}(\delta\theta+t)\, e^{-\sigma t}\, dt \Rightarrow \mu_r' = \frac{r!(\sigma\delta\theta+r+1)}{\sigma^r(\sigma\delta\theta+1)} \quad ; r = 1,2,\ldots \quad \ldots\ldots. (16)$$

Therefore, the first four non-central moments about origin are define as follows:

$$\mu_1' = \frac{\sigma\delta\theta+2}{\sigma(\sigma\delta\theta+1)},\ \mu_2' = \frac{2\sigma\delta\theta+6}{\sigma^2(\sigma\delta\theta+1)},\ \mu_3' = \frac{6\sigma\delta\theta+24}{\sigma^3(\sigma\delta\theta+1)},\ \mu_4' = \frac{24\sigma\delta\theta+120}{\sigma^4(\sigma\delta\theta+1)}$$

The central moments about the mean μ_r are:

$$\mu_r = E(t-\mu)^r = \int_0^\infty (t-\mu)^r \frac{\sigma^2}{\sigma\delta\theta+1}(\delta\theta+t)\, e^{-\sigma t}\, dt \quad ; r = 1,2,\ldots$$

$$\therefore E(t-\mu)^r = \frac{1}{\sigma\delta\theta+1}\left[\delta\theta \sum_{i=0}^{r} \binom{r}{i}(-\mu)^i \mu_{r-i}' + \frac{1}{\sigma^2}\sum_{i=0}^{r}\binom{r}{i}(-\mu)^i\, \mu_{r-i+1}'\right];\ r = 1,2,\ldots \quad \ldots\ldots. (17)$$

The three most common measures of tendency are the mean, median and mode can be obtained as follows:

$$mean = \frac{\sigma^2\delta\theta+2\sigma}{\sigma\delta\theta+1} \quad \ldots\ldots. \ (18)$$

$$Median = \int_0^{t_{med}} \frac{\sigma^2}{\sigma\delta\theta+1}(\delta\theta+t)\, e^{-\sigma t}dt = 0.5$$

$$\Rightarrow 1-\left[\frac{\sigma\delta\theta+\sigma t+1}{\sigma\delta\theta+1}\right] e^{-\sigma t_{med}} = 0.5$$

$$t_{median} = \left[\frac{(\sigma\delta\theta)\ln(0.5)}{\sigma(\sigma\delta\theta)+\sigma t+1\,)}\right] \quad \ldots\ldots. \ (19)$$

$$f(t) = \frac{\sigma^2}{\sigma\delta\theta+1}(\delta\theta+t)\, e^{-\sigma t} \quad ; \quad f'(t) = \frac{\sigma^2}{\sigma\delta\theta+1}\left[-\sigma(\delta\theta+t)e^{-\sigma t}+e^{-\sigma t}\,\right] = 0$$

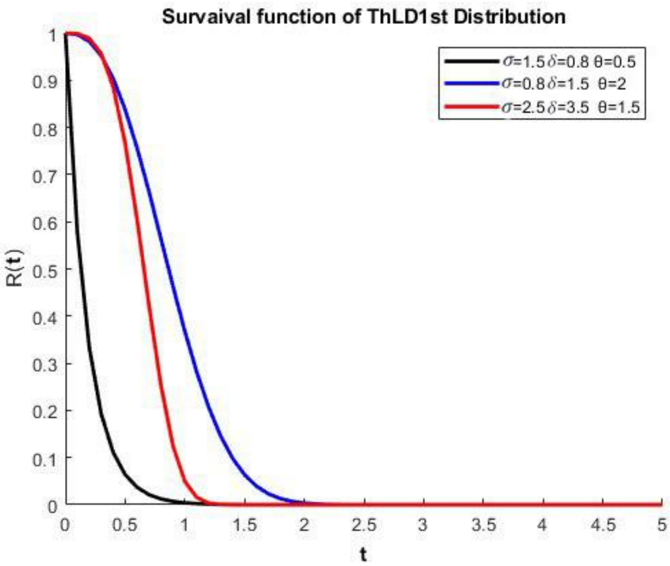
$$\therefore Mo = \begin{cases} \frac{1-\sigma\delta\theta}{\sigma} & , \ |\,\sigma\delta\theta| < 1 \\ zero, & otherwise \end{cases} \quad \ldots\ldots. \ (20)$$

$$The\ variance:\ var(t) = \frac{\sigma^2\delta^2\theta^2+4\sigma\delta\theta+2}{\sigma^2(\sigma\delta\theta+1)^2} \quad \ldots\ldots. \ (21)$$

$$The\ standard\ deviation:\ Std = \sqrt{\frac{\sigma^2\delta^2\theta^2+4\sigma\delta\theta+2}{\sigma^2(\sigma\delta\theta+1)^2}} \quad \ldots\ldots. \ (22)$$

The coefficient of variation: $C.V = \sqrt{\frac{\sigma^2\delta^2\theta^2+4\sigma\delta\theta+2}{(\sigma\delta\theta+2)^2}}$ ……. (23)

To estimate the parameters of (ThLD1st) and the survival function, we will use two methods of estimation: the moments method and the maximum likelihood method, as follows:



Figur.2. plot for Survival function of (ThLD1st)

3.1. The Method of Moments (MOM) for (ThLD1st)

$$\frac{\sigma\delta\theta+2}{\sigma(\sigma\delta\theta+1)} = \frac{1}{n}\sum_{i=1}^{n} t_i \quad \ldots\ldots. \ (24)$$

$$\frac{2\sigma\delta\theta+6}{\sigma^2(\sigma\delta\theta+1)} = \frac{1}{n-1}\sum_{i=1}^{n}(t_i-\bar{t}\,)^2 \quad \ldots\ldots. \ (25)$$

$$\frac{6\sigma\delta\theta+24}{\sigma^3(\sigma\delta\theta+1)} = \frac{1}{(n-1)(n-2)\,S^3}\sum_{i=1}^{n}(t_i-\bar{t}\,)^3 \quad \ldots\ldots. \ (26)$$

It can be seen that equations (24), (25), and (26) are non-linear and cannot be solved directly. So we will use the Lindley approximation method to solve equations, and then use the estimated parameters to estimate ThLD1st's survival function as follows:

$$\hat{S}_{mom}(t_i) = \left[\frac{\hat{\sigma}_{mom}\,\hat{\delta}_{mom}\,\hat{\theta}_{mom}+\hat{\sigma}_{mom}t_i+1}{(\hat{\sigma}_{mom}\,\hat{\delta}_{mom}\,\hat{\theta}_{mom}+1)}\right] e^{-\hat{\sigma}_{mom}t_i} \quad \ldots\ldots. \ (27)$$

3.2 The Maximum Likelihood Estimation (M.L.E) for (ThLD1st)

Let $t_1, t_2, t_3, \ldots. t_n$ constitute a random sample of size n selected from (ThLD1st), and then the likelihood function is defined as:

$$L = \left(\frac{\sigma^2}{\sigma\delta\theta+1}\right)^n \prod_{i=1}^{n}(\delta\theta + t)\ e^{-n\sigma\bar{t}}$$

$$\frac{\partial LnL}{\partial\hat{\sigma}} = \frac{2n}{\hat{\sigma}} - \frac{n\widehat{\delta\theta}}{\left(\hat{\sigma}\widehat{\delta\theta}+1\right)} - n\bar{t} = 0$$

……. (28)

$$\frac{\partial LnL}{\partial\hat{\delta}} = \sum_{i=1}^{n}\left[\frac{\hat{\theta}}{\widehat{\delta\theta}+ti}\right] - \frac{n\,\hat{\sigma}\hat{\theta}}{\hat{\sigma}\widehat{\delta\theta}+1} = 0$$

……. (29)

$$\frac{\partial LnL}{\partial\hat{\theta}} = \sum_{i=1}^{n}\left[\frac{\hat{\delta}}{\widehat{\delta\theta}+ti}\right] - \frac{n}{\hat{\sigma}\widehat{\delta\theta}+1} = 0$$

……. (30)

The first derivatives are non-linear equations that do not seem to have a straightforward solution. So we will solve the equations using the Fisher's scoring method to get (MLEs) for $\hat{\sigma}$, $\hat{\delta}$ and $\hat{\theta}$ of (ThLD1st) as follows:

$$\begin{bmatrix} \frac{\partial^2 LnL}{\partial\sigma^2} & \frac{\partial^2 LnL}{\partial\sigma\partial\delta} & \frac{\partial^2 LnL}{\partial\sigma\partial\theta} \\ \frac{\partial^2 LnL}{\partial\delta\partial\sigma} & \frac{\partial^2 LnL}{\partial\delta^2} & \frac{\partial^2 LnL}{\partial\delta\partial\theta} \\ \frac{\partial^2 LnL}{\partial\theta\partial\sigma} & \frac{\partial^2 LnL}{\partial\theta\partial\delta} & \frac{\partial^2 LnL}{\partial\theta^2} \end{bmatrix}_{\substack{\hat{\sigma}=\sigma_0\\ \hat{\delta}=\delta_0\\ \hat{\theta}=\theta_0}} \begin{bmatrix} \hat{\sigma}-\sigma_0 \\ \hat{\delta}-\delta_0 \\ \hat{\theta}-\theta_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial LnL}{\partial\sigma} \\ \frac{\partial LnL}{\partial\delta} \\ \frac{\partial LnL}{\partial\theta} \end{bmatrix}_{\substack{\hat{\sigma}=\sigma_0\\ \hat{\delta}=\delta_0\\ \hat{\theta}=\theta_0}}$$

$$\frac{\partial^2 LnL}{\partial\sigma^2} = \frac{-2n}{\sigma^2} + \frac{n\delta^2\theta^2}{(\sigma\delta\theta+1)^2} \quad , \quad \frac{\partial^2 LnL}{\partial\sigma\partial\delta} = \frac{-n\theta}{(\sigma\delta\theta+1)^2} = \frac{\partial^2 LnL}{\partial\delta\partial\sigma}$$

$$\frac{\partial^2 LnL}{\partial\sigma\partial\theta} = \frac{n\delta}{(\sigma\delta\theta+1)^2} = \frac{\partial^2 LnL}{\partial\theta\partial\sigma} \quad , \quad \frac{\partial^2 LnL}{\partial\delta^2} = \sum_{i=1}^{n}\frac{-\theta^2}{(\delta\theta+ti)^2} - \frac{n\,\sigma^2\theta^2}{(\sigma\delta+\theta)^2}$$

$$\frac{\partial^2 LnL}{\partial\delta\partial\theta} = -\sum_{i=1}^{n}\frac{ti}{(\delta\theta+ti)^2} + \frac{n\sigma}{(\sigma\delta\theta+1)^2} = \frac{\partial^2 LnL}{\partial\theta\partial\delta},$$

$$\frac{\partial^2 LnL}{\partial\theta^2} = -\sum_{i=1}^{n}\frac{\delta^2}{(\delta\theta+ti)^2} + \frac{n\sigma^2\delta^2}{(\sigma\delta\theta+1)^2}$$

The starting values σ_0, δ_0 and θ_0, respectively. Solving these equations iteratively, we get values of $\hat{\sigma}$, $\hat{\delta}$, and $\hat{\theta}$. To get the estimate of survival function of ThLD1st we substitute the estimate parameters as follows:

$$\hat{S}_{\text{ThLD1st}ml}(t_i) = \left[\frac{\hat{\sigma}_{ml}\,\hat{\delta}_{ml}\ \hat{\theta}_{ml}+\hat{\sigma}_{ml}t_i+1}{(\hat{\sigma}_{ml}\,\hat{\delta}_{ml}\ \hat{\theta}_{ml}+1)}\right] e^{-\hat{\sigma}_{ml}t_i}$$

……. (31)

4 Propose Three Parameters Lindley Distribution 2nd version(ThPL2nd)

We propose the 2nd version three parameters Lindley distribution **(ThLD2nd)** throws using the following assumptions:

i. Defined the exponential distribution in equation 9. And the gamma distribution in equation10.
ii. By using the mixing proportion in equation 1
iii. Define the mixing equation as follows:

$$f(t;\sigma,\delta,\theta) = (1-P)f_1(t) + P\,f_2(t) \quad …….$$

(32)

Then we can get the **(ThLD2nd)** as follows:

$$\therefore f(t,\sigma,\delta,\theta) = \frac{\sigma}{\sigma\delta+\theta}(\theta + \sigma^2\delta t)\,e^{-\sigma t} \quad ;$$

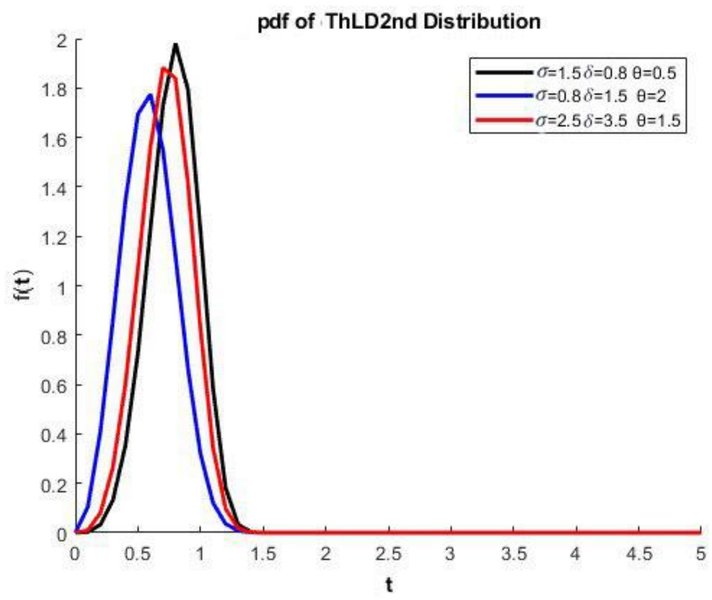
t, $\sigma,\delta,\theta > 0$ ……. (33)

δ,θ : Shape parameters, : Rate parameter,

where, $\delta,\theta > 0$.

We can prove that: $\int_0^\infty \frac{\sigma}{\sigma\delta+\theta}(\theta + \sigma^2\delta t)\,e^{-\sigma t}\ dt = 1$

So the **(ThLD2nd)** is a p.d.f



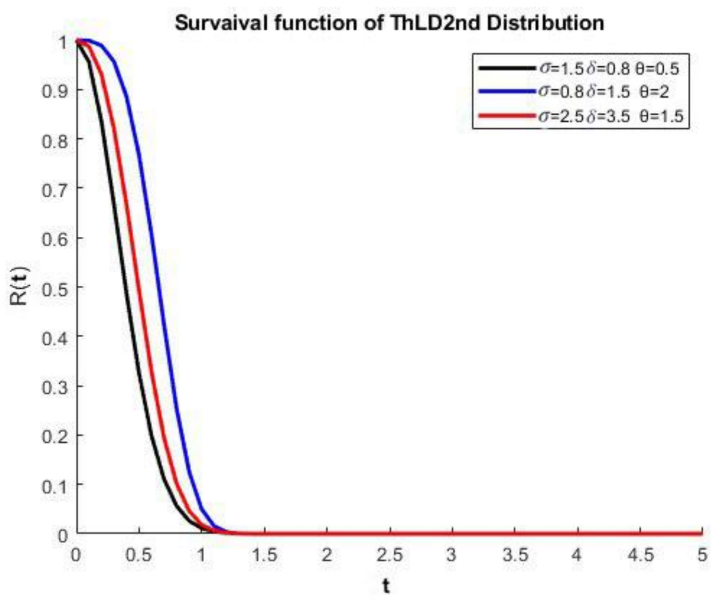
Figur.3. Plot for probability density function of (ThLD2nd)

The corresponding cumulative distribution function (c.d.f.) is defined as follows:

$$F_{(t)} = 1 - \left[\frac{\theta+\sigma^2\delta t+\sigma\delta}{\sigma\delta+\theta}\right] e^{-\sigma t} \qquad \ldots\ldots (34)$$

The Survival function S(t) is given by:

$$S(t,\sigma,\delta,\theta) = \left[\frac{\theta+\sigma^2\delta t+\sigma\delta}{\sigma\delta+\theta}\right] e^{-\sigma t};\ t,\sigma,\delta,\theta > 0 \qquad \ldots\ldots (35)$$



Figur.4. Plot for Survival function of (ThLD2nd)

And the hazard function can be shown as follows:

$$h(t,\sigma,\delta,\theta) = \frac{\sigma(\theta+\sigma^2\delta t)}{\theta+\sigma\delta(\sigma t+1)} \quad ;t,\sigma,\delta,\theta > 0 \qquad \ldots\ldots (36)$$

The r^{th} non-central moments about origin $\mu_r' = E(t)^r$ are:

$$\mu_r' = \int_0^\infty t^r \frac{\sigma}{\sigma\delta+\theta}(\theta+\sigma^2\delta t)\, e^{-\sigma t}\, dt \Rightarrow \mu_r' = \frac{r!\,[\theta+\sigma\delta(r+1)]}{\sigma^r(\sigma\delta+\theta)}\ ; r = 1,2,\ldots \qquad \ldots\ldots (37)$$

Therefore, the first four non-central moments about origin are defined as follows:

$\mu_1' = \frac{\theta+2\sigma\delta}{\sigma(\sigma\delta+\theta)}$, $\mu_2' = \frac{2(\theta+3\sigma\delta)}{\sigma^2(\sigma\delta+\theta)}$, $\mu_3' = \frac{6(\theta+4\sigma\delta)}{\sigma^3(\sigma\delta+\theta)}$, $\mu_4' = \frac{24(\theta+5\sigma\delta)}{\sigma^4(\sigma\delta+\theta)}$

The central moments about the mean μ_r are:

$$\mu_r = E(t-\mu)^r = \int_0^\infty (t-\mu)^r \frac{\sigma}{\sigma\delta+\theta}(\theta + \sigma^2\delta t)\, e^{-\sigma t}\, dt$$

$$= \frac{\sigma}{\sigma\delta+\theta}\left[\theta \sum_{i=0}^{r} \binom{r}{i}\left(-\frac{(\theta+2\sigma\delta)^i (r-i)!}{\sigma^{r+1}(\sigma\delta+\theta)^i}\right) + \delta \sum_{i=0}^{r} \binom{r}{i}\left(-\frac{(\theta+2\sigma\delta)^i (r-i+1)!}{\sigma(\sigma\delta+\theta)^i}\right)\right] \quad \ldots\ldots (38)$$

The three most common measures of tendency are the mean, median and mode can be obtained as follows:

$$mean = \frac{\theta+2\sigma\delta}{\sigma(\sigma\delta+\theta)} \quad \ldots\ldots (39)$$

$$Median = \int_0^{t_{med}} \frac{\sigma}{\sigma\delta+\theta}(\theta+\sigma^2\delta t)\, e^{-\sigma t} dt = 0.5 \Rightarrow\ =1 - \left[\frac{\theta+\sigma^2\delta t+\sigma\delta}{\sigma\delta+\theta}\right] e^{-\sigma t_{med}} = 0.5$$

$$t_{median} = \left[\frac{(\sigma\delta+\theta)\ln(0.5)}{\sigma(\theta+\sigma^2\delta t)+\sigma\delta}\right] \quad \ldots\ldots (40)$$

$$f(t) = \frac{\sigma}{\sigma\delta+\theta}(\theta+\sigma^2\delta t)\, e^{-\sigma t}\ ; f'(t) = \frac{\sigma}{\sigma\delta+\theta}\left[-\sigma(\theta+\sigma^2\delta t)e^{-\sigma t} + \sigma^2\delta\theta e^{-\sigma t}\right] = 0$$

$$\therefore Mo = \begin{cases} \frac{\sigma\delta-\theta}{\sigma^2\delta}, & |\sigma\delta| > \theta \\ zero, & otherwise \end{cases} \quad \ldots\ldots (41)$$

$$The\ variance:\ var(t) = \frac{2\sigma^2\delta^2+4\sigma\delta\theta+\theta^2}{\sigma^2(\sigma\delta+\theta)^2} \quad \ldots\ldots (42)$$

$$The\ standard\ deviation: Std = \sqrt{\frac{2\sigma^2\delta^2+4\sigma\delta\theta+\theta^2}{\sigma^2(\sigma\delta+\theta)^2}} \quad \ldots\ldots (43)$$

The coefficient of variation: $C.V = \sqrt{\frac{2\sigma^2\delta^2+4\sigma\delta\theta+\theta^2}{(\theta+2\sigma\delta)^2}}$ ……. (44)

To estimate the parameters of (**ThLD2nd**) and the survival function, we will use two methods of estimation: the moments method and the maximum likelihood method, as follows:

4.1 The Method of Moments (MOM) for (ThLD2nd)

$$\frac{\theta+2\sigma\delta}{\sigma(\sigma\delta+\theta)} = \frac{1}{n}\sum_{i=1}^{n} t_i \quad \ldots\ldots (45)$$

$$\frac{2(\theta+3\sigma\delta)}{\sigma^2(\sigma\delta+\theta)} = \frac{1}{n-1}\sum_{i=1}^{n}(t_i - \bar{t}\,)^2 \quad \ldots\ldots (46)$$

$$\frac{6(\theta+4\sigma\delta)}{\sigma^3(\sigma\delta+\theta)} = \frac{1}{(n-1)(n-2)\, S^3}\sum_{i=1}^{n}(t_i - \bar{t}\,)^3 \quad \ldots\ldots (47)$$

It can be seen that equations (45), (46), and (47) are non-linear and cannot be solved directly. So we will use the Lindley approximation method to solve equations, and then use the estimated parameters to estimate ThLD2nd's survival function as follows:

$$\hat{S}_{mom}(t_i) = \left[\frac{\hat{\theta}_{mom}+\widehat{\sigma^2}_{mom}\hat{\delta}_{mom}\mathrm{t} + \hat{\sigma}_{mom}\hat{\delta}_{mom}}{(\hat{\sigma}_{mom}\,\hat{\delta}_{mom} + \hat{\theta}_{mom})}\right] e^{-\hat{\sigma}_{mom} t_i} \quad \ldots\ldots (48)$$

4.2 The Maximum Likelihood Estimation(M.L.E) for (ThLD2nd)

Let $t_1, t_2, t_3, \ldots . t_n$ constitute a random sample of size n selected from (**ThLD2nd**), and then the likelihood function is defined as:

$$L = \left(\frac{\sigma}{\sigma\delta+\theta}\right)^n \prod_{i=1}^{n}(\theta+\sigma^2\delta t\,)\ e^{-n\sigma\bar{t}}$$

$$\frac{\partial LnL}{\partial\hat{\sigma}} = \frac{\mathrm{n}}{\hat{\sigma}} - \frac{\mathrm{n}\hat{\delta}}{(\hat{\sigma}\hat{\delta}+\hat{\theta})} + \sum_{i=1}^{n}\left[\frac{2\hat{\sigma}\hat{\delta}t_i}{\hat{\theta}+\sigma^2\delta ti}\right] - n\bar{t} = 0 \quad \ldots\ldots (49)$$

$$\frac{\partial LnL}{\partial\hat{\delta}} = \sum_{i=1}^{n}\left[\frac{\widehat{\sigma^2}ti}{\hat{\theta}+\widehat{\sigma^2}\hat{\delta}ti}\right] - \frac{n\,\hat{\sigma}}{\hat{\sigma}\hat{\delta}+\hat{\theta}} = 0 \quad \ldots\ldots (50)$$

$$\frac{\partial LnL}{\partial\hat{\theta}} = \sum_{i=1}^{n}\left[\frac{1}{\hat{\theta}+\sigma^2\hat{\delta}ti}\right] - \frac{n}{\hat{\sigma}\hat{\delta}+\hat{\theta}} = 0 \quad \ldots (51)$$

The first derivatives are non-linear equations that do not seem to have a straightforward solution. So we will solve the equations using the Fisher's scoring method to get (MLEs) for $\hat{\sigma}$, $\hat{\delta}$ and $\hat{\theta}$ of (**ThLD2nd**) as follows:

$$\begin{bmatrix} \frac{\partial^2 LnL}{\partial \sigma^2} & \frac{\partial^2 LnL}{\partial \sigma \partial \delta} & \frac{\partial^2 LnL}{\partial \sigma \partial \theta} \\ \frac{\partial^2 LnL}{\partial \delta \partial \sigma} & \frac{\partial^2 LnL}{\partial \delta^2} & \frac{\partial^2 LnL}{\partial \delta \partial \theta} \\ \frac{\partial^2 LnL}{\partial \theta \partial \sigma} & \frac{\partial^2 LnL}{\partial \theta \partial \delta} & \frac{\partial^2 LnL}{\partial \theta^2} \end{bmatrix}_{\substack{\hat{\sigma}=\sigma_0 \\ \hat{\delta}=\delta_0 \\ \hat{\theta}=\theta_0}} \begin{bmatrix} \hat{\sigma}-\sigma_0 \\ \hat{\delta}-\delta_0 \\ \hat{\theta}-\theta_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial LnL}{\partial \sigma} \\ \frac{\partial LnL}{\partial \delta} \\ \frac{\partial LnL}{\partial \theta} \end{bmatrix}_{\substack{\hat{\sigma}=\sigma_0 \\ \hat{\delta}=\delta_0 \\ \hat{\theta}=\theta_0}}$$

$$\frac{\partial^2 LnL}{\partial \sigma^2} = \frac{-\mathrm{n}}{\sigma^2} + \frac{\mathrm{n}\delta^2}{(\sigma\delta+\theta)^2} + \sum_{i=1}^{n} \frac{2\delta\theta ti - 2\sigma^2\delta^2\, ti^2}{(\theta+\sigma^2\delta t)^2} \quad ,$$

$$\frac{\partial^2 LnL}{\partial \sigma \partial \theta} = \frac{\mathrm{n}\delta}{(\sigma\delta+\theta)^2} - \sum_{i=1}^{n} \frac{2\sigma\delta ti}{(\theta+\sigma^2\delta t)^2} = \frac{\partial^2 LnL}{\partial \theta \partial \sigma} \quad ,$$

$$\frac{\partial^2 LnL}{\partial \delta^2} = \frac{n\,\sigma^2}{(\sigma\delta+\theta)^2} - \sum_{i=1}^{n} \frac{\sigma^4\, ti^2}{(\theta+\sigma^2\delta t)^2}$$

$$\frac{\partial^2 LnL}{\partial \delta \partial \theta} = \frac{n\,\sigma}{(\sigma\delta+\theta)^2} - \sum_{i=1}^{n} \frac{\sigma^2 ti}{(\theta+\sigma^2\delta t)^2} = \frac{\partial^2 LnL}{\partial \theta \partial \delta} \quad ,$$

$$\frac{\partial^2 LnL}{\partial \theta^2} = \frac{\mathrm{n}}{(\sigma\delta+\theta)^2} - \sum_{i=1}^{n} \frac{1}{(\theta+\sigma^2\delta t)^2}$$

The starting values σ_0, δ_0, andθ_0, respectively. Solving these equations iteratively, we get values of $\hat{\sigma}$, $\hat{\delta}$ and $\hat{\theta}$. To get an estimate of ThLD2nd's survival function, we substitute the estimated parameters as follows:

$$\hat{S}_{\mathrm{ThLD2nd}ml}(t_i) = \left[\frac{\hat{\theta}_{ml}+\widehat{\sigma^2}_{ml}\,\hat{\delta}_{ml}\mathrm{t}+\hat{\sigma}_{ml}\hat{\delta}_{ml}}{(\hat{\sigma}_{ml}\,\hat{\delta}_{ml}\,+\hat{\theta}_{ml})}\right] e^{-\hat{\sigma}_{ml}t_i} \quad \ldots\ldots. (52)$$

5 Propose Three Parameters Lindley Distribution 3rdversion (ThPL3rd)

We propose the 3rd version three parameters Lindley distribution **(ThLD3rd)** throws using the following assumptions:

i. Defined the exponential distribution in equation 9. And the gamma distribution in equation10.
ii. By using the mixing proportion in equation (11).
iii. Through the use of mixing equation in (32).

Then we can get the **(ThLD3rd)** as follows:

$$\therefore f(t,\sigma,\delta,\theta) = \frac{\sigma}{\sigma\delta\theta+1}(1+\sigma^2\delta\theta t)\, e^{-\sigma t} \quad ; \quad t,\sigma,\delta,\theta > 0 \quad \ldots\ldots. (53)$$

σ: Rate parameter

δ,θ : Shape parameters

We can prove that:

$$\int_0^\infty \frac{\sigma}{\sigma\delta\theta+1}(1+\sigma^2\delta\theta t)\, e^{-\sigma t}\; dt = 1$$

So the **(ThLD3rd)** is a p.d.f

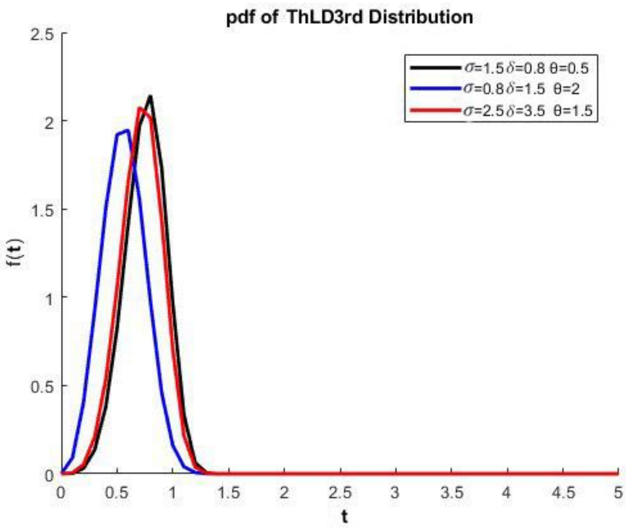


Figure 5.Plot for probability density function of (ThLD3rd)

The corresponding cumulative distribution function (c.d.f.) is defined as follows:

$$F_{(t)} = 1 - \left[\frac{1+\sigma^2\delta\theta t+\sigma\delta\theta}{\sigma\delta\theta+1}\right] e^{-\sigma t} \qquad \ldots\ldots (54)$$

The Survival function S(t) is given by:

$$S(t,\sigma,\delta,\theta) = \left[\frac{1+\sigma^2\delta\theta t+\sigma\delta\theta}{\sigma\delta\theta+1}\right] e^{-\sigma t};\ t,\sigma,\delta,\theta > 0 \qquad \ldots\ldots (55)$$

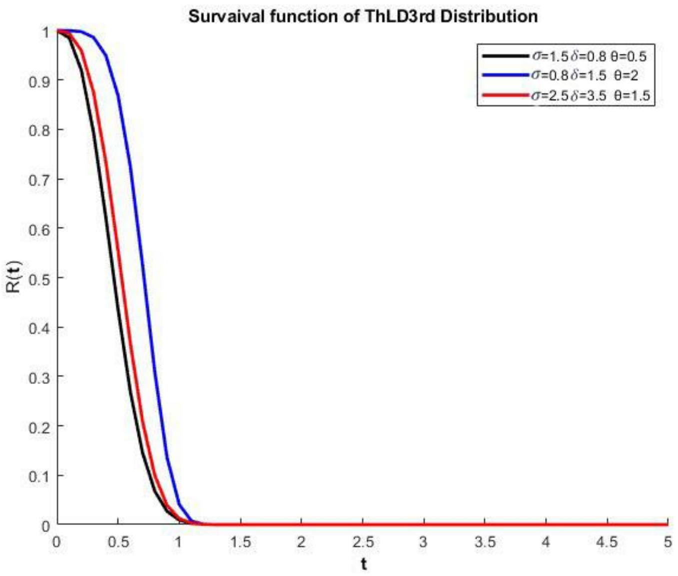


Figure 6. Plot for Survival function of (ThLD3rd)

And the hazard function can be shown as follows:

$$h(t,\sigma,\delta,\theta) = \frac{\sigma+\sigma^3\delta\theta t}{1+\sigma^2\delta\theta t+\sigma\delta\theta} \quad ; t,\sigma,\delta,\theta > 0 \qquad \ldots\ldots (56)$$

The r^{th} non-central moments about origin $\mu_r^{\prime} = E(t)^r$ are:

$$\mu_r^{\prime} = \int_0^{\infty} t^r \frac{\sigma}{\sigma\delta\theta+1}(1+\sigma^2\delta\theta t)\, e^{-\sigma t}\, dt$$

$$\mu_r^{\prime} = \frac{r![1+\sigma\delta\theta(r+1)]}{\sigma^r(\sigma\delta\theta+1)} \quad ; r = 1,2,\ldots \qquad \ldots\ldots (57)$$

Therefore, the first four non-central moments about origin are define as follows:

$$\mu_1^{\prime} = \frac{1+2\sigma\delta\theta}{\sigma(\sigma\delta\theta+1)} \quad , \quad \mu_2^{\prime} = \frac{2(1+3\sigma\delta\theta)}{\sigma^2(\sigma\delta\theta+1)}$$

$$\mu_3^{\prime} = \frac{6(1+4\sigma\delta\theta)}{\sigma^3(\sigma\delta\theta+1)} \quad , \quad \mu_4^{\prime} = \frac{24(1+5\sigma\delta\theta)}{\sigma^4(\sigma\delta\theta+1)}$$

The central moments about the mean μ_r are:

$$\mu_r = E(t-\mu)^r$$

$$= \int_0^{\infty} (t-\mu)^r \frac{\sigma}{\sigma\delta\theta+1}(1+\sigma^2\delta\theta t)\, e^{-\sigma t}\, dt$$

$$\therefore E(t-\mu)^r = \frac{\sigma}{\sigma\delta\theta+1}\left[\sum_{i=0}^{r}\binom{r}{i}\left(-\frac{1+2\sigma\delta\theta(r-i)!}{\sigma^{r-i+2}(\sigma\delta\theta+1)}\right) + \delta\theta\sum_{i=0}^{r}\binom{r}{i}\left(-\frac{1+2\sigma\delta\theta(r-i+1)!}{\sigma^{r-i+1}(\sigma\delta\theta+1)}\right)\right] \qquad \ldots\ldots (58)$$

The three most common measures of tendency are the mean, median and mode can be obtained as follows:

$$mean = \frac{1+2\sigma\delta\theta}{\sigma(\sigma\delta\theta+1)} \qquad \ldots\ldots (59)$$

$$Median = \int_0^{t_{med}} \frac{\sigma}{\sigma\delta\theta+1}(1+\sigma^2\delta\theta t)\ e^{-\sigma t}dt = 0.5 \Rightarrow = 1 - \left[\frac{\sigma\delta\theta+\sigma t+1}{\sigma\delta\theta+1}\right] e^{-\sigma t_{med}} = 0.5$$

$$t_{median} = \left[\frac{(\sigma\delta\theta+1)\ln(0.5)}{\sigma(\sigma\delta\theta+\sigma t+1)}\right] \qquad \ldots\ldots (60)$$

$$f(t) = \frac{\sigma}{\sigma\delta\theta+1}(1+\sigma^2\delta\theta t)\, e^{-\sigma t}\ ; f^{\prime}(t) = e^{-\sigma t}[-\sigma - \sigma^3\delta\theta t + \sigma^2\delta\theta] = 0$$

$$\therefore Mo = \begin{cases} \frac{\sigma\delta\theta-1}{\sigma^2\delta\theta}, |\ \sigma\delta\theta| > 1 \\ zero, \quad otherwise \end{cases} \qquad \ldots\ldots (61)$$

$$The\ variance:\ var(t) = \frac{2\sigma^2\delta^2\theta^2+4\sigma\delta\theta+1}{\sigma^2(\sigma\delta\theta+1)^2} \qquad \ldots\ldots (62)$$

$The\ standard\ deviation$: $Std = \sqrt{\frac{2\sigma^2\delta^2\theta^2+4\sigma\delta\theta+1}{\sigma^2(\sigma\delta\theta+1)^2}}$ ……. (63)

The coefficient of variation: $C.V = \sqrt{\frac{2\sigma^2\delta^2\theta^2+4\sigma\delta\theta+1}{(1+2\sigma\delta\theta)^2}}$ ……. (64)

To estimate the parameters of (ThLD1st) and the survival function, we will use two methods of estimation: the moments method and the maximum likelihood method, as follows:

5.1 The Method of Moments (MOM) for (ThLD3rd)

$$\frac{1+2\sigma\delta\theta}{\sigma(\sigma\delta\theta+1)} = \frac{1}{n}\sum_{i=1}^{n} t_i \quad ……. (65)$$

$$\frac{2(1+3\sigma\delta\theta)}{\sigma^2(\sigma\delta\theta+1)} = \frac{1}{n-1}\sum_{i=1}^{n}(t_i - \bar{t}\,)^2 \quad ……. (66)$$

$$\frac{6(1+4\sigma\delta\theta)}{\sigma^3(\sigma\delta\theta+1)} = \frac{1}{(n-1)(n-2)\,S^3}\sum_{i=1}^{n}(t_i - \bar{t}\,)^3 \quad ……. (67)$$

It can be seen that equations (65), (66), and (67) are non-linear and cannot be solved directly. So we will use the Lindley approximation method to solve equations, and then use the estimated parameters to estimate **ThLD3rd'**s survival function as follows:

$$\hat{S}_{mom}(t_i) = \left[\frac{1+\widehat{\sigma^2}_{mom}\,\hat{\delta}_{mom}\hat{\theta}_{mom}\mathrm{t} + \hat{\sigma}_{mom}\hat{\delta}_{mom}\hat{\theta}_{mom}}{(\hat{\sigma}_{mom}\,\hat{\delta}_{mom}\hat{\theta}_{mom} + 1)}\right] e^{-\hat{\sigma}_{mom}t_i} \quad ……. (68)$$

5.2 The Maximum Likelihood Estimation(M.L.E) for (ThLD3rd)

Let $t_1, t_2, t_3, \ldots . t_n$ constitute a random sample of size n selected from (**ThLD3rd**), and then the likelihood function is defined as:

$$L = \left(\frac{\sigma}{\sigma\delta\theta+1}\right)^n \prod_{i=1}^{n}(1+\sigma^2\delta\theta t\,)\ e^{-\,n\sigma\bar{t}}$$

$$\frac{\partial LnL}{\partial\hat{\sigma}} = \frac{2\mathrm{n}}{\hat{\sigma}} - \frac{-\mathrm{n}\hat{\delta}}{(\hat{\sigma}\hat{\delta}+\hat{\theta})} - n\bar{t} = 0 \quad ……. (69)$$

$$\frac{\partial LnL}{\partial\hat{\delta}} = \sum_{i=1}^{n}\left[\frac{1}{\widehat{\hat{\delta}+\hat{\theta}ti}}\right] - \frac{n\,\hat{\sigma}}{\hat{\sigma}\hat{\delta}+\hat{\theta}} = 0 \quad ……. (70)$$

$$\frac{\partial LnL}{\partial\hat{\theta}} = \sum_{i=1}^{n}\left[\frac{ti}{\widehat{\hat{\delta}+\hat{\theta}ti}}\right] - \frac{n}{\hat{\sigma}\hat{\delta}+\hat{\theta}} = 0 \quad ……. (71)$$

The first derivatives are non-linear equations that do not seem to have a straightforward solution. So we will solve the equations using the Fisher's scoring method to get (MLEs) for $\hat{\sigma}$, $\hat{\delta}$ and $\hat{\theta}$ of (**ThLD3rd**) as follows:

$$\begin{bmatrix} \frac{\partial^2 LnL}{\partial\sigma^2} & \frac{\partial^2 LnL}{\partial\sigma\partial\delta} & \frac{\partial^2 LnL}{\partial\sigma\partial\theta} \\ \frac{\partial^2 LnL}{\partial\delta\partial\sigma} & \frac{\partial^2 LnL}{\partial\delta^2} & \frac{\partial^2 LnL}{\partial\delta\partial\theta} \\ \frac{\partial^2 LnL}{\partial\theta\partial\sigma} & \frac{\partial^2 LnL}{\partial\theta\partial\delta} & \frac{\partial^2 LnL}{\partial\theta^2} \end{bmatrix}_{\substack{\hat{\sigma}=\sigma_0 \\ \hat{\delta}=\delta_0 \\ \hat{\theta}=\theta_0}} \begin{bmatrix} \hat{\sigma}-\sigma_0 \\ \hat{\delta}-\delta_0 \\ \hat{\theta}-\theta_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial LnL}{\partial\sigma} \\ \frac{\partial LnL}{\partial\delta} \\ \frac{\partial LnL}{\partial\theta} \end{bmatrix}_{\substack{\hat{\sigma}=\sigma_0 \\ \hat{\delta}=\delta_0 \\ \hat{\theta}=\theta_0}}$$

$$\frac{\partial^2 LnL}{\partial\sigma^2} = \frac{-2\mathrm{n}}{\sigma^2} + \frac{\mathrm{n}\delta^2}{(\sigma\delta+\theta)^2}$$

$$\frac{\partial^2 LnL}{\partial\sigma\partial\delta} = \frac{-\mathrm{n}\theta}{(\sigma\delta+\theta)^2} = \frac{\partial^2 LnL}{\partial\delta\partial\sigma}$$

$$\frac{\partial^2 LnL}{\partial\sigma\partial\theta} = \frac{-\mathrm{n}\delta}{(\sigma\delta+\theta)^2} = \frac{\partial^2 LnL}{\partial\theta\partial\sigma}$$

$$\frac{\partial^2 LnL}{\partial\delta^2} = \frac{-n\,\sigma^2}{(\sigma\delta+\theta)^2} - \sum_{i=1}^{n}\frac{1}{(\hat{\delta}+\hat{\theta}ti)^2}$$

$$\frac{\partial^2 LnL}{\partial\delta\partial\theta} = \frac{n\,\sigma}{(\sigma\delta+\theta)^2} - \sum_{i=1}^{n}\frac{ti}{(\hat{\delta}+\hat{\theta}ti)^2} = \frac{\partial^2 LnL}{\partial\theta\partial\delta}$$

$$\frac{\partial^2 LnL}{\partial\theta^2} = \frac{\mathrm{n}}{(\sigma\delta+\theta)^2} - \sum_{i=1}^{n}\frac{ti}{(\hat{\delta}+\hat{\theta}ti)^2}$$

The starting values σ_0, δ_0 and θ_0, respectively. Solving these equations iteratively, we get values of $\hat{\sigma}$, $\hat{\delta}$ and $\hat{\theta}$. To get an estimate of **ThLD3rd'**s survival function, we substitute the estimated parameters as follows:

$$\hat{S}_{\mathrm{ThLD3rd}ml}(t_i) = \left[\frac{1+\widehat{\sigma^2}_{ml}\,\hat{\delta}_{ml}\hat{\theta}_{ml}\mathrm{t}+\hat{\sigma}_{ml}\hat{\delta}_{ml}\hat{\theta}_{ml}}{(\hat{\sigma}_{ml}\,\hat{\delta}_{ml}\hat{\theta}_{ml} + 1)}\right] e^{-\hat{\sigma}_{ml}t_i} \quad ……. (72)$$

6 Application to covid-19 patient

The data on patients infected with coronavirus were obtained from the statistics division of the Al Hussein Educational Hospital in Nasiriyah. These data represent the number of days from hospitalization until death due to coronavirus disease for a total of 100 patients during the month of July 2020. The chi-square for goodness of fit test was used for confirmation. Using the MATLAB program, we tested the following null hypothesis, which aligns with the proposed distribution:

H_0:The data have ThLD3rd Distribution

H_1:The data don't have ThLD3rd Distribution

Table 2. Presents the results of the ThLD3rd distribution data fit test.

Distribution	χ_c^2	χ_t^2	Sig.	Decision
ThLD3rd	0.14856	123.226	0.46755	Accept the H_0

Table (2) shows that the estimated value of χ_c^2 (0.14855) is lower than the computed value of χ_t^2 (123.225), and that the value of Sig = 0.46754 above the significance threshold of 0.05. Therefore, it is not feasible to refute the null hypothesis, indicating that the observed data are distributed in accordance with the planned (ThLD3rd).

Table 3. Presents the results of comparison and accuracy tests conducted on actual data.

Distribution	Parameters estimation			-2LnL	AIC	CAIC	BIC
	$\hat{\sigma}_{ml}$	$\hat{\delta}_{ml}$	$\hat{\theta}_{ml}$				
ThLD	1.678	0.997	0.776	46.7784	46.9984	46.2484	46.2784
ThLD1st	1.678	0.967	0.766	45.7984	45.0184	45.2284	45.2684
ThLD2nd	1.654	0.899	0.673	43.7684	43.1084	43.2584	43.3784
ThLD3rd	1.635	0.989	0.711	39.7584	39.0984	39.2084	39.3884

Table (3) shows the Propose Three Parameters Lindley Distribution3rdversion (ThPL3rd) had superior performance in the specific tests, as shown by lower requirements. This indicates that it is more appropriate for the actual data in comparison to the other distributions.The experimental results demonstrated the superiority of the (ThPL3rd) and suggested that the maximum likelihood method was the most effective for big sample sizes. Therefore,

the data will be adjusted to the (ThPL3rd) and the MLH will be used to calculate the parameters of this distribution based on the given theoretical values (σ=0.8, δ=1.5, θ =2). The survival function was calculated using the MLH method in the MATLAB application. The estimation results may be seen in table (4) below:

Table 4. The survival function is estimated using (MLH) estimation using actual data.

t	S_Real	S-ML	t	S_Real	S-ML	T	S_Real	S-ML	t	S_Real	S-ML
0.51	0.87887	0.76050	**2.12**	0.11316	0.15849	**0.34**	0.94201	0.87589	**0.18**	0.97211	0.94155
0.97	0.60765	0.52408	**0.66**	0.79946	0.69764	**7.01**	0.00007	0.00005	**0.87**	0.66897	0.59504
0.51	0.73024	0.62721	**0.5**	0.98352	0.93891	**0.68**	0.79417	0.69587	**2.14**	0.039967	0.15675
0.18	0.98211	0.93153	**0.66**	0.86976	0.76646	**0.33**	0.96385	0.88584	**0.92**	0.66099	0.56889
0.87	0.66895	0.57401	**0.29**	0.98777	0.89617	**0.93**	0.65189	0.56586	**0.77**	0.79773	0.69595
2.05	0.07814	0.14471	**0.12**	0.98598	0.96377	**0.27**	0.97157	0.95055	**0.13**	0.97829	0.97187
0.89	0.61096	0.55881	**0.78**	0.79179	0.73162	**0.16**	0.98158	0.95455	**0.56**	0.86066	0.76886
0.77	0.72770	0.62589	**0.55**	0.86686	0.74426	**1.8**	0.17448	0.19946	**0.33**	0.95385	0.92081
0.13	0.98024	0.965381	**0.92**	0.65295	0.56885	**2.5**	0.04406	0.08892	**0.89**	0.67699	0.58387
0.56	0.86005	0.73683	**0.34**	0.95179	0.86577	**0.43**	0.91973	0.89981	**1.1**	0.49679	0.44271
0.35	0.95584	0.88385	**0.7**	0.73513	0.66549	**0.8**	0.68688	0.58986	**0.97**	0.63866	0.54419
4.1	0.00117	0.01657	**0.43**	0.91575	0.81886	**1.1**	0.58576	0.53476	**0.59**	0.88025	0.77778
1.98	0.12804	0.16851	**0.75**	0.75615	0.63648	**2.5**	0.05413	0.10410	**0.18**	0.98219	0.96158
2.34	0.03887	0.11764	**0.42**	0.92368	0.86416	**0.8**	0.68599	0.58588	**0.17**	0.98392	0.96579
0.73	0.75044	0.64731	**0.21**	0.98508	0.96845	**0.74**	0.75907	0.65659	**0.77**	0.78721	0.64591
0.12	0.98395	0.98379	**0.5**	0.86527	0.76255	**2.11**	0.12543	0.16622	**0.1**	0.97010	0.92722
0.88	0.68294	0.58694	**0.77**	0.74674	0.65594	**1.7**	0.19878	0.23428	**3.4**	0.01339	0.03303
0.18	0.97012	0.95163	**1.5**	0.26826	0.26277	**0.22**	0.97079	0.92887	**1.12**	0.52006	0.45260
0.52	0.86709	0.77435	**1.7**	0.19798	0.24557	**6.4**	0.00028	0.00119	**0.7**	0.72699	0.67979
0.22	0.99520	0.96441	**0.47**	0.89618	0.77848	**0.12**	0.98198	0.98277	**0.77**	0.73774	0.67594
0.46	0.91575	0.82685	**6.2**	0.00122	0.00123	**0.32**	0.95101	0.89587	**0.42**	0.93179	0.85418
0.78	0.97120	0.98340	**1.54**	0.31181	0.29888	**0.97**	0.62867	0.55407	**0.42**	0.93269	0.85515
0.82	0.66478	0.57489	**0.3**	0.93113	0.84895	**2.6**	0.03513	0.04959	**0.61**	0.84515	0.71179
1.55	0.39617	0.39476	**0.23**	0.96739	0.95844	**2.7**	0.02585	0.07029	**0.54**	0.87964	0.75968
0.51	0.87358	0.78498	**0.43**	0.93674	0.83783	**0.22**	0.98191	0.93393	**0.85**	0.70187	0.69420

Table 4 displays the actual data for patients' survival times. The survival function values clearly decrease with time. Reduced hospitalization durations increase the probability of patient survival. The patient who had a hospital stay of 10 minutes had a 97% likelihood of surviving, whereas the patient who remained for 7 days and 1 minute had a 0.005% likelihood of survival.

From Table (4-4) and Figure (4-1), the following is clear:

1. The values of the survival function decrease clearly with time, and this is consistent with the behavior of this function as it decreases with time.
2. The estimated values of the survival function according to the maximum likelihood method are closer to the true values, and this indicates the accuracy of the estimate.
3. The shorter the patient stays in the hospital, the more likely he or she will survive. The patient who remained in the hospital for (0.11) from day had a probability of surviving (97%), and the patient whose length of stay in the hospital was (7.21) from day had a probability of surviving (0.002%).

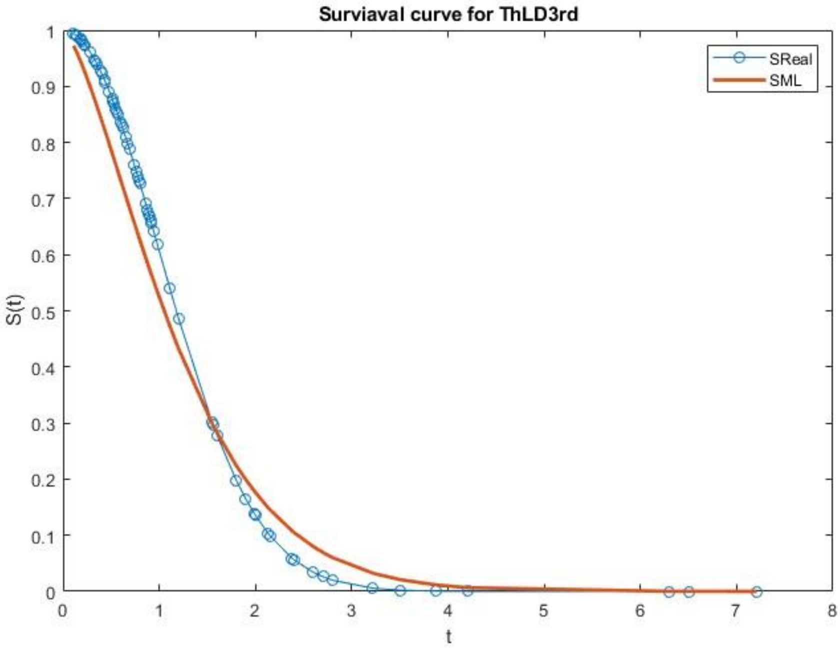


Figure 7.the curve of the theoretical survival function estimated according to the maximum likelihood

Conclusion

The (ThLD3rd) exhibits lower-level testing requirements in practical application have compared to previous distributions, showing a better fit to the data. The findings of the study revealed that the survival function, which was predicted using the maximum likelihood approach, closely approximates the real data function. Furthermore, the survival function values obtained from the maximum likelihood technique showed a decreasing behavior as time progressed. The study indicates a negative correlation between extended durations of hospitalization and a probability of patient survival.

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