



APPROXIMATING THE NASH SOCIAL WELFARE WITH INDIVISIBLE ITEMS

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Abstract

We study the problem of allocating a set of indivisible items among agents with additive valuations, with the goal of maximizing the geometric mean of the agents' valuations, i.e., the Nash social welfare. This problem is known to be NP-hard, and our main result is the first efficient constant-factor approximation algorithm for this objective. We first observe that the integrality gap of the natural fractional relaxation is exponential, so we propose a different fractional allocation which implies a tighter upper bound and, after appropriate rounding, yields a good integral allocation. An interesting contribution of this work is the fractional allocation that we use. The relaxation of our problem can be solved efficiently using the Eisenberg-Gale program, whose optimal solution can be interpreted as a market equilibrium with the dual variables playing the role of item prices. Using this market-based interpretation, we define an alternative equilibrium allocation where the amount of spending that can go into any given item is bounded, thus keeping the highly priced items underallocated, and forcing the agents to spend on lower priced items. The resulting equilibrium prices reveal more information regarding how to assign items so as to obtain a good integral allocation.

keywords : Approximating - Nash – With – Indivisible- Items

تقريب ناش للرعاية الاجتماعية مع عناصر غير قابلة للتجزئة

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الملخص:

دراسة مشكلة توزيع مجموعة من البنود غير القابلة للتجزئة بين الوكلاء ذوي التقييمات المضافة ، وذلك بهدف تعظيم المتوسط الهندسي لتقييمات الوكلاء ، أي ناش للرعاية الاجتماعية من المعروف أن هذه المشكلة هي NP-hard ، والنتيجة الرئيسية التي توصلنا إليها هي أول خوارزمية فعالة لتقريب العامل الثابت لهذا الهدف. نلاحظ أولاً أن فجوة التكامل في الاسترخاء الجزئي الطبيعي هي أسية ، لذلك نقترح تخصيصاً كسرياً مختلفاً مما يعني ضمناً حداً أعلى أكثر إحكاماً، وبعد التقريب المناسب، يؤدي إلى تخصيص تكاملي جيد. إحدى المساهمات المثيرة للاهتمام في هذا العمل هي التخصيص الجزئي الذي نستخدمه. يمكن حل بقية مشكلتنا بكفاءة باستخدام برنامج آيزنبرغ-غيل، والذي يمكن تفسير الحل الأمثل له على أنه توازن السوق حيث تلعب المتغيرات المزدوجة دور أسعار العناصر. باستخدام هذا التفسير القائم على السوق، نحدد توزيعاً متوازناً بدلاً من حيث يتم تحديد مبلغ الإنفاق الذي يمكن أن يذهب إلى أي بند معين، وبالتالي إبقاء العناصر ذات الأسعار المرتفعة غير مخصصة، وإجبار الوكلاء على الإنفاق على العناصر ذات الأسعار الأقل . تكشف أسعار التوازن الناتجة عن مزيد من المعلومات حول كيفية تخصيص العناصر للحصول على توزيع متكامل جيد

الكلمات المفتاحية: تقريب - ناش - مع - عناصر - غير مجزئة



Introduction.

We study the problem of allocating a collection of m indivisible items among a set of n agents ($n \leq m$), aiming to maximize the Nash social welfare (NSW). Since the items are indivisible, an allocation x assigns each item to a single agent. We assume that the agents have additive valuations, i.e., each agent i has a non-negative value v_{ij} for each item j , and her value for an allocation x that assigns to her some bundle of items B_i , is $v_i(x) = \sum_{j \in B_i} v_{ij}$. The NSW objective is to compute an allocation maximizing the geometric mean of the agents' values, i.e.,

$$\max_x \left(\prod_i v_i(x) \right)^{1/n}.$$

The motivation for this problem is closely related to the well-studied Santa Claus problem [11, 8, 6, 26, 10, 17, 7, 5], where the objective is to compute an allocation that maximizes the minimum value across all agents, i.e., $\max_x \min_i v_i(x)$. The story behind the Santa Claus problem is that Santa is carrying presents (the items) which will be given to children (the agents) and his goal is to allocate the presents in a way which ensures that the least happy child is as happy as possible. As we discuss later on, the geometric mean objective, just like the max-min objective, aims to reach a balanced distribution of value, so both these problems belong to the long literature on fair division. Social Choice Theory. Allocating resources among a set of agents in a fair manner is one of the fundamental goals in economics and, in particular, social choice theory. Before embarking on computing fair allocations, one first needs to ask what is the right objective for fairness. This question alone has been the subject of long debates

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in both social science and game theory, leading to a very rich literature. At the time of writing this paper, there are at least five academic books [46, 13, 43, 33, 9] written on the topic of fair division, providing an overview of various proposed solutions for fairness. Over the last decade, the computer science and operations research communities have contributed to this literature, mostly focusing on the tractability of computing allocations that maximize specific fairness objectives. The result of this work has been a deeper understanding of the extent to which some of these objectives can be optimized or approximated in polynomial time which, in turn, serves as a signal regarding the appropriateness and applicability of these objectives. Nash Social Welfare. The objective we seek to maximize in this work, the Nash social welfare (also known as Bernoulli-Nash social welfare), dates back to the fifties [34, 29], when it was proposed by Nash as a solution for bargaining problems, using an axiomatic approach. This objective, like other



standard welfare objectives, is captured by the following family of functions known as generalized means, or power means:

$$M_p(x) = \left(\frac{1}{n} \sum_i [v_i(x)]^p \right)^{1/p}.$$

In particular, the NSW corresponds to $M_0(x)$, i.e., the limit of $M_p(x)$ as p goes to zero. Other important examples of welfare functions captured by $M_p(x)$ include the max-min objective that the Santa Claus problem studies, known as egalitarian social welfare, as well as the utilitarian social welfare, which maximizes the average value across the agents. The former corresponds to the limit of $M_p(x)$ as p goes to $-\infty$, while the latter corresponds to $M_1(x)$. Two of the most notable properties that the NSW objective satisfies follow. (Additional appealing properties are discussed by Moulin [33] and Caragiannis et al. [16].) • The optimal allocation with respect to the NSW objective is scale-free, i.e., it is independent of the scale of each agent's valuations. • The NSW objective provides a natural compromise between fairness and efficiency. The fact that the NSW objective is scale-free means that choosing the desired allocation does not require interpersonal comparability of the individual's preferences. The agents only need to report relative valuations, i.e., how much more they like an item compared to another. In other words, maximizing the NSW using the v_{ij} values is equivalent to using $\alpha_i v_{ij}$ as values instead, where $\alpha_i > 0$ is some constant for each agent i . This property is particularly useful in settings where the agents are not paying for the items they are allocated, in which case the scale in which their valuations are expressed may not have any real meaning¹. Note that neither the egalitarian nor the utilitarian social welfare objectives are scale-free. As a result, the former could end up allocating most of the items to an agent who has a low value for all the items, while the latter could end up allocating all the items to just one agent. Regarding the second property, the two alternative welfare objectives mentioned above, i.e., the egalitarian and the utilitarian objectives, correspond to extreme fairness and efficiency considerations respectively. The former objective maximizes the happiness of the least satisfied agent, irrespective of how much inefficiency this might be causing and, on the other extreme, the utilitarian social welfare approach maximizes efficiency while disregarding how unsatisfied some agents might become.

Approximation Algorithm

1. Integrality Gap. The accepted way to create an approximation algorithm is to take a fractional mapping and "round" it the right way to produce an integer. This is because the Eisenberg-Gale program can be used to compute the fractional



relaxation of the IP program. It is hoped that the fractional allocation would provide some light on what a suitable integral allocation might include.

Fractional relaxation of IP provides proposals for item distributions and upper bound constraints on the geometric mean of the optimal integer solution ' x^* '. Instead, proving this upper bound is the normative way to prove the approximate bound of the rounding procedure. Unfortunately, it is possible to show that the integer gap in IP, or the difference between the geometric mean of the fractional solution and the geometric mean of x^* , is not constant.

The integrality gap of IP is $\Omega(2^m)$.

Consider the situation of m objects and n identical agents. In the first $m - 1$ entries, each agent has a value of 1, and in the last entry it has a value of $2m$. Partial allocation allows you to distribute each item evenly. The agents are distributed, giving each a clump of values equal to $(2m + m - 1)/n$, resulting in a geometric mean equal to $(2m + m - 1)/n$. However, the high price object only has one. Each integral-related agent assigned, so that the geometric mean is at most

$$2m/n[(m-1)/(n-1)]^{(n-1)/n} \leq 2m/n. \text{ These instances have minimal IP consistency gaps } \frac{2^m}{2^{m/nmn}} = \frac{2^{\frac{n-1}{n}m}}{mn}$$

As a result, the integration gap grows exponentially with m for any constant value $n \geq 2$.

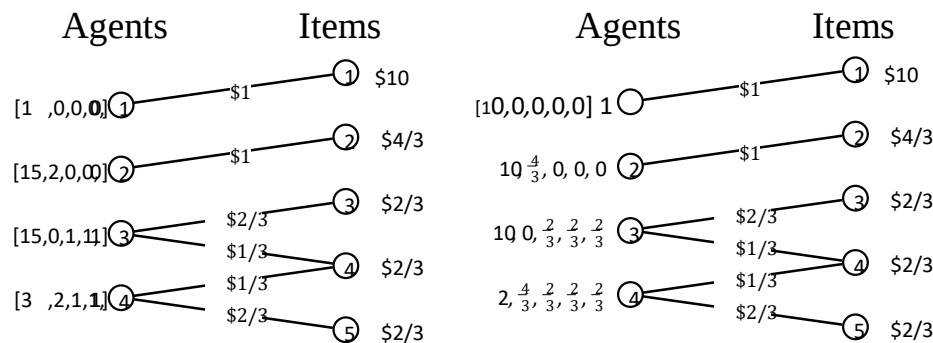
Lemma 3.1 means that the traditional methods for demonstrating a constant-factor approximation guarantee cannot be used to the fractional solution of IP. Additionally, there are times when this partial solution offers very little guidance as to how the goods need to be distributed. For example, agents 2 and 3 in the example in Figure 1b have completely different preferences, but are the same with respect to partial allocation. So, whenever agent 1 gets item 1, the partial solution gives no information that these two other agents can use to decide what they should receive. (Alireza Farhadi, 2021).

2. Spending-Restricted Equilibrium.

Encouraged by the fact that IP's fractional solution failed to provide enough detail for rounding, they provide a new interesting constraint on the fractional solution to fill the completeness gap. Notably, he relaxes the requirement that things must be fully allocated, and instead limits the total money spent on each item to \$1, the maximum amount he can allocate to one agent. The principal (x_{ij}) and dual (p_j) P-variables of the Eisenberg-Gale program for each element j . The answer must satisfy the condition $\sum_{i \in N} x_{ij} p_j \leq 1$, that satisfies both of these requirements.



Definition . The definition of a spending-constrained (SR) equilibrium is partial allocation (x_{-j}) and price vector p , where each agent spends its entire budget on MBB items at price (P_{-j}) , and each total spending is min" is equal to $\{1, (P_{-j})\}$. Example 3. They go back to the issue instance they looked at in Example 2.1 to provide further context for this spending-restricted market equilibrium.



(a) A spending-restricted equilibrium (x, p) . (b) Valuations that are scaled for p .

1 A spending-restricted equilibrium $((x_{-j}) \bar{00} p_{-j})$ for the instance of Example 2.

Three agents spend all of their money on the highly desired Item 1 at its price of \$3 in the unconstrained equilibrium of this situation. This would not be acceptable in the spending-restricted equilibrium; so the price of this item should be raised to the extent that only Agent 1 will be able to spend her money on it because she doesn't have other options. Figure 2a depicts the expenditure graph for this SR equilibrium. It is evident in this figure, that the preferences of Agents 2 and 3 are described with more information than they are shown in the unconstrained market equilibrium spending graph.

Normalizing the Valuations. The magnitude of every agent's appraisals doesn't have any effect on the conclusion since, as they noted previously, the NSW aim is scalefree. They now assume that every agent's values are normalized in the following order to simplify the assertions and proofs of this section and take use of this fact.

Definition 3.4. Given an SR price (p_{-j}) , the MBB element j of each agent i is p and v_{ij} of all other items.

Given an SR price (p_{-j}) and some agents i of valuations v_i that don't ensure this property, the agents are given all v_{ij} . Obtain a scaled estimate v_i against p by counting.}, i.e., H . The reciprocal of the maximum bang-per-buck ratio. Each element j which is her MBB of i in p satisfies $v_{ij}' / (P_{-j}) = 1/\alpha_i$, $v_{ij} = \alpha_i v_{ij}' = (P_{-j})$. While, for all other elements $v_{ij}' / (P_{-j}) < 1/\alpha_i$, ie $v_{ij} < \alpha_i \cdot (P_{-j})$. will be.

3. Upper Bound. The following theorem offers a new upper limit for the geometric mean of the ideal integral solution x^* based on the prices p using the scaled values.



Let $H(\bar{P})$ or $L(\bar{P})$ be the collection of items j with $\bar{P}_j > 1$ or $\bar{P}_j \leq 1$ respectively. Due to the fact that each item's total cost in $H(\bar{P})$ is 1 and the agents' total budget is n , it should be noted that $|H(\bar{P})| \leq n$.

Theorem 3.5. *The best geometric mean is upper bounded in the following way, given SR prices $\bar{P}$ and agent valuations v scaled for $\bar{P}$.*

$$(\prod_{i \in N} v_i(x^*))^{1/n} \leq (\prod_{j \in H(\bar{p})} \bar{p}_j) \quad (3.1)$$

Proof. Given an SR price $(p_-)^-$, the MBB element j of each agent i is p and v_{ij} of all other items.

If we keep the SR price p at the current level, increasing the value of the agent will only increase the left side of the inequality (3.1), not the right side. So, $v_{ij} \leq (P_-)_j^-$ (v is scaled with respect to $(P_-)_j^-$) for every agent i and every element j , so $v_{ij} = (P_-)_j^-$ for every i and j . For $^-$, it is sufficient to show that (3.1) holds. . Throughout the rest of the proof, let $v_{ij} = (P_-)_j^-$ for all i and j . In fact, they relax the completeness criterion for the optimal assignment of items $j \in L((P_-)_j^-)$ later in the proof. In other words, make x' an optimal assignment if each element j is divisible by $(P_-)_j^- \leq 1$. Since this is a relaxation of the original problem, it suffices to prove that there are $\sum_{i \in N} v_i(x') \geq \sum_{i \in N} v_i(x^*)$ such that the inequality (3.1) is true for x' . is.

Q

First, assume that $Q H(\bar{P})$ is empty, means that each item j has price $\bar{p}_j \leq 1$.

Then, it suffices to prove that $\sum_{i \in N} v_i(x') \leq 1$. But, since $v_{ij} = \bar{P}_j$ for all i and j , they know that $\sum_{i \in N} v_i(x') = \sum_{j \in M} \bar{P}_j$. Also, since $\bar{P}_j \leq 1$, the spending on every item is P equal to its price, and so $\sum_{j \in M} \bar{P}_j = n$. Then, $\prod_{i \in N} v_i(x') = 1$.

Given an SR price $(p_-)^-$, the MBB element j of each agent i is p and v_{ij} of all other items.

Assume that $H n((P_-)_j^-)$ is not empty at this point. Each of the items Q in $H((P_-)_j^-)$ is assigned to at most 1 agent, since x' of these things is an integer. That is, at maximum $|H((P_-)_j^-)|$ the agent receives one of the items in x' . Let $N_L \subseteq N$ be the set of remaining agents that assign only those items in $L((P_-)_j^-)$ where x' has at least $n - |H((P_-)_j^-)|$. The cost of each item in the SR equilibrium is exactly 1, and the price of $L((P_-)_j^-)$ is added to the remaining budget or $n - |H((P_-)_j^-)|$. Because $v_{ij} = (P_-)_j^-$ for all i and j , the result is the maximum value that can be distributed to the agents in N_L . (Mohammad Ghodsi M. T., 2021):

$$\sum_{i \in N_L} v_i(x') \leq \sum_{i \in N_L} \bar{p}_j = n - |H(\bar{p})|$$



So, at least there are $n - |H(\bar{P})|$ agents in N_L , and the total value of them in x' is at maximum $n - |H(\bar{P})|$, so $\min \sum_{i \in N_L} \{v_i(x')\} \leq 1$.

They now demonstrate that any agent that x' allots a portion of some item $j \in L(\bar{P})$ has value in x' of no more than 1. This suggests that each agent in N_L has a value of no more than one. Assume, in order to create a paradox, that there is an agent α with $v_\alpha(x') > 1$ who obtains value $v_\beta > 0$ from a portion of an object with $j \in L(\bar{P})$. As they shown previously, some agent exists in x' with $v_\beta(x') = 1$. They assume that each agent is the same, therefore agent β has also a value of v_β for the portion of item j that is assigned to. As a result, they would get an allocation x that is incongruent with the optimality of x' if they were to remove a portion of item j of some positive value $v < v_\alpha(x') - v_\beta(x')$ from α and allocate it to β instead:

$$\frac{\prod_{i \in N} v_i(x'')}{\prod_{i \in N} v_i(x')} = \frac{[v_\alpha(x') - v][v_\beta(x') + v]}{v_\alpha(x')v_\beta(x')} = 1 + \frac{[v_\alpha(x') - v_\beta(x') - v]v}{v_\alpha(x')v_\beta(x')} > 1$$

It is similarly simple to demonstrate that every agent gets a maximum of one item from $H(\bar{P})$ in x' using a nearly identical set of inputs. If not, reassigning one of these goods to the agent with the lowest value would result in a higher NSW once again. There are precisely $|H(\bar{P})|$ agents, and each of them is given a single item with the value $\bar{P}_j$ in the form of the function $j \in H(\bar{P})$. Since these agents' value in x' is larger than 1 (since $\bar{P}_j > 1$), they are not given any percentage of items $j \in L(\bar{P})$.

As a result, "They showed that the product of the agent values of N_L in x is at most 1 and that the product of the agents associated with the elements of $j \in H(\bar{P})$ is equal. This implies $\prod_{i \in N} v_i(x') \leq \prod_{j \in H(\bar{P})} \bar{P}_j$ and completes the proof (Kurokawa, 2017).

4. Spending-Restricted Rounding Algorithm (SRR).

SRR is an approximation approach that first computes the SR allocation $(x_{\bar{P}})$ and the price $(p_{\bar{P}})$, then divides every item into Appropriately assign to one. Its neighborhood in the output graph $Q((x_{\bar{P}}))$. In this way, we want the majority of agents to at least get 1/2 of the value given by the partial solution, and all other agents to get a good chunk of that value to approximate a constant coefficient. (Ian A. Kash, 2014) .

They assume that the cost graph of partial allocation $(x_{\bar{P}})$ is the forest given in Section 2. Also, since each object has at least one agent (partially) associated with it, each tree in this forest has a vertex relative to an agent. After computing this forest in step 1, the SRR method selects one of the vertices of each tree corresponding to an agent as the root of the tree. As a result, all vertices at depth 1 correspond to things the root agent spends money on, vertices at depth 2 correspond to things the agent at depth 1 spends money on, and so on. They refer to the agent corresponding to the parent vertex of any vertex in the root tree



corresponding to the element as the vertex's parent, and its descendants, if any, as the vertex's children. (Martin Hoefer, 2021).

The first integral assignments are made in the first two stages of the SRR algorithm. As with all items corresponding to leaves of the root tree of the output graph, items with a price (P_j) of at most $1/2$ are also assigned to the parent agent. The first form of rounding is trivial because the parent agent is the only agent that outputs a leaf entry in the output graph. Multiple subagents of this object in x may spend money on it. The second is less obvious, because each child agent will only spend half of its budget on the item given to it at this stage, as it is price capped at $1/2$. The rest of the money will be spent on products that match the children in the tree (Jiafan He, 2019). Specifically, in the output subdiagram created by agents and remaining unassigned elements, each of these subagents becomes the root of a new tree after step 4.

Each remaining expensive item is passed to another nearby agent in the final stage of the algorithm. In other words, it computes the element-agent match constrained by the edges of the output graph. Considering the element assignments that occurred in steps 3 and 4, this method computes the best possible matching (Daniel Halpern, 2020). In a suitable weighted bipartite network this is achieved using a simple maximal weighted matching technique. A set of vertices on one side of this bisection graph correspond to a set of agents, and another set of vertices on the other side correspond to the remaining objects. Agent i and item j have an edge that exists only if a matching edge exists in the output graph. Let $v_i(x')$ be the value of the term previously assigned to agent i in steps 3 and 4 ("if any") to set the edge weights. Then set the edge weight between agent i and element j in the bipartite graph to $w_{ij} = \log(v_{ij} + v_i(x'))$. This is equal to the logarithm of the final value of i when an item is assigned to j . Lastly, add another item vertex that is only associated with agent i and set the weight to $\log(v_i(x'))$. By using the maximum weighted matching procedure on this weighted bipartite graph, we can determine which matching of the remaining elements leads to the final value of the agent that maximizes the sum of the logarithms. This is the same as maximizing social welfare. Nash.

Algorithm 1: SRR.

- 1 Computing a spending-restricted equilibrium $(\bar{x}, \bar{p})$.
 - 2 Choosing a root-agent for every tree in the spending graph $Q(\bar{x})$.
 - 3 Assigning any leaf-item in the trees to its parent-agent.
 - 4 Assigning any item j with $\bar{P}_j \leq 1/2$ to its parent-agent.
 - 5 Computing the optimal matching of the remained items to adjacent agents.
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Theorem 3. . *The integral allocation $\tilde{x}$ computed by the Spending-Restricted Rounding algorithm satisfies always:*



$$\left(\prod_{i \in N} v_i(x^*)\right)^{1/n} \leq 2.889 \left(\prod_{i \in N} v_i(\tilde{x})\right)^{1/n}$$

Since CE computes the best match of the remaining terms in the final step of the SRR method, it provides the approximation guarantees necessary to prove Theorem 3.6, showing that matches between terms usually occur. is enough. We start by constructing like matching by creating a subgraph of the original output graph $Q((x_{-})^{\bar{}})$. "Beginning with $Q((x_{-})^{\bar{}})$, build the pruned output graph $P((x_{-})^{\bar{}})$ by removing everything assigned in steps 3 and 4. Step 3 Since all the leaf elements have been assigned in , each remaining element has at least 1 child agent, and finally, they either have every element j as multiple subagents, or the one that spends the most money on element j . It intersects the edge connected to subagent i with the largest $(x_{-})^{\bar{}}_{ij}$ value among all but one subagent.

The rest of this part uses only the trimmed edges of the graph to prove that there exists a matching that satisfies the required inequality. To demonstrate this, they first demonstrate Lemma 3.7 and 3.8.

Lemma 3.7 . For any matching-tree T with k agents

$$\sum_{j \in M_T} \min\{1, \bar{P}_j\} \geq k - 1/2$$

Proof. The first thing they notice is that there is only one object $j \in MT$ in x that an agent $i \in T$ can spend money on. To demonstrate this, considering the fact that the absence of j in MT is either because j was assigned to a parent agent in step 4 or because edge (i,j) was removed by the pruning process. please give me. Both the original output graph $Q(x)$. In both scenarios, the agent that "loses" j is its child agent, indicating that i rises to the position of the matching root. Hence, such an element If j exists, a subagent of T must act as the root, and there can be only one such element.

Since the total cost of T 's agents is k , it is sufficient to show that T 's root spends less than half of its budget on only one item ($j \in MT$). If j is lost in step 4, its cost, or the amount of money I spent on it, is both at most $1/2$. However, if edge (i,j) is pruned away, the highest output child of j is not i (Maria Kyropoulou, 2020). This proves the lemma because it states that i spends only $1/2$ on j and specifies that each item of x spends at most 1.

Lemma 3.8. For any matching-tree T with k agents, there exists an agent $i \in T$ who, during Steps 3 and 4 received one or more items that she values at least $1/(2k)$ (David Kurokawa, 2018).

Proof. Since the maximum amount spent on an item is 1, the maximum amount spent on all $k-1$ in T is $k-1$. However, Lemma 3.7 states that the total amount spent on things in MT is at least $k/2$, implying that the amount spent on items in $MT \setminus T$ is at least $1/2$. increase. The total value that T 's agents got from these things



is also at least $1/2$. This is because every one of these objects is assigned to an agent which is her MBB. Therefore, at least one of these agents received a score of at least $1/(2k)$.

They can now demonstrate their primary conclusion using Lemma 3.8.

Proof. [of Theorem 3.6] Each matching tree T in $P((x_{-})^{\bar{}})$ has exactly 2 neighbors and has exactly 1 more agent than Ding. As a result, for any matching tree T , edge-restricted matching in $P(x)$ must leave one agent unmatched. Considering Lemma 3.8, a simple way to perform matching is to choose a particular agent and exclude it from matching if its value is already at least $1/(2k)$. As a result, her value is her $v_{ij} = (P_{-j})^{\bar{}} > 1/2$, and each additional agent I is assigned to her one of her MBB entries j . If $N_T \subseteq N$ is the set of k agents in tree T , then this matching is (Suksompong, 2021):

$$\prod_{i \in N_T} v_i(x) \geq \left(\frac{1}{2}\right)^{k-1} \frac{1}{2k} = \frac{1}{2^k k}$$

In order to perform at least as well, their algorithm's last phase selects the greatest feasible matching with regard to the geometric mean. Consequently, their algorithm ensures

$$\left(\prod_{i \in N} v_i(\tilde{x})\right)^{1/n} \geq \left(\prod_T \frac{1}{2^k k(T)}\right)^{1/n} = \frac{1}{2} \left(\prod_T \frac{1}{k(T)}\right)^{1/n}$$

Such that $k(T)$ is the quantity of agents in every tree T , and the equality uses the fact that

$2^{k(T)} = 2^n$, since $P k(T) = n$. If the quantity of matching-trees is t , thus the inequality of geometric and arithmetic means shows that $\prod_T k(T)^{1/t} \leq \frac{\sum_T k(T)}{t} = \frac{n}{t}$

So, their algorithm guarantees

$$\left(\prod_{i \in N} v_i(\tilde{x})\right)^{1/n} \geq \frac{1}{2^{(n/t)t/n}} \geq \frac{1}{2e^{1/e}} \quad (3.2)$$

Theorem 3.5's upper limit is equal to 1 if $H(\bar{P})$ is empty, and Inequality (3.2) suggests that their approximation factor is at maximum $2e^{1/e} \approx 2.889$ in this situation. The matching gives every item in $H(\bar{P})$ to a unique agent for whom this item is MBB if it is not empty. The analysis presented implies merely that these agents get values greater than $1/2$, but they really receive values greater than $\bar{P}_j/2$, or $p_j > 1$. These agents get the following when $\bar{P}_j/2$ is substituted for $1/2$ and the same parameters are used (Richard J. Lipton E. M., 2004):

$$\left(\prod_{i \in N} v_i(\tilde{x})\right)^{1/n} \geq \frac{1}{2e^{1/e}} \left(\prod_{j \in H(\bar{p})} \bar{p}_j\right)^{1/n}$$



In addition to employing EF and its relaxations as a fairness metric throughout the length of this thesis, we want to employ Nash welfare and Pareto-optimality as efficiency metrics in our study. This is due to the fact that we consider these two measurements to be the most accurate depictions of the ideal state. The literature on equitable division has a variety of various definitions of fairness, some of which were briefly discussed in this chapter.

This chapter reports the results for each of the methods mentioned in the previous chapter. First, we discuss and analyze the results of the first method that is proposed by Bhaskar Ray Chaudhur, where he investigated if EFX allocations occur when agents have broad values. Then we review the results of the second method proposed by RICHARD COLE where he investigated how to maximize the Nash social welfare by solving the APX-hard combinatorial optimization issue of distributing a group of indivisible goods among agents under additive values.

Results of Method

To take full advantage of Nash's welfare, RICHARD COLE examines the APX-hard combinatorial optimization problem of distributing an indivisible collection of commodities among agents with additive values. His most important achievement is the development of the first polynomial-time method that guarantees approximation of the geometric mean of agent values by constant coefficients. In particular, he shows that his method can only achieve approximation coefficients up to $2e^{1/e} \approx 2.889$.

He first states that the Eisenberg-Gale convex program can be used to optimally solve the natural fractional relaxation of the problem, despite the fact that the object is not convex, but the size of the problem As, this relaxation completeness gap widens. This takes advantage of the fact that the Eisenberg-Gale program's solution can be viewed as a market equilibrium where agents pay to acquire a portion of the goods to avoid consistency gaps. (Suksompong., 2021).

Richard Cole suggests a brand-new kind of market equilibrium as a result of this interpretation of the fractional solution from the perspective of the market. He imposes a limitation that limits the total amount of money the agents may spend on any one thing in particular. Some agents may be influenced by the "spending-restricted" equilibrium to forego purchasing products that are in great demand in favor of cheaper alternatives. Because of this, the fractional allocation of this equilibrium reveals important information about how the less desirable items should be distributed, and our rounding approach makes use of this information to determine a suitable integral distribution. This fractional allocation provides for the proof of the constant factor guarantee since it not only implies an upper limit for the ideal integral solution that closely approximates it but also acts as a guide toward an integral solution.



It's noteworthy to note that the expenditure presented restriction combines the dual and primal variables of the Eisenberg-Gale program. As a consequence, concepts from combinatorial methods for solving the Eisenberg-Gale problem were used to calculate this answer. Both a straightforward strongly polynomial time technique and a more complex weakly polynomial time algorithm were provided.

REFERENCES

1. Budish, E. (2011). The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6), 1061-1103.
2. Cole, R., & Gkatzelis, V. (2018). Approximating the Nash social welfare with indivisible items. *SIAM Journal on Computing*, 47(3), 1211-1236.
3. Kurokawa, D., Procaccia, A. D., & Wang, J. (2018). Fair enough: Guaranteeing approximate maximin shares. *Journal of the ACM (JACM)*, 65(2), 1-27.
4. Amanatidis, G., Markakis, E., & Ntokos, A. (2020). Multiple birds with one stone: Beating $1/2$ for EFX and GMMS via envy cycle elimination. *Theoretical Computer Science*, 841, 94-109.
5. Amanatidis, G., Markakis, E., & Ntokos, A. (2020). Multiple birds with one stone: Beating $1/2$ for EFX and GMMS via envy cycle elimination. *Theoretical Computer Science*, 841, 94-109.
6. Amanatidis, G., Birmpas, G., Christodoulou, G., & Markakis, E. (2017, June). Truthful allocation mechanisms without payments: Characterization and implications on fairness. In *Proceedings of the 2017 ACM Conference on Economics and Computation* (pp. 545-562).
7. Kash, I., Procaccia, A. D., & Shah, N. (2014). No agent left behind: Dynamic fair division of multiple resources. *Journal of Artificial Intelligence Research*, 51, 579-603.
8. Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A. D., Shah, N., & Wang, J. (2019). The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation (TEAC)*, 7(3), 1-32.
9. oannis Caragiannis, N. G. (2019). Envy-freeness up to any item with high nash welfare: The virtue of donating items. 20th ACM Conference on Economics and Computation (EC), (pp. 527–545).
10. He, J., Procaccia, A. D., Psomas, C. A., & Zeng, D. (2019). Achieving a fairer future by changing the past. *IJCAI'19*.



- 11.Amanatidis, G., Markakis, E., & Ntokos, A. (2020). Multiple birds with one stone: Beating $1/2$ for EFX and GMMS via envy cycle elimination. *Theoretical Computer Science*, 841, 94-109.
- 12.Gourvès, L., Lesca, J., & Wilczynski, A. (2017, August). Object allocation via swaps along a social network. In *26th International Joint Conference on Artificial Intelligence (IJCAI'17)* (pp. 213-219).
- 13.Kyropoulou, M., Suksompong, W., & Voudouris, A. A. (2020). Almost envy-freeness in group resource allocation. *Theoretical Computer Science*, 841, 110-123.
- 14.Hoefer, M., Schmalhofer, M., & Varricchio, G. (2021). Approximating Nash Social Welfare in 2-Valued Instances. *arXiv e-prints*, arXiv-2107.
- 15.Bentert, M., Chen, J., Froese, V., & Woeginger, G. J. (2019). Good things come to those who swap objects on paths. *arXiv preprint arXiv:1905.04219*.
- 16.Lipton, R. J., Markakis, E., Mossel, E., & Saberi, A. (2004, May). On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce* (pp. 125-131).
- 17.Lipton, R. J., Markakis, E., Mossel, E., & Saberi, A. (2004, May). On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce* (pp. 125-131).
- 18.Sengupta, R. (2021). *Fair allocation of operations and makespan minimization for multiple robotic agents* (Doctoral dissertation).
- 19.Manurangsi, P., & Suksompong, W. (2022). Almost envy-freeness for groups: Improved bounds via discrepancy theory. *Theoretical Computer Science*.
- 20.Segal-Halevi, E., & Suksompong, W. (2019). Democratic fair allocation of indivisible goods. *Artificial Intelligence*, 277, 103167.
- 21.Suksompong, W. (2021). Constraints in fair division. *ACM SIGecom Exchanges*, 19(2), 46-61.
- 22.Garg, J., & Taki, S. (2021). An improved approximation algorithm for maximin shares. *Artificial Intelligence*, 300, 103547.
- 23.Garg, J., & Taki, S. (2021). An improved approximation algorithm for maximin shares. *Artificial Intelligence*, 300, 103547.
- 24.Feige, U., Sapir, A., & Tauber, L. (2021, December). A tight negative example for MMS fair allocations. In *International Conference on Web and Internet Economics* (pp. 355-372). Springer, Cham.



25. Aleksandrov, M., & Walsh, T. (2020, April). Online fair division: A survey. In *Proceedings of the AAAI Conference on Artificial Intelligence* (Vol. 34, No. 09, pp. 13557-13562).
26. Procaccia, A. D., & Wang, J. (2014, June). Fair enough: Guaranteeing approximate maximin shares. In *Proceedings of the fifteenth ACM conference on Economics and computation* (pp. 675-692).
27. Chevaleyre, Y., Endriss, U., & Maudet, N. (2017). Distributed fair allocation of indivisible goods. *Artificial Intelligence*, 242, 1-22.