The Graph Of Some Types Of Topological Spaces

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Abstract

A foundation for the relationship between graph theory and topology can be found in Euler's solution to the Königsberg bridge problem. In topological graph theory, if we describe the open sets in the topological space as vertices and define the relationships on the open sets as edges at the vertices, we can get a new construction called a topological graph. In this paper, we introduce graphs for several types of topological spaces: discrete topology indiscrete topology, co-finite topology, co-countable topology and nested topological spaces. Also, in this research we presented a method to find the topological space if we only have the graph of the space.

Keywords: Topological space, Graph Theory, Topological Graph, indiscrete topology space graph

الرسم البياني لبعض أنواع الفضاءات الطوبولوجية

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المستخلص

يمكن العثور على أساس العلاقة بين نظرية الرسم البياني والطوبولوجيا في حل أويلر لمشكلة جسر كونيجسبيرج. في نظرية الرسم البياني الطوبولوجي، إذا وصفنا المجموعات المفتوحة في الفضاء الطوبولوجي بالرؤوس وقمنا بتعريف العلاقات على المجموعات المفتوحة كحواف عند القمم، فيمكننا الحصول على بناء جديد يسمى الرسم البياني الطوبولوجي. في هذا البحث، قمنا بتقديم رسوم بيانية لعدة أنواع من الفضاءات الطوبولوجية: الفضاءات الطوبولوجية المنفصلة والمبعثره ،وتوبولوجي المتتمات المنتهية ،وتوبولوجي المتتمات المعدودة والمتداخلة. كما قدمنا في هذا البحث طريقة لإيجاد الفضاء الطوبولوجي إذا كان لدينا الرسم البياني للفضاء فقط.

الكلمات المفتاحية: الفضاء الطوبولوجي، نظرية الرسم البياني، الرسم البياني الطوبولوجي، الرسم البياني الفضائي غير المنفصل.

1.Introduction

The 2016 Nobel Prize in Physics was awarded to scientists who study phase transitions and topological stages of matter. This incident brought attention to more topological knowledge. Topology is a branch of mathematics, and its concepts not only exist in almost all branches of mathematics, but also penetrate into other disciplines such as biology, computer science, communications, and medicine [5].

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It is a science that studies the properties of objects and is not based on bias. This means it can enlarge and smooth wrinkles, but not eliminate them. Graph theory has become a discipline in its own right and is an important mathematical tool in disciplines as

A graph $G_\tau = (V, E)$ is an ordered pair of disjoint sets (V, E), where $V \neq \emptyset$ and E is a subset of unordered pairs of V. The elements $V = V(G)$ and $E = E(G)$ respectively are the vertices and edges of a graph G. A graph G is finite if the set $V(G)$ is finite. The degree of a vertex $u \in V\,(G)$ is the number of edges containing u. The vertices u is called isolated point if the degree of u is zero, which indicates the fact that there is no edge in a graph G that contains u. The maximum degree for any vertex of $V(G)$ is denoted by $\Delta(G)$ and the minimum degree is denoted by $\partial(G)$. An empty graph is called a simple graph; a simple graph is called a complete graph if two distinct vertices are connected by an edge; edges with the same vertices at the endpoints are called cycles; edges with different endpoints are called connections; and no pair of links connects the same pair of nodes [2,3,5,8].

One of the most active fields in mathematics is topology. It is considered as one of the key disciplines of mathematics. As a result of the widespread use of topological concepts by mathematicians and scientists to simulate and comprehend real-world structures and events, topology has recently grown in importance within applied mathematics [1]. The ordered pair (X, τ) of nonempty set X and collection τ on X is called topological space if $X, \emptyset \in \tau$ and τ is closed under finite union and intersection. If $X$is non empty set and $\tau_{indis} = \{X, \emptyset\}$ the pair $(X, \tau_{indis})$is called indiscrete topology space. And where X be non empty set andτ is the power set of X, i.e $\tau_{dis} = \{u: u \subseteq X\}$, the pair $(X, \tau_{dis})$is called discrete topology space. Co-finite topology space defines by (X, τ) where τ is the empty set and the co-finite subsets of X but if τconsists ofthe empty set and the countable subsets of X then (X, τ) is called Co- countable topological space [6,7,9], A complete graph K_n that has order n such that each vertex in it is adjacent to $n - 1$ of the remaining vertices.

2. Topological Graph

Abbreviated topology can be described as a family of sets that satisfy topological properties. Therefore, it is possible to write an algorithm to convert a topology into a graph, as we will see in the definition.

Definition 1

A graph G_τ is said to be a topological graph if there exists a topology τ generated by G_τ [1].

Definition 2

Let $\mathcal{A} \subseteq P(X)$, $\mathcal{B} \subseteq P(Y)$, $\mathcal{A}$ is parallel to $\mathcal{A}$ (say $X \sim Y$), if there exists a bijective function $F : X \to Y$ such that for each $c_1 \in \mathcal{A}$, there exist $c_2 \in \mathcal{B}$ and $F(c_1) = c_2$. Let $= Y$. Then $\mathcal{A}$ and $\mathcal{B}$ are parallel if there exists a transformation $F : X \to X$, such that $F(\mathcal{A}) = \mathcal{B}$ [5].

Definition 3

Let G be a graph, and let (X, τ) be a topological space. Let $v_i, v_j \in V(G)$. In a set graph G, if the vertices v_i and v_j are represented by sets A and B, respectively, then $N(v_i, v_j) = r$, where $|A \cap B| = r$, $r \in \mathbb{N}$, is the number of edges connecting the vertices. $(v_i, X) = |A|$, $N(v_j, X) = |B|$ and $N(v_i, \emptyset) = 0$are all evident.

2.1 The graph of indiscrete topology space

A topological space is said to be non-discrete if it satisfies the following equivalent conditions: It has an empty basis. It has a base that just covers the entire room. The only open subsets are the entire space and the empty subset. The indiscrete topological graph denoted by $G_\tau = (V, E)$ is a graph of the vertex set $V = \{A;\ A \in \tau \ and \ A = \emptyset, X\}$, and the edge set $E = \emptyset$, The vertices $X, \emptyset$ is called isolated point since the degree of $X, \emptyset$ is zero.

Example 1

Let $X = \{a, b\}$ be anon empty set and $\tau_{indis} = \{X, \emptyset\}$ The graph of indiscrete topology, see figure (1).

Figure 1: indiscrete topology space

The rank of each vertical in the indiscrete topology is zero, because the vertices in this space do not have edges in the previous example, note that the vertical of group X is equal to zero, and also ∅

2.2 The graph of discrete topology space

A discrete topology is the best topology that can be specified on a set. Every subset in a discrete topology is an open set, so in particular every singleton subset in a discrete topology is an open set. Let τ be a discrete topology on X and X be a non-empty set. $G_\tau = (V, E)$ represents a discrete topological graph that consists of the edge set $V = \{A;\ A \in \tau\ and\ A \neq \emptyset, X\}$ and the vertex set $V = \{A;\ A \in \tau\ and\ A \neq \emptyset, X\}$.

Proposition 1: Let X be a non-empty set of order n and τ be a discrete topology of X. If $n = 2$, then $G\tau \cong K_2$.

Proof: Let $X = \{a, b\}$, then $\tau = \{\emptyset, X, \{a\}, \{b\}\}$, so $V = \{\{a\}, \{b\}\}$.

Since $\{a\} \nsubseteq \{b\}$ and $\{b\} \nsubseteq \{a\}$, then there is an edge between them. Therefore, the graph G_τ isomorphic to K_2as shown in Figure 2

Figure 2: The graph $G_\tau \equiv K_2$.

$\Delta(G) = 2$ and $\partial(G) = 0$

Proposition 2: Let $|X| = n$ and G_τ be a discrete topological graph. Then, the graph G_τ has $n - 1$ complete induced subgraphs K_t such that $t \geq n$.

Theorem 1: Let G_τ be a discrete topological graph of a non-empty set X. Then, $G_\tau$is a connected graph [11].

Proof. Let v_1 and v_2 are any two vertices in a graph G_τ, let S be a set of all vertices of singleton element. Then there are three cases as follows:

Case 1: If $v_1, v_2 \in S$

since $G\,[S] = K_n$from proof of Proposition 2, Then v_1 adjacent to v_2 for all elements of S. So, there is an edge $v_1, v_2 \in E\,(G_\tau)$ in a graph G_τ.

Case 2: If $v_1 \in S$ and $v_2 \notin S$,

if $v_1 \nsubseteq v_2 \wedge v_2 \nsubseteq v_1$ then $v_1, v_2 \in E\,(G_\tau)$. If v_1 not adjacent to v_2

Then, there is at least one vertex in S say v adjacent to v_2 such that $v \nsubseteq v_2$ and $v_2 \nsubseteq v$. Since v adjacent with v_1 from proof of Proposition 2, so that v adjacent to v_1 and v_2. Thus, $v_1 - v - v_2$ is a path in a graph G_τ. Case 3: If $v_1, v_2 \notin S$ and v_1 not adjacent to v_2. If there is a vertex $t \in S$ such that v_1, also $v_2 \nsubseteq t \wedge t \nsubseteq v_2$. Then, v_1t $\in E\,(G_\tau)$ and $v_2\, t \in E\,(G_\tau)$ and $v_1 - t - v_2$ is a path in G_τ. Otherwise, there is $t_1, t_2 \in S$ where $v_1\, t_1 \in E\,(G_\tau)$ and $t_2\, v_2 \in E\,(G_\tau)$, then $v_1 - t_1 - t_2 - v_2$is a path in G_τ. Hence, $G_\tau$is a connected graph.

Proposition 3: Let $|X| = n$, then the order of discrete topological graph G_τ is

$2n - 2$.

Theorem 2: Let $|X| = n\,(n \geq 4)$ and $G_\tau$be a discrete topological graph. Then, $G_\tau$has bi-dominating set and$\gamma_{bi}\,(G_\tau) = \sum_{i=1}^{n-1}\binom{n}{i} - 4$ [12].

Proposition 4: Let $|X| = n\,(n \geq 4)$ and $G_\tau$be a discrete topological graph. Then, G_τ has no inverse bi-dominating set.

Proposition 5: Let $|X| = n$, and $G_\tau$be a discrete topological graph, then the graph G_τ has $n - 1$ complete induced subgraphs $K_t$such that $t \geq n$.

Proof: Let W be a set of all vertices of singleton elements such that $|w| = n$, let $u, v \in S$. Since u is not a subset of v and v is not a subset of u for all elements of S, then u is adjacent to v.

Proposition 6: Let X be a non-empty set of order n and let τ be a discrete topology on X. If $n = 3$, then $G_\tau \cong \bar{G}_6$

Proof: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

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so $V = \{\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$. Let $u = \{\delta\}$ and $v = \{\beta\}$ be two vertices of singleton elements. Since $\{\delta\}$ is not subset of $\{\beta\}$ and $\{\beta\}$ is not subset of $\{\delta\}$ for all vertices of singleton element, then u is adjacent with v. Hence, the vertices of singleton element $\{\{a\},\{b\},\{c\}\}$ form a complete induced subgraph K_3 of G_τ as shown in Figure 3. Now, let $u_1$and $u_2$be any two vertices that have two elements. Since $u_1$is not a subset of $u_2$and $u_2$is not a subset of $u_1$for all vertices that have two elements, then $u_1$is adjacent to u_2. So, the vertices that have two elements $\{\{a,b\},\{a,c\},\{b,c\}\}$. Also, form a complete induced subgraph $K_3$of G_3. Since every vertex of two elements such as $\{\delta,\beta\}$ is adjacent to only one vertex has a singleton element say $\{\alpha\}$ if $\alpha \neq \delta$ and $\alpha \neq \beta$. Where $\{\delta,\beta\}$ is not subset of $\{\alpha\}$ and $\{\alpha\}$ is not subset of $\{\delta,\beta\}$. Therefore, $G_\tau \cong \bar{G}_6$ as shown in Figure 3.

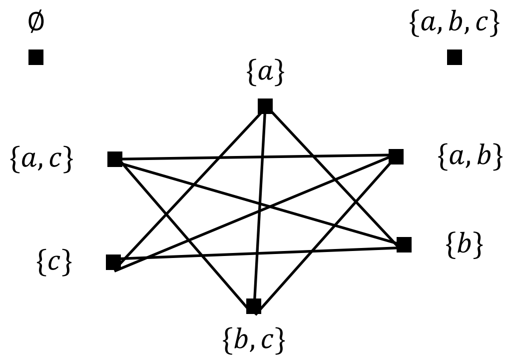

Figure 3: The graph $G_\tau \cong \bar{G}_6$

$\Delta(G) = 3$ and $\partial(G) = 0$

Proposition 7: Let $|X| = n\ (n \geq 4)$, then G_τ has no inverse doubly connected bi-dominating set.

Proof: Given that G_τ is a discrete topological graph and that for $n \geq 4$, it lacks an inverse bi-dominating set (let $X| = n\ (n \geq 4)$. Then, according to [10], G_τ lacks an inverse doubly connected bi-dominating set.

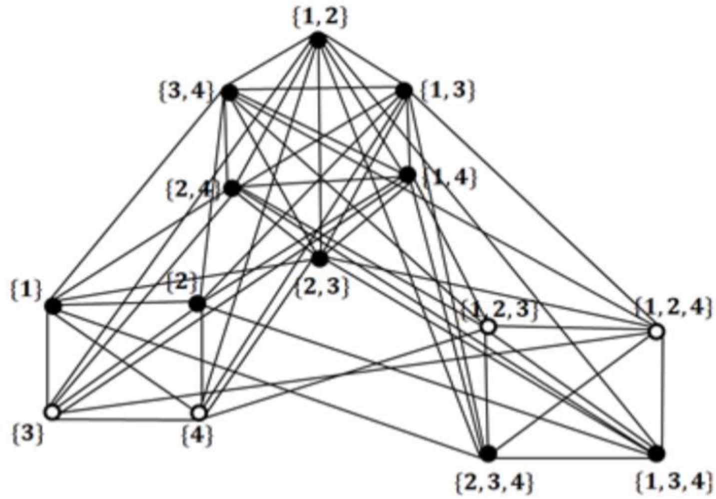

Figure 4: The minimum doubly connected bi-domination when $|X| = 4$.

Example 2

If $X = \{a, b, c, d\}$ and $|X| = 3$, then
$\tau =$
$\{ \emptyset, X, \{c\}, \{b\}, \{a\}, \{d\}, \{a, d\}, \{a, c\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\},$
$\{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$

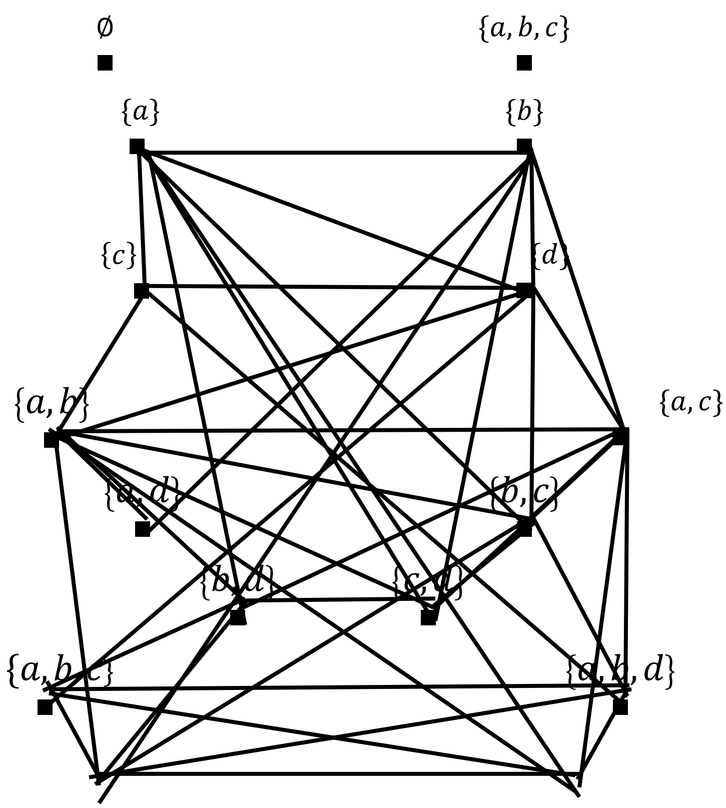

Figure 5: The graph of topology with set X and
$\tau =$
$\{\, \emptyset, X, \{c\}, \{b\}, \{a\}, \{d\}, \{a,d\}, \{a,c\}, \{a,b\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\},$
$\{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$

$\Delta(G) = 3$ and $\partial(G) = 0$

3. The graph of co-countable topology space

Let X is non empty set $\tau_{co} = \emptyset \cup \{A \subseteq X, X - A \ is \ \ countable\}$ then $(X, \tau_{coun})$is called co-countable topology space. If we define a collection τ_{co} on integers set $\mathbb{Z}$ by

$$\tau = \{\emptyset\ , \mathbb{Z}_e, \mathbb{Z}_o, \mathbb{Z}\}$$

Where $\mathbb{Z}_e$ and $\mathbb{Z}_o$ are the set of even and odd set numbers, respectively. Since $\mathbb{Z}$ is countable then the complement of $\mathbb{Z}_e$ and $\mathbb{Z}_o$ respect to $\mathbb{Z}$ are countable sets, there for $(\mathbb{Z}, \tau_{coun})$ is co-countable topological space. The topological graph G_τ of τ_{coun} shown in the figure.

Figure 6: graph the co-countable topology space with the set Z and $\tau_{coun} =$
$\{\emptyset, Z, Z_e\ , Z_o\ \}$.

In the previous example (Figure 6), we notice that the degree of $\mathbb{Z}_e$ and $\mathbb{Z}_o$vertical in the graph is one.

4. The graph of co-finite topology space

Let X is non empty set $\tau_{inf} = \emptyset \cup \{A \subseteq X, X - A \ is \ $ finite$\}$ then $(X, \tau_{inf})$is called co-finite topology space. If we define a collection τ_{inf} on integers set $X = N$ by

When $\tau_{inf} = \ \{\emptyset, N, N - \{3\},\ \ N - \{5\}, N - \{3,5\}\ \}$ see Figure 7.

Figure 8: The graph infinite topology space with the set N and

$$\tau_{inf} = \{\emptyset, N, N-\{3\}, \qquad N-\{5\}, N-\{3,5\}\}$$

5. The nested topological space graph

Let F be defined as $F = \{A_n = \{1, \dots, n\}, n \in \mathbb{N}\}$, a family of increasing Nested sets on $\mathbb{N}$. $A_i \subseteq A_{i+1}$, $\forall i \in \mathbb{N}$, and if $i \le j$, then $A_i \cap A_j = A_i$ and $\left|A_i \cap A_j\right| = i$, for each $i \le j$ in N. Given $\tau = F \cup \{\emptyset, \mathbb{N}\}$, the topological space $(\mathbb{N}, \tau)$ is referred to as nested topological space. If $i \le j$, then there are no edges connecting A_i and A_j, making the topological graph G simple, with

$$V(G) = \{V_0, V_1, \dots, V_n\}$$

Figure 8 illustrates this situation, where$v_0 = \emptyset$, $v_1 = A_1$, $v_2 = A_2$, … , $v_n = \mathbb{N}$, and $|V(G)|$ =0

Because τ is nested, there are 0 edges between the vertices V_n and V_m for each $n, m \in \mathbb{N}, A_n \cap A_m \neq \emptyset$, for some $k \in N$, let $|A_n \cap A_m| = k$. $\Delta(G) = 0$ and $\partial(G) = 0$ for a graph of nested topological space.

Example 3

If we define a collection τ on Natural set N by

$$\tau = \{\emptyset, N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{3\}, N \setminus \{1\}, N \setminus \{1,2\}, N \setminus \{1,3\}, N \setminus \{2,3\}\}$$

Where $N \setminus \{1\}$ Natural set numbers except $\{1\}$, respectively. Since N is infinite then the complement of respect $N \setminus \{1\}$ to N are finite set, there for (N, τ) is infinite topological space. The topological graph G of τ shown in the figure 9.

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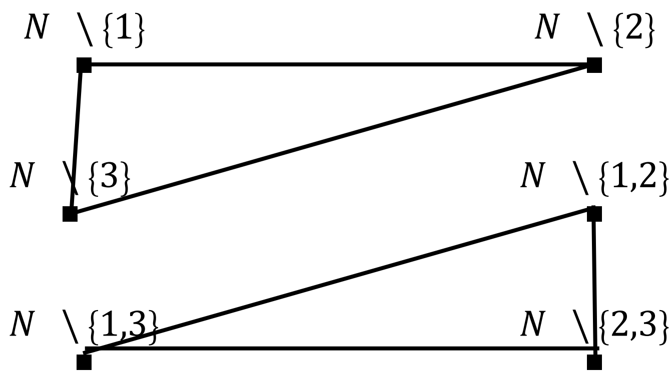

Figure 9: The graph topology space with the set N and $\tau = \{\emptyset, N, N \quad \{1\}, N \setminus \{2\}, N \setminus \{3\}, N \quad \{1\}, N \setminus \{1,2\}, N \setminus \{1,3\},$

$N \setminus \{2,3\}\}$

In the previous example, we notice that the degree of each vertical in the graph is 0 the number of edges between N and $N \setminus \{1\}$ is 0.

6. Knowing the topological value of a graph

We can know the structure of the topological space if we have the graph of the topological space. We know that the vertices in the graph represent the open sets in sub collection τ and the edge between tow vertical represent two non-partial sets.

If we have the graph topology in Figure10

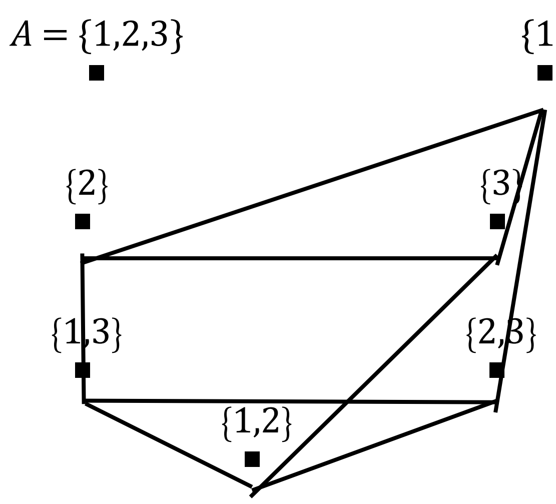

Figure10: graph

Then $V = \{A, \{1\}, \{2\}, \{3\}, \{12\}, \{1,3\}, \{23\}\}$ from Vwe can writ
$\tau = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{12\}, \{1,3\}, \{23\}\}$

Since

1) $\emptyset, X \in \tau$
2) $u, v \in \tau$ then $u \cap v \in \tau$
3) $u_i \in \tau$ then $\bigcup_{i=1}^{n} u_i \in \tau$

From 1,2 and 3 $(A, \tau)$is topology space. In this way, it is very easy to find every topological space by knowing the graph and checking the conditions of the topological space.

7. Conclusions

In light of the results of the research, the following can be concluded:

1- Topological spaces can be described in the form of a graph.
2- study many properties of this graph.
3- There are some topological spaces that are difficult to describe with a graph.
4- The sums of the topological space can be known through the graph, and the opposite is also true.
5- In a graph, it is necessary to know how to write vertices and edges and what is the relationship between them and the sets of the topology space

8. Suggestions of forth works

In this thesis we studied the most important characteristics of graph of some types of topological spaces and presented new ways to know the graph of some topological spaces. We can offer a suggestion to researchers interested in this field that they should study the rest of the types of topological spaces and describe them graphically, take some applications and examples in topological spaces, and try to draw them in a graphical form. Also, pay attention to setting the basic views when drawing the topological space in a graph form. Study more properties of topological graphs and apply the advantages of numerous parameters it. Such as: independent domination, complete domination, weak domination, strong domination, joint independent domination, two-way domination, etc.

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