DOI: 10.24996/ijs.2025.66.6.18

ISSN: 0067-2904

A New Modification of Adomian Decomposition Method for Solving Nonlinear Parabolic Partial Differential Equations

Shaheed N. Huseen*, **Mustafa Q. Karim**

Department of Mathematics, College of Computer Science and Mathematics, University of Thi-Qar, Iraq

Received: 16/12/2023 Accepted: 5/7/2024 Published: 30/6/2025

Abstract

In this article, a new modification of the Adomian decomposition method (ADM) called Adomian decomposition 𝕁-transform method (AD𝕁TM) is presented for finding analytical and approximate solutions of nonlinear parabolic partial differential equations. The AD𝕁TM is an aggregation of Adomian decomposition method and 𝕁-transform. Comparison of obtained result with exact solutions, modified variational iteration algorithm-II (MVIA-II), Laplace Adomian decomposition method (LADM), B-spline technique, Adomian decomposition method (ADM), modified variational iteration technique (MVIA) and homotopy perturbation transform method (HPTM) show that the AD𝕁TM is an accurate, efficient, and reliable method. All the iterative process in this work implemented via Wolfram Mathematica 13.

Keywords: Adomian decomposition method, 𝕁-transform, Nonlinear parabolic partial differential equations.

تعديل جديد لطريقة تفكيك أدومين لحل المعادلات التفاضلية الجزئية المكافئة غير الخطية

شهيد ناصر حسين* , مصطفى قاسم كريم

قسم الرياضيات كلية علوم الحاسوب والرياضيات، جامعة ذي قار ، ذي قار ، العراق.

الخلاصة

في هذه البحث، تم تقديم تعديل جديد لطريقة تفكيك أدومين يسمى طريقة تفكيك أدومين تحويل جي لإيجاد الحلول التقريبية للمعادلات التفاضلية الجزئية المكافئة غيرالخطية. إن طريقة تفكيك أدومين تحويل جي هي تركيب من طريقة تفكيك أدومين(ADM) وتحويل جي . مقارنة النتائج التي تم الحصول عليها مع الحلول المضبوطة وخوارزمية التكرار المتغيرة المعدلة (MVIA-II) وطريقة التجميع المثلثية B-spline وطريقة لابلاس تفكيك أدومين (LADM) وطريقة تحويل الاضطراب الهوموتوبي (HPTM) وطريقة تفكيك أدومين (ADM) وطريقة التكرار المتغيرة المعدلة (VITM) أظهرت أن الطريقة الجديدة هي طريقة دقيقة وفعالة وموثوقة. تم تنفيذ جميع العمليات التكرارية في هذا العمل عبر Wolfram Mathematica 13.

* Email: shn_n2002@yahoo.com

1. Introduction

Adomian's decomposition method [1, 2] homotopy perturbation method [3, 4] homotopy analysis method [5, 6], variational iteration method [7, 8] and q-homotopy analysis method [9, 10] and other methods have all been used to solve linear or nonlinear, ordinary or partial differential equations. In this work we proposed a new modification of Adomian decomposition method namely Adomian decomposition 𝕁-transform method (AD𝕁TM) which is a combination of Adomian decomposition method and 𝕁-transform [11] for solving a nonlinear parabolic PDEs of the general form [12].

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \alpha\phi + \beta\phi^r. \qquad (1)$$

Where α and β are non-zero real constants and r is a positive integer. Eq. (1) leads to three well known models. It becomes Allen-Cahn (AC) equation when $r = 3, \alpha = 1$ and $\beta = -1$, which has different applications in plasma physics, biology and quantum mechanics [13]. If β is replaced by $-\beta$ and $r = 3$, then Eq. (1) end up being Newell–Whitehead (NW) equation. While the equation turn out to be Fishers equation when $r = 2$ and $\beta = -\alpha$. The Allen-Cahn model is the essential model for various physical phenomena and helps as a model for the investigation of separation of phase in isotropic, binary and isothermal mixtures. Analytical and approximate solutions of this model can be found using various approaches like the Haar wavelet or transform technique [12], finite difference technique [14] and finite element scheme [15].

The NW equation which describes the dynamical behaviour near the bifurcation point for the Rayleigh–Benard convection of binary fluid mixtures [16]. Different approaches have been employed to solve (NW) equation like Adomian decomposition methods [17, 18], Variational iteration method [19], homotopy perturbation method [20] and Laplace Adomian Decomposition Method [21].

Fisher's equation is defined as the nonlinear reaction diffusion equation that describes the relationship between the diffusion and nonlinear multiplication of a species [22]. Numerical ,analytical and approximate techniques were utilized and implemented to solve this equation like, Galerkin and finite elements scheme [23], homotopy perturbation approach [24], Laplace q-Homotopy analysis method [25], modified variational iteration scheme [26] and Adomian decomposition method [27] .

This article is organized as following: Section 2 presents the definition and properties of the 𝕁-transform. Section 3 presents the basic idea of the AD𝕁TM. Section 4 is devoted to the convergence analysis. The numerical examples with numerical comparison presented in section 5. The numerical results and discussion are presented in section 6. Finally, conclusion is presented in Section 7.

2. 𝕁 –Transform

This section introduces fundamental concepts about the 𝕁 – transform that are utilized in this paper.

Definition 2.1: [11] The 𝕁-transform denoted by the symbol 𝕁 (.) is defined by:

$$\mathbb{J}[v(t)] = V[s,u] = u\int_0^\infty e^{\frac{-st}{u}}\, v(t)\ dt\ ,s > 0, u > 0.$$

$$\mathbb{J}[v(t)] = \lim_{\alpha\to\infty} u\int_0^\alpha e^{\frac{-st}{u}}\, v(t)\ dt\ ,s > 0, u > 0. \qquad (2)$$

Remark 2.2: [11] The 𝕁-transform possesses the following properties

i. $\mathbb{J}[1] = \frac{u^2}{s}$.

ii. $\mathbb{J}[t] = \frac{u^3}{s^2}$.

iii. $\mathbb{J}[t^n] = n!\frac{u^{n+2}}{s^{n+1}}$.

iv. $\mathbb{J}[e^{at}] = \frac{u^2}{s-au}$.

v. $\mathbb{J}[sin\ at] = \frac{au^3}{s^2+a^2u^2}$.

vi. $\mathbb{J}[cos\ at] = \frac{su^3}{s^2+a^2u^2}$.

Theorem 2.3: [11] Let $V[s,u]$ is the 𝕁 - transform of the $\phi(t)$ then

i. $\mathbb{J}\left[\frac{\partial\phi(x,t)}{\partial t}\right] = \frac{s}{u}V[x,s,u] - u\phi(x,0)$. (3.a)

ii. $\mathbb{J}\left[\frac{\partial^2\phi(x,t)}{\partial t^2}\right] = \frac{s^2}{u^2}V[x,s,u] - s\,\phi(x,0) - u\,\phi'(x,0)$. (3.b)

iii. $\mathbb{J}\left[\frac{\partial^n\phi(x,t)}{\partial t^n}\right] = \frac{s^n}{u^n}V[x,s,u] - \sum_{i=0}^{n-1}\frac{s^{n-(i+1)}}{u^{n-(i+2)}}\phi^{(i)}(x,0)$. (3.c)

where $V(x,s,u)$ is the 𝕁 - transform for $\phi(x,t)$.

3. The fundamental concepts of AD𝕁TM

Let us consider the general nonlinear parabolic Eq. (1) with initial conditions

$$\phi(x,0) = \varphi(x) \tag{4}$$

utilizing the 𝕁- transform on both sides of Eq. (1), we have

$$\mathbb{J}\left[\frac{\partial\phi}{\partial t}\right] = \mathbb{J}\left[\frac{\partial^2\phi}{\partial x^2}\right] + \mathbb{J}[\alpha\phi] + \mathbb{J}[\beta\phi^r]. \tag{5}$$

Using Eq. (3.a) we obtain

$$\frac{s}{u}\mathbb{J}\big(\phi(x,t)\big) - u\phi(x,0) = \mathbb{J}\left[\frac{\partial^2\phi}{\partial x^2}\right] + \mathbb{J}[\alpha\phi] + \mathbb{J}[\beta\phi^r]. \tag{6}$$

Hence,

$$\mathbb{J}[\phi(x,t)] = \frac{u^2}{s}\phi(x,0) + \frac{u}{s}\left(\mathbb{J}\left[\frac{\partial^2\phi}{\partial x^2}\right] + \alpha\,\mathbb{J}[\phi] + \beta\mathbb{J}[\phi^r]\right). \tag{7}$$

If we perform the inverse of the J-transform on Eq. (7), we get

$$\phi(x,t) = G(x,t) + \mathbb{J}^{-1}\left[\frac{u}{s}\left(\mathbb{J}\left[\frac{\partial^2\phi}{\partial x^2}\right] + \alpha\,\mathbb{J}[\phi] + \beta\mathbb{J}[\phi^r]\right)\right], \tag{8}$$

where

$$G(x,t) = \mathbb{J}^{-1}\left[\frac{u^2}{s}\phi(x,0)\right]. \tag{9}$$

The AD𝕁TM admits the decomposition into an infinite series of components

$$\phi(x,t) = \sum_{m=0}^{\infty}\phi_m(x,t). \tag{10}$$

The nonlinear term $N(\phi) = \phi^r(x,t)$ be equated to an infinite series polynomial

$$N(\phi) = \sum_{m=0}^{\infty}A_m, \tag{11}$$

where A_m are Adomian polynomials, which can be determined by

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[F\left(\sum_{i=0}^{m}\lambda^i\phi_i(x,t)\right)\right]_{\lambda=0}, m = 0,1,2,\ldots \tag{12}$$

Substituting Eq. (10) and Eq. (11) in Eq. (8) gives

$$\sum_{m=0}^{\infty}\phi_m(x,t) = G(x,t) + \mathbb{J}^{-1}\left[\frac{u}{s}\left(\mathbb{J}\left[\frac{\partial^2}{\partial x^2}\sum_{m=0}^{\infty}\phi_m(x,t)\right] + \alpha\,\mathbb{J}\left[\sum_{m=0}^{\infty}\phi_m\right] + \beta\mathbb{J}\left[\sum_{m=0}^{\infty}A_m\right]\right)\right]. \tag{13}$$

The recursive relationship is found to be

$$\begin{cases}\phi_0(x,t) = G(x,t)\\ \phi_{m+1}(x,t) = \mathbb{J}^{-1}\left[\frac{u}{s}\left(\mathbb{J}\left[\frac{\partial^2}{\partial x^2}\phi_m\right] + \alpha\,\mathbb{J}[\phi_m] + \beta\mathbb{J}[A_m]\right)\right].\end{cases} \tag{14}$$

The AD𝕁TM series solution is

$$\Phi^k(x,t) = \sum_{m=0}^{k} \phi_m(x,t). \tag{15}$$

As $k \longrightarrow \infty$ we can get an accurate approximation to Eq. (1).

4. Convergence analysis

This section introduces the sufficient conditions to ensure the existence of a unique solution and discusses the convergence of this solution. We will study the convergence analysis as same manner in [28].

Theorem 4.1: (Uniqueness Theorem) Eq. (14) has a unique solution whenever $0 < \gamma < 1$ where, $\gamma = (L_1 + L_2)t$.

Proof. Let $X = (C[I], \|.\|)$be the Banach space of all continuous functions on I = [0, T] with the norm $\|\phi\| = \max_{t\in I}|\phi|$, we define a mapping $F: X \to X$ where

$$\phi_{m+1}(x,t) = G(x,t) + \mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}[\, R[\phi_m(x,t)] + N[\phi_m(x,t)]]\right].$$

Where $R(\phi(x,t)) = \left(\frac{\partial^2}{\partial x^2} + \alpha\right)(\phi(x,t))$ and $N(\phi(x,t)) = \beta\phi^r(x,t)$. Now, assume N and R are Lipschitzian with $|R(\phi) - R(\hat{\phi})| \le L_1|\phi - \hat{\phi}|$ and $|N(\phi) - N(\hat{\phi})| \le L_2|\phi - \hat{\phi}|$ for all $\phi, \hat{\phi} \in C[I]$ where L_1 and L_2 are Lipschitz constants.

$$\begin{aligned}
\|F\phi - F\hat{\phi}\| &= \max_{t\in I}\left|\begin{array}{l} \mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}[\, R[\phi(x,t)] + N[\phi(x,t)]]\right] \\ -\mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}[\, R[\hat{\phi}(x,t)] + N[\hat{\phi}(x,t)]]\right] \end{array}\right| \\
&= \max_{t\in I}\left|\begin{array}{l} \mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}[\, R[\phi(x,t)] - R[\hat{\phi}(x,t)]]\right] \\ +\mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}[N[\phi(x,t) - N[\hat{\phi}(x,t)]]\right] \end{array}\right| \\
&\le \max_{t\in I}\left[\begin{array}{l} L_1\mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}|\phi(x,t) - \hat{\phi}(x,t)|\right] \\ + L_2\,\mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}|\phi(x,t) - \hat{\phi}(x,t)|\right] \end{array}\right] \\
&\le \max_{t\in I}(L_1 + L_2)\mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}|\phi(x,t) - \hat{\phi}(x,t)|\right] \\
&\le (L_1 + L_2)\mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}\|\phi(x,t) - \hat{\phi}(x,t)\|\right] \\
&= (L_1 + L_2)t\|\phi(x,t) - \hat{\phi}(x,t)\|.
\end{aligned}$$

Under the condition $0 < \gamma < 1$, the mapping is contraction. Thus, by Banach fixed point theorem for contraction, there exists a unique solution to Eq. (1).

□

Theorem 4.2: (Convergence Theorem) The solution of Eq. (1) using ADJTM is convergent.

Proof. Let $\delta_n = \sum_{i=0}^{n} \phi_i(x,t)$, be the n^{th} partial sum; let δ_n and δ_m be arbitrary partial sums with $n \ge m$. We are going to prove that $\{\delta_n\}$ is a Cauchy sequence in the Banach space X.Using a new formulation of Adomian polynomials we gain

$$\hat{N}(\delta_n) = \sum_{i=0}^{n} \hat{A}_i \,.$$

Now,

$$\begin{aligned}
\|\delta_n - \delta_m\| &= \max_{t\in I}|\delta_n - \delta_m| = \max_{t\in I}\left|\sum_{i=m+1}^{n} \hat{\phi}_i(x,t)\right| \\
&= \max_{t\in I}\left|\mathbb{J}^{-1}\left(\frac{u}{s}\mathbb{J}[R(\textstyle\sum_{i=m+1}^{n}\phi_{i-1})]\right) + \mathbb{J}^{-1}\left(\frac{u}{s}\mathbb{J}[(\textstyle\sum_{i=m+1}^{n}A_{i-1})]\right)\right| \\
&= \max_{t\in I}\left|\mathbb{J}^{-1}\left(\frac{u}{s}\mathbb{J}[R(\textstyle\sum_{i=m}^{n-1}\phi_i)]\right) + \mathbb{J}^{-1}\left(\frac{u}{s}\mathbb{J}[\textstyle\sum_{i=m}^{n-1}(A_i)]\right)\right| \\
&= \max_{t\in I}\left|\mathbb{J}^{-1}\left(\frac{u}{s}\mathbb{J}[R(\delta_{n-1}) - R(\delta_{m-1})]\right) + \mathbb{J}^{-1}\left(\frac{u}{s}\mathbb{J}[N(\delta_{n-1}) - N(\delta_{m-1})]\right)\right| \\
&\le \max_{t\in I}(L_1 + L_2)\left[\mathbb{J}^{-1}\left(\frac{u}{s}\mathbb{J}[|\delta_{n-1} - \delta_{m-1}|]\right)\right] \\
&= (L_1 + L_2)t\,\|\delta_{n-1} - \delta_{m-1}\|.
\end{aligned}$$

Let $n = m + 1$; then

$$\|\delta_{m+1} - \delta_m\| \le \gamma\|\delta_m - \delta_{m-1}\| \le \gamma^2\|\delta_{m-1} - \delta_{m-2}\| \le \cdots \le \gamma^m\|\delta_1 - \delta_0\|.$$

Where $\gamma = (L_1 + L_2)t$ also, from the triangle inequality we have

$$\begin{aligned}\|\delta_n - \delta_m\| &\le \|\delta_{m+1} - \delta_m\| + \|\delta_{m+2} - \delta_{m+1}\| + \cdots + \|\delta_n - \delta_{n-1}\| \\ &\le (\gamma^m + \gamma^{m+1} + \cdots + \gamma^{n-1})\|\delta_1 - \delta_0\| \\ &\le \gamma^m\left(\frac{1-\gamma^{n-m}}{1-\gamma}\right)\|\phi_1\|.\end{aligned}$$

Since $0 < \gamma < 1$, so $1 - \gamma^{n-m} < 1$, then

$$\|\delta_n - \delta_m\| \le \left(\frac{\gamma^m}{1-\gamma}\right)\max_{t\in I}|\phi_1|$$

However, $|\phi_1| < \infty$ therefore, as $m \to \infty$ then $\|\delta_n - \delta_m\| \to 0$, hence $\{\delta_n\}$ is a Cauchy sequence in X thus, the series $\sum_{i=0}^{\infty}\phi_i(x,t)$ converges and the proof is complete. □

Theorem 4.3: (Error estimate) The maximum absolute truncation error of Eq. (15) to Eq. (1) is estimated to be:

$$\max_{t\in I}\left|\phi(x,t) - \sum_{i=0}^{m}\phi_i(x,t)\right| \le \frac{\gamma^m}{1-\gamma}\max_{t\in I}|\phi_1|.$$

Proof. From Eq. (15) and Theorem 4.2 we have

$$\|\delta_n - \delta_m\| \le \frac{\gamma^m}{1-\gamma}\max_{t\in I}|\phi_1|.$$

as $n \to \infty$ then $\delta_n \to \phi(x,t)$ so we have

$$\|\phi(x,t) - \delta_m\| \le \frac{\gamma^m}{1-\gamma}\max_{t\in I}|\phi_1|.$$

Finally, the maximum absolute truncation error in the interval I is

$$\max_{t\in I}|\phi(x,t) - \sum_{i=0}^{m}\phi_i(x,t)| \le \frac{\gamma^m}{1-\gamma}\max_{t\in I}|\phi_1|.$$ □

5. Numerical examples

In this section, the approximate solutions of five various kinds of nonlinear parabolic partial differential equations have been presented. Two of them are for AC model, two for NW model and one for the Fisher model.

Example 5.1: Consider the Eq. (1) with $r = 3, \alpha = 1$ and $\beta = -1$, which gives the Allen-Cahn equation of the form [29].

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + \phi - \phi^3, \quad (16)$$

subject to the following initial-boundary conditions

$$\phi(x,0) = -0.5 + 0.5\tanh(0.3536\,x). \quad (17)$$

$$\phi(0,t) = -0.5 + 0.5\tanh(0.75t). \quad (18)$$

$$\phi(1,t) = -0.5 + 0.5\tanh(0.3536 - 0.75t). \quad (19)$$

The exact solution of this problem is [29]

$$\phi(x,t) = -0.5 + 0.5\tanh(0.3536x - 0.75t). \quad (20)$$

Applying $\mathbb{J}$- transform on Eq. (16) subject to the condition (17), we obtain

$$\frac{s}{u}\mathbb{J}[\phi(x,t)] - u\phi(x,0) = \mathbb{J}[\frac{\partial^2\phi}{\partial x^2}] + \mathbb{J}[\phi] - \mathbb{J}[\phi^3]. \quad (21)$$

$$\mathbb{J}[\phi(x,t)] = \frac{u^2}{s}[-0.5 + 0.5\tanh(0.3536\,\mathrm{x})] + \frac{u}{s}\mathbb{J}[\frac{\partial^2\phi}{\partial x^2} + \phi - \phi^3]. \quad (22)$$

Apply the inverse $\mathbb{J}$- transform on Eq. (22), we gain

$$\phi(x,t) = -0.5 + 0.5\tanh(0.3536\,x) + \mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}\left[\frac{\partial^2\phi}{\partial x^2} + \phi - \phi^3\right]\right]. \quad (23)$$

From the AD$\mathbb{J}$TM, rewrite Eq. (23) as follows

$$\sum_{m=0}^{\infty}\phi_m(x,t) = -0.5 + 0.5\tanh(0.3536\,x) + \mathbb{J}^{-1}\left[\frac{u}{s}\,\mathbb{J}\left[\frac{\partial^2}{\partial x^2}\sum_{m=0}^{\infty}\phi_m(x,t) + \sum_{m=0}^{\infty}\phi_m(x,t) - \sum_{m=0}^{\infty}A_m\right]\right]. \quad (24)$$

Where, A_m are Adomian polynomials. Using Eq. (12) A_m can be deduced as follows
$A_0 = \phi_0^3$, $A_1 = 3\phi_1\phi_0^2$, $A_2 = 3\phi_0\phi_1^2 + 3\phi_0^2\phi_2)$, ...
Using Eq. (14) we have

$$\phi_0 = -0.5 + 0.5\text{Tanh}[0.3536x]$$

$$\phi_1 = t(-0.375 + (0.125 - 0.12503296000000003\text{Sech}[0.3536x]^2)\text{Tanh}[0.3536x] + 0.375\text{Tanh}[0.3536x]^2 - 0.125\text{Tanh}[0.3536x]^3)$$

$$\phi_2 = 0.015645604345446 42t^2(-2.996049175540014 - 16.977611994726747\text{Tanh}[0.3536x] + 17.976295053240083\text{Tanh}[0.3536x]^2 + \cdots$$

$$\phi_3 = 0.006502170962456641t^3(10.21293509251416 - 25.031703658122943\text{Tanh}[0.3536x] - 28.23576172636267\text{Tanh}[0.3536x]^2 + 69.48800935494928\text{Tanh}[0.3536x]^3 - \cdots$$

$$\vdots$$

Hence, By Eq. (15) we have the 3rd order approximate solution of ADJTM:

$$\Phi^{(3)} = -0.5 - 0.375t - 0.046875t^2 + 0.06640625t^3 + t^3\text{Sech}[0.3536x]^6(-0.0234498616295424 + 0.006502170962456641\text{Tanh}[0.3536x]) + (0.5 + 0.125t - 0.265625t^2 - 0.16276041666666669t^3)\text{Tanh}[0.3536x] + \cdots$$

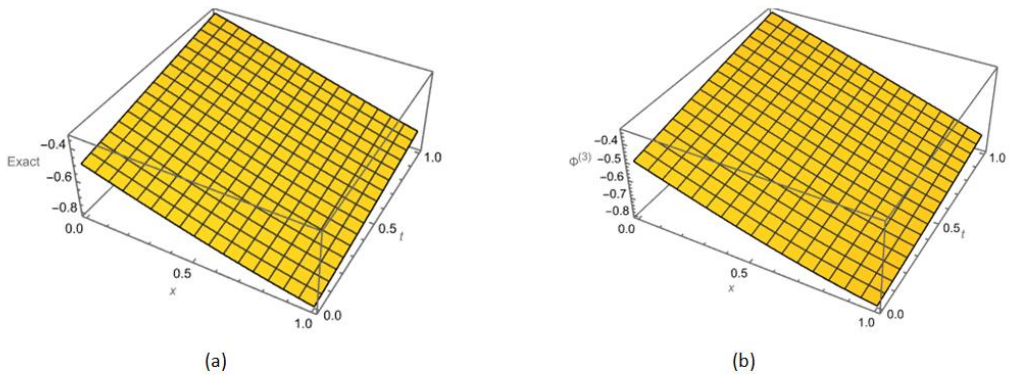


Figure 1: (a) The exact solution (b) $\Phi^{(3)}$ of ADJTM for Example 5.1.

Table 1: Comparison of exact solution, $\Phi^{(3)}$ of ADJTM, MVIA-II and TBS of Example 5.1.

x	t	ADJTM $\Phi^{(3)}$	Exact solution	TBS [29]	MVIA-II [30]
0.1	0.001	-0.48270190	−0.48270191	−0.48294667	−0.48268570
0.2	0.002	-0.46544517	−0.46544517	−0.46584916	−0.46541289
0.3	0.003	-0.44827076	−0.44827076	−0.44881129	−0.44822262
0.4	0.004	-0.43121886	−0.43121885	−0.43186126	−0.43115521
0.5	0.005	-0.4143285	−0.41432848	−0.41504090	−0.41424978
0.6	0.006	-0.39763721	−0.39763717	−0.39839276	−0.39754397
0.7	0.007	-0.38118069	−0.38118064	−0.38195995	−0.38107357
0.8	0.008	-0.36499256	−0.36499250	−0.36571227	−0.36487227
0.9	0.009	-0.34910408	−0.34910400	−0.34910129	−0.34897140

Table 2: Comparison of the absolute errors of $\Phi^{(3)}$ of ADJTM, MVIA-II and TBS of Example 5.1.

x	$t = 0.001$			$t = 0.005$			$t = 0.01$		
	ADJTM $\Phi^{(3)}$	**TBS** [29]	**MVIA-II** [30]	**ADJTM** $\Phi^{(3)}$	**TBS** [29]	**MVIA-II** [30]	**ADJTM** $\Phi^{(3)}$	**TBS** [29]	**MVIA-II** [30]
0.1	1.150E-9	2.448E-4	1.620E-5	5.502E-9	9.687E-4	8.101E-5	1.040E-8	1.606E-3	1.621E-4
0.2	2.302E-9	2.000E-4	1.614E-5	1.126E-8	1.016E-3	8.073E-5	2.194E-8	1.935E-3	1.615E-4
0.3	3.431E-9	1.797E-4	1.604E-5	1.690E-8	9.082E-4	8.024E-5	3.327E-8	1.826E-3	1.606E-4
0.4	4.527E-9	1.594E-4	1.590E-5	2.239E-8	8.052E-4	7.956E-5	4.427E-8	1.632E-3	1.593E-4
0.5	5.578E-9	1.410E-4	1.572E-5	2.765E-8	7.124E-4	7.869E-5	5.485E-8	1.444E-3	1.576E-4
0.6	6.577E-9	1.242E-4	1.551E-5	3.266E-8	6.280E-4	7.764E-5	6.491E-8	1.273E-3	1.555E-4
0.7	7.514E-9	1.091E-4	1.526E-5	3.736E-8	5.520E-4	7.643E-5	7.437E-8	1.116E-3	1.531E-4
0.8	8.383E-9	9.502E-5	1.499E-5	4.172E-8	4.824E-4	7.505E-5	8.316E-8	9.434E-4	1.504E-4
0.9	9.177E-9	8.886E-5	1.468E-5	4.572E-8	3.725E-4	7.535E-5	9.121E-8	6.534E-4	1.441E-4

Example 5.2: Consider the Eq. (1) with $r = 3, \alpha = 1$ and $\beta = -1$, which gives the Allen-Cahn equation of the form [29].

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^3 , \tag{25}$$

subject to the following initial-boundary conditions

$$\phi(x,0) = \left(1 + e^{-\frac{\sqrt{2}}{2}x}\right)^{-1}. \tag{26}$$

$$\phi(0,t) = \left(1 + e^{-\frac{3}{2}t}\right)^{-1}. \tag{27}$$

$$\phi(1,t) = \left(1 + e^{-\frac{\sqrt{2}}{2}(1+\frac{3\sqrt{2}}{2}t)}\right)^{-1}. \tag{28}$$

The exact solution of this problem is [29].

$$\phi(x,t) = \left(1 + e^{-\frac{\sqrt{2}}{2}(x+\frac{3\sqrt{2}}{2}t)}\right)^{-1}. \tag{29}$$

Applying $\mathbb{J}$- transform on Eq. (25) subject to the condition (26), we obtain

$$\mathbb{J}[\phi(x,t)] = \frac{u^2}{s}[\left(1 + e^{-\frac{\sqrt{2}}{2}x}\right)^{-1}] + \frac{u}{s}\mathbb{J}[\frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^3]. \tag{30}$$

Apply the inverse $\mathbb{J}$- transform on Eq. (30), we gain

$$\phi(x,t) = \left(1 + e^{-\frac{\sqrt{2}}{2}x}\right)^{-1} + \mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}\left[\frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^3\right]\right]. \tag{31}$$

From the ADJTM, rewrite Eq. (31) as follows

$$\sum_{m=0}^{\infty} \phi_m(x,t) = \left(1 + e^{-\frac{\sqrt{2}}{2}x}\right)^{-1} + \mathbb{J}^{-1}\left[\frac{u}{s}\mathbb{J}\left[\frac{\partial^2}{\partial x^2}\sum_{m=0}^{\infty} \phi_m(x,t) + \sum_{m=0}^{\infty} A_m\right]\right]. \tag{32}$$

Where, A_m are Adomian polynomials. Using Eq. (12) A_m can be deduced as follows
$A_0 = \phi_0^3,\ A_1 = 3\phi_0^2\phi_1,\ A_2 = 3\phi_0(\phi_1^2 + \phi_0\phi_2),\ A_3 = \phi_1^3 + 6\phi_0\phi_1\phi_2 + 3\phi_0^2\phi_3, \ldots$
Using Eq. (14) we have

$$\phi_0 = \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}} \quad , \quad \phi_1 = \frac{3e^{\frac{x}{\sqrt{2}}}t}{2(1+e^{\frac{x}{\sqrt{2}}})^2}, \quad \phi_2 = -\frac{9e^{\frac{x}{\sqrt{2}}}(-1+e^{\frac{x}{\sqrt{2}}})t^2}{8(1+e^{\frac{x}{\sqrt{2}}})^3}, \quad \phi_3 = \frac{9e^{\frac{x}{\sqrt{2}}}(1-4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x})t^3}{16(1+e^{\frac{x}{\sqrt{2}}})^4},$$

$$\phi_4 = -\frac{27e^{\frac{x}{\sqrt{2}}}(-1+e^{\frac{x}{\sqrt{2}}})(1-10e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x})t^4}{128(1+e^{\frac{x}{\sqrt{2}}})^5}.$$

Hence, By Eq. (15) we have

$$\Phi^{(4)} = \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}} + \left(-\frac{1}{\left(1+e^{-\frac{x}{\sqrt{2}}}\right)^3} + \frac{e^{-\sqrt{2}x}}{\left(1+e^{-\frac{x}{\sqrt{2}}}\right)^3} - \frac{e^{-\frac{x}{\sqrt{2}}}}{2\left(1+e^{-\frac{x}{\sqrt{2}}}\right)^2} + \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}}\right)t$$
$$-\frac{9e^{\frac{x}{\sqrt{2}}}\left(-1+e^{\frac{x}{\sqrt{2}}}\right)t^2}{8\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3} + \frac{9e^{\frac{x}{\sqrt{2}}}\left(1-4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x}\right)t^3}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4}$$
$$-\frac{27e^{\frac{x}{\sqrt{2}}}\left(-1+e^{\frac{x}{\sqrt{2}}}\right)\left(1-10e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x}\right)t^4}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}.$$

We note that the absolute error is largely dropped when modifying the solution by taking further terms. Table 3 and Figure 3 are explained this fact.

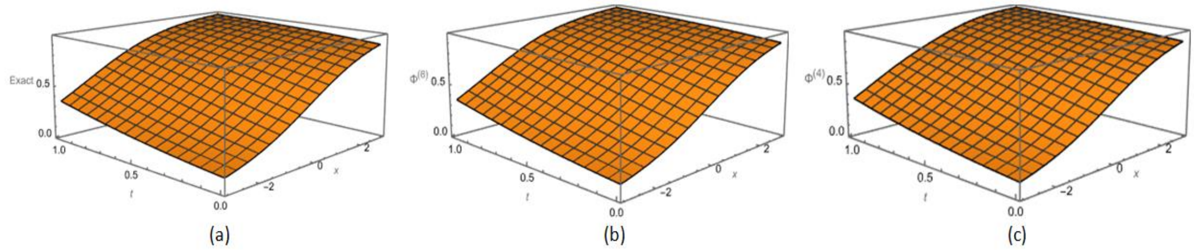


Figure 2: (a) Exact solution, (b) $\Phi^{(6)}$ of ADJTM and (c) $\Phi^{(4)}$ of ADJTM for Example 5.2.

Table 3: Comparison of the absolute errors of $\Phi^{(4)}$ and $\Phi^{(6)}$ of ADJTM, MVIA-II and TBS of Example 5.2.

x	$t = 0.001$			$t = 0.005$			$t = 0.01$			
	ADJTM Φ^4	**TBS** [29]	**MVIA-II** [30]	**ADJTM Φ^4**	**TBS** [29]	**MVIA-II** [30]	**ADJTM Φ^4**	**TBS** [29]	**MVIA-II** [30]	**ADJTM Φ^6**
0.1	4.250E-17	2.509E-4	3.765E-5	4.891E-14	1.127E-3	2.882E-4	1.564E-12	2.108E-3	5.763E-4	2.880E-17
0.2	7.478E-17	3.351E-4	5.744E-5	4.739E-14	1.715E-3	2.871E-4	1.513E-12	3.446E-3	5.738E-4	1.636E-17
0.3	8.824E-17	4.389E-4	5.708E-5	4.477E-14	2.233E-3	2.852E-4	1.431E-12	4.555E-3	5.700E-4	1.761E-17
0.4	2.625E-17	5.456E-4	5.658E-5	4.136E-14	2.768E-3	2.827E-4	1.321E-12	5.639E-3	5.648E-4	1.026E-16
0.5	2.807E-17	6.542E-4	5.595E-5	3.707E-14	3.314E-3	2.795E-4	1.187E-12	6.735E-3	5.582E-4	5.848E-17
0.6	3.061E-17	7.626E-4	5.519E-5	3.242E-14	3.857E-3	2.756E-4	1.033E-12	7.830E-3	5.505E-4	7.851E-17
0.7	5.534E-17	8.697E-4	5.432E-5	2.712E-14	4.399E-3	2.712E-4	8.642E-13	8.850E-3	5.415E-4	1.726E-16
0.8	2.070E-17	9.653E-4	5.333E-5	2.147E-14	4.902E-3	2.662E-4	6.862E-13	9.353E-3	5.314E-4	7.802E-17
0.9	3.703E-17	1.161E-3	5.224E-5	1.599E-14	4.617E-3	2.607E-4	5.041E-13	7.685E-3	5.203E-4	7.222E-17

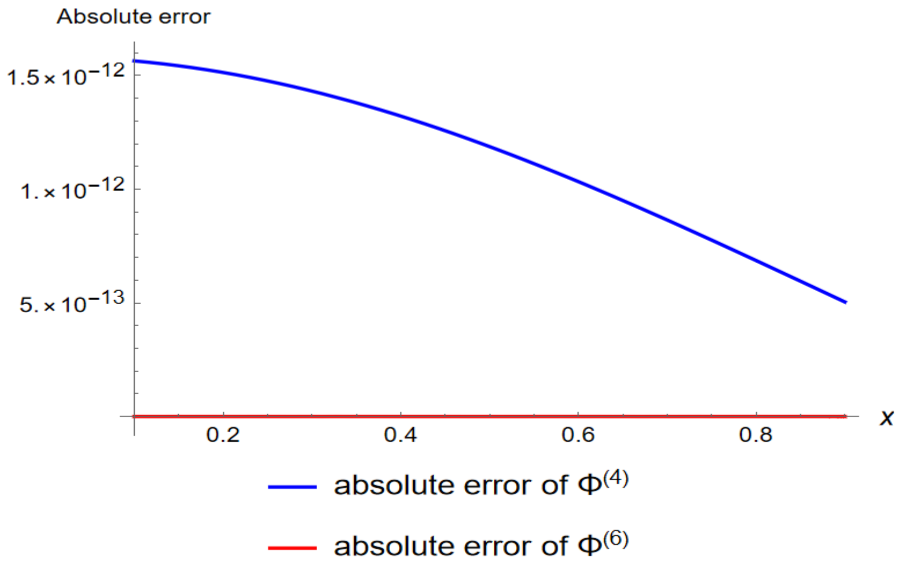


Figure 3: Comparison of the absolute errors of $\Phi^{(4)}$ and $\Phi^{(6)}$ of Example 5.2.

Example 5.3: Consider the Eq. (1) with $r = 2, \alpha = 1$ and $\beta = -1$, which gives the Newell–Whitehead equation of the form [31].

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^2. \tag{33}$$

Subject to the following initial conditions

$$\phi(x,0) = \left(1 + e^{\frac{x}{\sqrt{6}}}\right)^{-2}. \tag{34}$$

The exact solution of this problem is [31]

$$\phi(x,t) = \left(1 + e^{\frac{x}{\sqrt{6}} - \frac{5}{6}t}\right)^{-2}. \tag{35}$$

Applying $\mathbb{J}$- transform on Eq. (33) subject to the condition (34), we obtain

$$\mathbb{J}[\phi(x,t)] = \frac{u^2}{s}\left[\left(1 + e^{\frac{x}{\sqrt{6}}}\right)^{-2}\right] + \frac{u}{s}\,\mathbb{J}\left[\frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^2\right]. \tag{36}$$

Apply the inverse $\mathbb{J}$- transform on Eq. (36), we gain

$$\phi(x,t) = \left(1 + e^{\frac{x}{\sqrt{6}}}\right)^{-2} + \mathbb{J}^{-1}\left[\frac{u}{s}\,\mathbb{J}\left[\frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^2\right]\right]. \tag{37}$$

From the AD$\mathbb{J}$TM, rewrite Eq. (37) as follows

$$\sum_{m=0}^{\infty} \phi_m(x,t) = \left(1 + e^{\frac{x}{\sqrt{6}}}\right)^{-2} + \mathbb{J}^{-1}\left[\frac{u}{s}\,\mathbb{J}\left[\frac{\partial^2}{\partial x^2}\sum_{m=0}^{\infty} \phi_m(x,t) + \sum_{m=0}^{\infty} \phi_m(x,t) - \sum_{m=0}^{\infty} A_m\right]\right]. \tag{38}$$

Where, A_m are Adomian polynomials. Using Eq. (12) A_m can be deduced as follows
$A_0 = \phi_0^2,\ A_1 = 2\phi_0\phi_1,\ A_2 = \phi_1^2 + 2\phi_0\phi_2,\ A_3 = 2\phi_1\phi_2 + 2\phi_0\phi_3, \ldots$
Using Eq. (14) we have

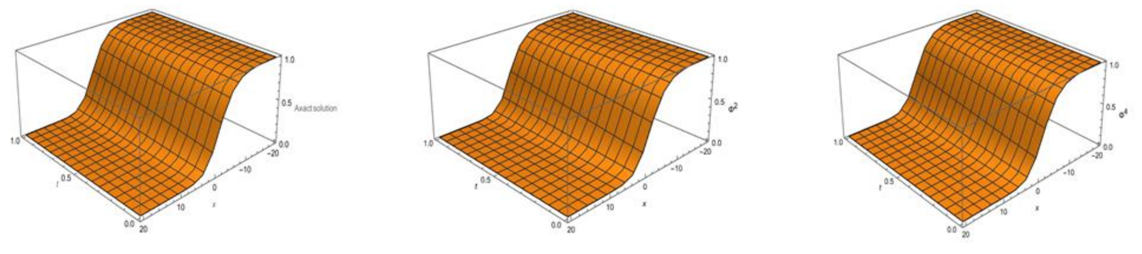
$$\phi_0 = \left(1 + e^{\frac{x}{\sqrt{6}}}\right)^{-2},\ \phi_1 = \frac{5e^{\frac{x}{\sqrt{6}}}t}{3(1+e^{\frac{x}{\sqrt{6}}})^3},\ \phi_2 = \frac{25e^{\frac{x}{\sqrt{6}}}(-1+2e^{\frac{x}{\sqrt{6}}})t^2}{36(1+e^{\frac{x}{\sqrt{6}}})^4},\ \phi_3 = \frac{125e^{\frac{x}{\sqrt{6}}}(1+4e^{\sqrt{\frac{2}{3}}x}-7e^{\frac{x}{\sqrt{6}}})t^3}{648(1+e^{\frac{x}{\sqrt{6}}})^5}$$

$$\phi_4 = \frac{625e^{\frac{x}{\sqrt{6}}}(-1-33e^{\sqrt{\frac{2}{3}}x}+8e^{\sqrt{\frac{3}{2}}x}+18e^{\frac{x}{\sqrt{6}}})t^4}{15552(1+e^{\frac{x}{\sqrt{6}}})^6},\ \ldots.$$

By Eq. (15) we have

$$\Phi^{(4)}=\frac{1}{(1+e^{\frac{x}{\sqrt{6}}})^2}+\left(-\frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^4}+\frac{e^{\sqrt{\frac{2}{3}}x}}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^4}-\frac{e^{\frac{x}{\sqrt{6}}}}{3\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3}+\frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2}\right)t+\frac{25e^{\frac{x}{\sqrt{6}}}\left(-1+2e^{\frac{x}{\sqrt{6}}}\right)t^2}{36\left(1+e^{\frac{x}{\sqrt{6}}}\right)^4}+$$
$$\frac{125e^{\frac{x}{\sqrt{6}}}(1+4e^{\sqrt{\frac{2}{3}}x}-7e^{\frac{x}{\sqrt{6}}})t^3}{648(1+e^{\frac{x}{\sqrt{6}}})^5}+\frac{625e^{\frac{x}{\sqrt{6}}}(-1-33e^{\sqrt{\frac{2}{3}}x}+8e^{\sqrt{\frac{3}{2}}x}+18e^{\frac{x}{\sqrt{6}}})t^4}{15552(1+e^{\frac{x}{\sqrt{6}}})^6}.$$

We note that the absolute error is largely dropped when modifying the solution by taking further terms. Table 4 explain this fact.



(a) (b) (c)

Figure 4: (a) Exact solution, (b) $\Phi^{(2)}$ of ADJTM and (c) $\Phi^{(4)}$ of ADJTM for Example 5.3.

Table 4: Comparison of the absolute errors of $\Phi^{(2)}$ and $\Phi^{(4)}$ of ADJTM, LTDM and VITM with $t = 0.0001$ of Example 5.3.

x	ADJTM $\Phi^{(2)}$	LTDM and VITM [31]	ADJTM $\Phi^{(4)}$
0.0	1.20956 E-14	1.0000000000E-10	3.87426 E-17
0.1	1.10653 E-14	5.0941566800E-11	1.18119 E-17
0.2	1.00293 E-14	8.1583004800E-11	1.21851 E-17
0.3	9.02068 E-15	1.3295068340E-10	6.56684 E-17
0.4	7.88704 E-15	1.7637132220E-10	1.25649 E-17
0.5	6.82862 E-15	1.5669454130E-10	4.56336 E-17
0.6	5.71687 E-15	1.0301314750E-10	2.89665 E-17
0.7	4.56864 E-15	2.3924337590E-10	2.77911 E-17
0.8	3.47356 E-15	2.9445508530E-10	4.20051 E-17
0.9	2.43401 E-15	3.1297190840E-10	1.79653 E-17
1.0	1.44329 E-15	2.6399135400E-10	2.77556 E-17

Example 5.4: Consider the Eq. (1) with $r = 4, \alpha = 1$ and $\beta = -1$, which gives the Newell–Whitehead equation of the form [32].

$$\frac{\partial \phi}{\partial t}=\frac{\partial^2 \phi}{\partial x^2}+\phi-\phi^4\,, 0<x<1, t>0, \tag{39}$$

subject to the following initial conditions

$$\phi(x,0)=(1+e^{\frac{3}{\sqrt{10}}x})^{\frac{-2}{3}}. \tag{40}$$

The exact solution of this problem is [32]

$$\phi(x,t)=(\tfrac{1}{2}\text{Tanh}[-\tfrac{3}{2\sqrt{10}}(x-\tfrac{7}{\sqrt{10}}t)]+\tfrac{1}{2})^{\frac{2}{3}}. \tag{41}$$

Applying $\mathbb{J}$- transform on Eq. (39) subject to the condition (40), we obtain

$$\mathbb{J}[\phi(x,t)]=\frac{u^2}{s}[(1+e^{\frac{3}{\sqrt{10}}x})^{\frac{-2}{3}}]+\frac{u}{s}\,\mathbb{J}[\frac{\partial^2 \phi}{\partial x^2}+\phi-\phi^4]. \tag{42}$$

Apply the inverse 𝕁- transform on Eq. (42), we gain

$$\phi(x,t) = (1+e^{\frac{3}{\sqrt{10}}x})^{\frac{-2}{3}} + \mathbb{J}^{-1}\left[\frac{u}{s}\,\mathbb{J}\left[\frac{\partial^2\phi}{\partial x^2} + \phi - \phi^4\right]\right]. \tag{43}$$

From the AD𝕁TM, rewrite Eq. (43) as follows

$$\sum_{m=0}^{\infty}\phi_m(x,t) = \left(1+e^{\frac{x}{\sqrt{6}}}\right)^{-2} + \mathbb{J}^{-1}\left[\frac{u}{s}\,\mathbb{J}\left[\frac{\partial^2}{\partial x^2}\sum_{m=0}^{\infty}\phi_m(x,t) + \sum_{m=0}^{\infty}\phi_m(x,t) - \sum_{m=0}^{\infty}A_m\right]\right]. \tag{44}$$

Where, A_m are Adomian polynomials. Using Eq. (12) A_m can be deduced as follows
$A_0 = \phi_0^4,\ A_1 = 4\phi_0^3\phi_1, A_2 = 2\phi_0^2(3\phi_1^2 + 2\phi_0\phi_2), A_3 = 4\phi_0(\phi_1^3 + 3\phi_0\phi_1\phi_2 + \phi_0^2\phi_3), \ldots$
Using Eq. (14) we have

$$\phi_0 = (1+e^{\frac{3}{\sqrt{10}}x})^{\frac{-2}{3}}, \quad \phi_1 = \frac{7e^{\frac{3x}{\sqrt{10}}}t}{5(1+e^{\frac{3x}{\sqrt{10}}})^{5/3}}, \quad \phi_2 = \frac{49e^{\frac{3x}{\sqrt{10}}}(-3+2e^{\frac{3x}{\sqrt{10}}})t^2}{100(1+e^{\frac{3x}{\sqrt{10}}})^{8/3}}, \quad \phi_3 = \frac{343e^{\frac{3x}{\sqrt{10}}}(9+4e^{3\sqrt{\frac{2}{5}}x}-27e^{\frac{3x}{\sqrt{10}}})t^3}{3000(1+e^{\frac{3x}{\sqrt{10}}})^{11/3}}, \quad \phi_4 = \frac{2401e^{\frac{3x}{\sqrt{10}}}(-27-171e^{3\sqrt{\frac{2}{5}}x}+234e^{\frac{3x}{\sqrt{10}}}+8e^{\frac{9x}{\sqrt{10}}})t^4}{120000(1+e^{\frac{3x}{\sqrt{10}}})^{14/3}}, \ldots$$

Hence, By Eq. (15) we have

$$\Phi^{(4)} = \frac{1}{(1+e^{\frac{3x}{\sqrt{10}}})^{2/3}} + \left(-\frac{1}{(1+e^{\frac{3x}{\sqrt{10}}})^{8/3}} + \frac{e^{3\sqrt{\frac{2}{5}}x}}{(1+e^{\frac{3x}{\sqrt{10}}})^{8/3}} - \frac{3e^{\frac{3x}{\sqrt{10}}}}{5(1+e^{\frac{3x}{\sqrt{10}}})^{5/3}} + \frac{1}{(1+e^{\frac{3x}{\sqrt{10}}})^{2/3}}\right)t + \frac{49e^{\frac{3x}{\sqrt{10}}}(1+e^{\frac{3x}{\sqrt{10}}})^{1/3}(-3+2e^{\frac{3x}{\sqrt{10}}})t^2}{100(1+e^{\sqrt{\frac{2}{5}}x}-e^{\frac{x}{\sqrt{10}}})^3(1+e^{\frac{x}{\sqrt{10}}})^3} + \cdots$$

Table 5: Comparison of the absolute errors of $\Phi^{(4)}$ of AD𝕁TM, MVIA-II and HPTM of Example 5.4.

x	$t=0.1$			$t=0.3$			$t=0.5$		
	AD𝕁TM $\Phi^{(4)}$	**MVIA-II** [30]	**HPTM** [32]	**AD𝕁TM $\Phi^{(4)}$**	**MVIA-II** [30]	**HPTM** [32]	**AD𝕁TM $\Phi^{(4)}$**	**MVIA-II** [30]	**HPTM** [32]
0.0	4.555 E-7	9.258 E-6	1.221 E-4	1.004 E-4	3.857 E-5	3.022 E-3	1.133 E-3	1.621 E-3	1.245 E-2
0.2	5.137 E-7	1.135 E-5	1.328 E-4	1.199 E-4	1.519 E-4	3.415 E-3	1.438 E-3	1.870 E-4	1.462 E-2
0.4	5.020 E-7	1.268 E-5	1.352 E-4	1.233 E-4	2.918 E-4	3.602 E-3	1.554 E-3	1.673 E-3	1.597 E-2
0.6	4.268 E-7	1.315 E-5	1.296 E-4	1.106 E-4	3.631 E-4	3.570 E-3	1.467 E-3	2.614 E-3	1.636 E-2
0.8	3.077 E-7	1.281 E-5	1.168 E-4	8.553 E-5	3.652 E-4	3.333 E-3	1.205 E-3	2.933 E-3	1.580 E-2
1.0	1.704 E-7	1.188 E-5	9.893 E-5	5.367 E-5	3.133 E-4	2.929 E-3	8.335 E-4	2.708 E-3	1.440 E-2

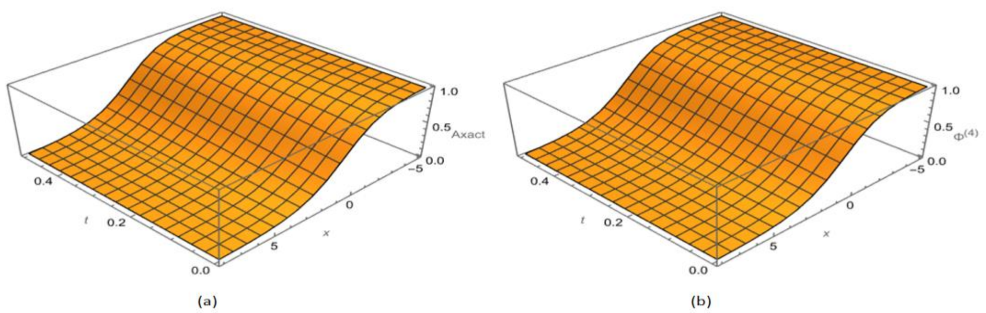


Figure 5: (a) Exact solution and (b) $\Phi^{(4)}$ of AD𝕁TM for Example 5.4.

Example 5.5: Consider the Eq. (1) with $r = 7, \alpha = 1$ and $\beta = -1$, which gives the Fisher's model of the form [26].

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^7, \quad (45)$$

subject to the following initial conditions

$$\phi(x,0) = (1 + e^{\frac{3x}{2}})^{\frac{-1}{3}}. \quad (46)$$

The exact solution of this problem is [26].

$$\phi(x,t) = (\tfrac{1}{2}\text{Tanh}[-\tfrac{3}{4}(x - \tfrac{5}{2}t)] + \tfrac{1}{2})^{\frac{1}{3}}. \quad (47)$$

Applying $\mathbb{J}$- transform on Eq. (45) subject to the condition (46), we obtain

$$\mathbb{J}[\phi(x,t)] = \frac{u^2}{s}[(1 + e^{\frac{3x}{2}})^{\frac{-1}{3}}] + \frac{u}{s}\,\mathbb{J}[\frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^7]. \quad (48)$$

Apply the inverse $\mathbb{J}$- transform on Eq. (48), we gain

$$\phi(x,t) = (1 + e^{\frac{3x}{2}})^{\frac{-1}{3}} + \mathbb{J}^{-1}\left[\frac{u}{s}\,\mathbb{J}\left[\frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^7\right]\right]. \quad (49)$$

From the AD$\mathbb{J}$TM, rewrite Eq. (49) as follows

$$\sum_{m=0}^{\infty} \phi_m(x,t) = \left(1 + e^{\frac{x}{\sqrt{6}}}\right)^{-2} + \mathbb{J}^{-1}\left[\frac{u}{s}\,\mathbb{J}\left[\frac{\partial^2}{\partial x^2}\sum_{m=0}^{\infty} \phi_m(x,t) + \sum_{m=0}^{\infty} \phi_m(x,t) - \sum_{m=0}^{\infty} A_m\right]\right]. \quad (50)$$

Where, A_m are Adomian polynomials. Using Eq. (12) A_m can be deduced as follows
$A_0 = \phi_0^7,\ A_1 = 7\phi_0^6\phi_1,\ A_2 = 7\phi_0^5(3\phi_1^2 + \phi_0\phi_2),\ \ldots$

Using Eq. (14) we have

$$\phi_0 = (1 + e^{\frac{3x}{2}})^{\frac{-1}{3}},\ \phi_1 = \frac{5e^{3x/2}t}{4(1+e^{3x/2})^{4/3}},\ \phi_2 = \frac{25e^{3x/2}(-3+e^{3x/2})t^2}{32(1+e^{3x/2})^{7/3}},\ \ldots$$

Hence, By Eq. (15) we have

$$\Phi^{(2)} = \frac{1}{(1 + e^{3x/2})^{1/3}} + (-\frac{1}{(1 + e^{3x/2})^{7/3}} + \frac{e^{3x}}{(1 + e^{3x/2})^{7/3}} - \frac{3e^{3x/2}}{4(1 + e^{3x/2})^{4/3}} + \frac{1}{(1 + e^{3x/2})^{1/3}})t + \frac{25e^{3x/2}(-3 + e^{3x/2})(1 + e^{3x/2})^{2/3}t^2}{32(1 + e^{x/2})^3(1 - e^{x/2} + e^x)^3}.$$

We note that the absolute error is largely dropped when modifying the solution by taking further terms. Table 6 explain this fact.

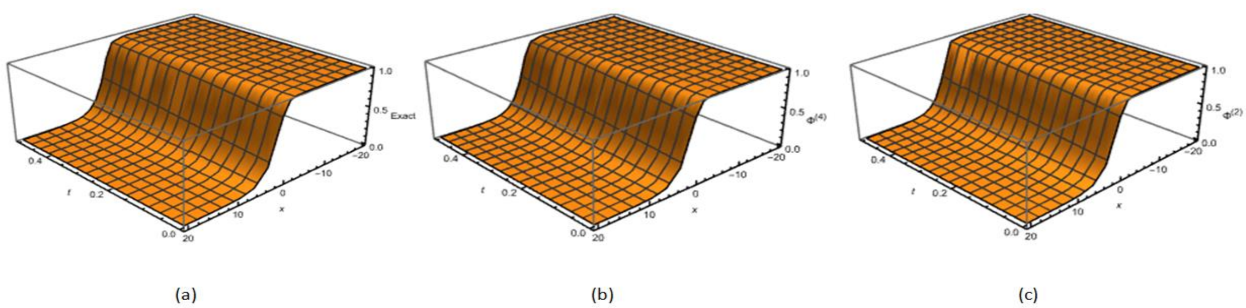


Figure 6: (a) Exact solution, (b) $\Phi^{(4)}$ of AD$\mathbb{J}$TM and (c) $\Phi^{(2)}$ of AD$\mathbb{J}$TM for Example 5.5.

Table 6: Comparison of the absolute errors of $\Phi^{(2)}$ and $\Phi^{(4)}$ of ADJTM, MVIA-II and HPTM of Example 5.5.

	t = 0.2				t = 0.4			
x	**ADJTM** $\boldsymbol{\Phi^{(4)}}$	**ADJTM** $\boldsymbol{\Phi^{(2)}}$	**ADM** [26]	**MVIM** [26]	**ADJTM** $\boldsymbol{\Phi^{(4)}}$	**ADJTM** $\boldsymbol{\Phi^{(2)}}$	**ADM** [26]	**MVIM** [26]
0.0	5.04358E-5	1.49970E-3	5.24926E−2	4.54137E−2	8.14718E-4	7.45300E-3	1.21845E−1	1.97465E−1
0.2	1.02130E-4	2.32570E-3	7.79547E−2	4.17460E−2	2.56843E-3	1.43093E-2	2.17494E−1	8.39974E−2
0.4	1.22688E-4	2.86477E-3	1.10805E−1	3.23276E−2	3.59434E-3	1.97309E-2	3.41710E−1	9.22231E−4
0.6	1.08280E-4	3.03127E-3	1.51375E−1	1.91936E−2	3.58249E-3	2.27394E-2	4.94354E−1	4.10631E−2
0.8	7.11481E-5	2.84591E-3	1.99601E−1	5.03821E−3	2.71657E-3	2.30245E-2	6.74017E−1	4.10631E−2
1.0	2.91972E-5	2.40914E-3	2.55137E−1	7.85833E−3	1.47529E-3	2.09547E-2	8.78892E−1	1.46625E−2

6. Discussion the results

The ADJTM is examined out to some nonlinear parabolic equations. The accuracy and efficiency of this technique was exemplified by five examples. In Figs.1-6 the comparison of the approximate solutions obtained by ADJTM with various order of approximations with the exact solution of examples 5.1-5.5 is presented. In tables 1-3, comparison of ADJTM, TBS and MVIA-II of example 5.1 and 5.2 are presented. In tables 2-6, comparison of absolute errors of ADJTM, MVIA-II, TBS, LTDM, VITM, HPTM and ADM are presented. We note that the absolute error is largely dropped when modifying the solution by taking further terms. Figure 3 and Tables 3, 4 and 6 are explained this fact.

7. Conclusions

In this paper, the ADJTM is successfully applied for solving AC, NW and Fisher equations with different parameters. The ADJTM is an aggregation of Adomian decomposition scheme and J -transform. The main advantage of the proposed technique is that it solves the problems without any form of differentiation, linearity or perturbation. Results show that the ADJTM is so helpful to gain approximate analytical solutions. Eventually, we can detect that the ADJTM is an efficient and accurate approach in solving partial differential equations that arises in physics, engineering and various areas of mathematics.

References

[1] R. Rajaram and M. Najafi, "Analytical treatment and convergence of the Adomian decomposition method for a system of coupled damped wave equations," *Applied mathematics and Computation,* vol. 212, no. 1, pp. 72-81, 2009.

[2] A. C. Varsoliwala and T. R. Singh, "Mathematical modeling of atmospheric internal waves phenomenon and its solution by Elzaki Adomian decomposition method," *Journal of Ocean Engineering and Science,* vol. 7, no. 3, pp. 203-212, 2022.

[3] S. Momani and Z. Odibat, "Homotopy perturbation method for nonlinear partial differential equations of fractional order," *Physics Letters A,* vol. 365, no. 5-6, pp. 345-350, 2007.

[4] P. M. A. Hasan and N. A. Sulaiman, " Homotopy perturbation method and convergence analysis for the linear mixed Volterra-Fredholm integral equations," *Iraqi Journal of Science,* vol. 61, no. 2, pp. 409-415, 2020.

[5] S. A. Altaie, N. Anakira, A. Jameel, O. Ababneh, A. Qazza and A. K. Alomari, "Homotopy Analysis Method Analytical Scheme for Developing a Solution to Partial Differential Equations in Fuzzy Environment," *Fractal and Fractional,* vol. 6, no. 8, p. 419, 2022.

[6] S. Maitama and W. Zhao, "Homotopy analysis Shehu transform method for solving fuzzy differential equations of fractional and integer order derivatives," *Computational and Applied Mathematics,* vol. 40, no. 3, p. 86, 2021.

[7] H. Ahmad, A. R. Seadawy and T. A. Khan, "Study on numerical solution of dispersive water wave phenomena by using a reliable modification of variational iteration algorithm," *Mathematics and Computers in Simulation,* vol. 177, pp. 13-23, 2020.

[8] A. A. Abdulsahib, F. S. Fadhel and J. H. Eidi, "Approximate Solution of Linear Fuzzy Random Ordinary Differential Equations Using Laplace Variational Iteration Method," *Iraqi Journal of Science,* vol. 65, no. 2, pp. 804-817, 2024.

[9] R. K. Jbr and A. Al-Rammahi, "Q-homotopy analysis method for solving nonlinear fredholm integral equation of the second kind," *International Journal of Nonlinear Analysis and Applications,* vol. 12, no. 2, pp. 2145-2152, 2021.

[10] L. Verma, R. Meher, Z. Avazzadeh and O. Nikan, "Solution for generalized fuzzy fractional Kortewege-de Varies equation using a robust fuzzy double parametric approach," *Journal of Ocean Engineering and Science,* vol. 8, no. 6, pp. 602-622, 2023.

[11] S. Maitama and W. Zhao, "Beyond sumudu transform and natural transform: J-transform properties and applications," *Journal of Applied Analysis and Computation,* vol. 10, no. 4, pp. 1223-1241, 2020.

[12] G. Hariharan and K. Kannan, "Haar wavelet method for solving some nonlinear parabolic equations," *Journal of mathematical chemistry,* vol. 48, pp. 1044-1061, 2010.

[13] G. Hariharan, "An efficient Legendre wavelet-based approximation method for a few Newell--Whitehead and Allen--Cahn equations," *The Journal of membrane biology,* vol. 247, pp. 371-380, 2014.

[14] A. Yokuş and H. Bulut, "On the numerical investigations to the Cahn-Allen equation by using finite difference method," *An International Journal of Optimization and Control: Theories & Applications (IJOCTA),* vol. 9, no. 1, pp. 18-23, 2019.

[15] Y. Chen, Y. Huang and N. Yi, "Recovery type a posteriori error estimation of adaptive finite element method for Allen–Cahn equation," *Journal of Computational and Applied Mathematics,* vol. 369, p. 112574, 2020.

[16] H. C. Rosu and O. Cornejo-Pérez, "Supersymmetric pairing of kinks for polynomial nonlinearities," *Physical Review E,* vol. 71, no. 4, p. 046607, 2005.

[17] R. Ezzati and K. Shakibi, "Using adomian's decomposition and multiquadric quasi-interpolation methods for solving newell-whitehead equation," *Procedia Computer Science,* vol. 3, pp. 1043-1048, 2011.

[18] S. Manaa, "An Approximate Solution to The Newell-Whitehead Equation by Adomian Decomposition Method," *AL-Rafidain Journal of Computer Sciences and Mathematics,* vol. 8, no. 1, pp. 171-170, 2011.

[19] A. Prakash and M. Kumar, "He's variational iteration method for the solution of nonlinear Newell-Whitehead-Segel equation," *J. Appl. Anal. Comput,* vol. 6, no. 3, pp. 738-748, 2016.

[20] S. S. Nourazar, M. Soori and A. Nazari-Golshan, "On the exact solution of newell-whitehead-segel equation using the homotopy perturbation method," *Australian Journal of Basic and Applied Sciences,* vol. 5, no. 8, pp. 1400-1411, 2011.

[21] P. Pue-On, "Laplace Adomian decomposition method for solving Newell-Whitehead-Segel equation," *Applied Mathematical Sciences,* vol. 7, no. 132, pp. 6593-6600, 2013.

[22] R. C. Mittal and G. Arora, "Efficient numerical solution of Fisher's equation by using B-spline method," *International Journal of Computer Mathematics,* vol. 87, no. 13, pp. 3039-3051, 2010.

[23] J. Roessler and H. Hüssner, "Numerical solution of the 1 + 2 dimensional Fisher's equation by finite elements and the Galerkin method," *Mathematical and Computer Modelling,* vol. 25, no. 3, pp. 57-67, 1997.

[24] D. Ağrseven and T. Öziş, "An analytical study for Fisher type equations by using homotopy perturbation method," *Computers & Mathematics with Applications,* vol. 60, no. 3, pp. 602-609, 2010.

[25] S. N. Huseen, "On Analytical Solution of Time-Fractional Type Model of the Fisher's Equation," *Iraqi Journal of Science,* vol. 61, no. 6, pp. 1419-1425, 2020.

[26] S. T. Mohyud-Din and M. A. Noor, "Modified variational iteration method for solving fisher's equations," *Journal of Applied Mathematics and Computing,* vol. 31, no. 1-2, pp. 295-308, 2009.

[27] A. A. Aswhad and A. F. Jaddoa, "The Approximate Solution of Newell-Whitehead-Segel and Fisher Equations Using The Adomian Decomposition Method," *Al-Mustansiriyah J. Sci.,* vol. 25, no. 4, pp. 45-56, 2014.

[28] H. Eltayeb, Y. T. Abdalla, I. Bachar and M. H. Khabir, "Fractional telegraph equation and its solution by natural transform decomposition method," *Symmetry,* vol. 11, no. 3, p. 334, 2019.

[29] W. Zahra, "Trigonometric B-spline collocation method for solving PHI-four and Allen--Cahn equations," *Mediterranean journal of mathematics,* vol. 14, pp. 1-19, 2017.

[30] H. Ahmad, A. R. Seadawy, T. A. Khan and P. Thounthong, "Analytic approximate solutions for some nonlinear Parabolic dynamical wave equations," *Journal of Taibah University for Science,* vol. 14, no. 1, pp. 346-358, 2020.

[31] M. Areshi, A. Khan, R. Shah and K. Nonlaopon, "Analytical investigation of fractional-order Newell-Whitehead-Segel equations via a novel transform," *Aims Math,* vol. 7, no. 4, pp. 6936-6958, 2022.

[32] M. Zellal and K. Belghaba, "Applications of Homotopy Perturbation Transform Method for Solving Newell-Whitehead-Segel Equation," *General Letters in Mathematics (GLM),* vol. 3, no. 1, pp. 35-46, 2017.