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e*Singular–Hollow Modules and e*Singular–Coclosed Submodules

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Abstract.
In this article, we will go over the basics of e*S–hollow modules, e*S–coessential submodules and e*S–coclosed submodules as a generalization of the concepts of hollow modules, coessential submodules and coclosed submodules, respectively. We shall demonstrate some characteristics of these ideas.

Keywords: e*_singular modules, e*S–small submodule, e*S–hollow module, e*S–coessential submodule, e*S–coclosed submodule.

المقاسات المجوفة المفردة من النمط *e والمقاسات الجزئية ضد المغلقة الأساسية المفردة من النمط e*

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الخلاصة.
في هذه البحث، سنتناول أساسيات المقاسات المجوفة المفردة من النمط *e والمقاسات الجزئية ضد الجوهرية الأساسية المفردة من النمط *e والمقاسات الجزئية ضد المغلقة الأساسية المفردة من النمط *e كتعميم لمفاهيم المقاسات المجوفة والمقاسات الجزئية ضد الجوهرية الاساسية والمقاسات الجزئية ضد المغلقة الاساسية على التوالي. وسنبين بعض خصائص هذه الأفكار.

1. Introduction.

In this paper C will be a unitary left R-module, and R be an associative ring with identity. Notationally, it is commonly known that a submodule D of an R-module C is small. D ≪ C if for every submodule L of C, D + L = C, then L = C, [1], [2]. A non-zero submodule D of C is considered to be an essential if and only if, for every submodule L of C, L = {0} whenever D ∩ L = {0}. Here, we denote $D \leq_e C$, where C is known as the essential extension of D [2] [3]. In a module C, a submodule D is closed if and only if has no proper essential extension [4], [5].

A new submodule type was created by Baanoon and Khaild in [6] and which is a generalization of the essential submodule and it is called an e∗-essential as follows: For any

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non-zero cosingular submodule B of C, if A ∩ B ≠ 0, we say that A is an e*-essential submodule in C. Denoted by $A \leq_{e*} C$.
Now, we will define the singular module: Z(C) = {m in C: ann(m) $\leq_e$R}. Notice, if Z(C) = C, then C is called a singular module and if Z(C) = 0, then C is called non-singular [4], [7]. We generalized Z(C) to Z_{e*}(C), by applying e*_essential submodule. Let C be a module and define $Z_{e*}(C) = \{n \text{ in } C: ann(n) \leq_{e*} R\}$, if $Z_{e*}(C) = C$, then C is called an e*_singular. As well as if $Z_{e*}(C) = 0$, then C is called an e*_non-singular module [6].

A non-zero module C is considered to be hollow if each proper submodule of C is small in C, see [8], [9]. Many authors present generalizations of a small submodule, see [10] [11] [12] [13]. In [6], the generalization of small submodule known as an e*S–small submodule which is introduced by A. Kabban and W. Khalid. A submodule D of C is called an e*S–small submodule of C (signified by $D \ll_{e*S} C$) if whenever C = D + H, with $Z_{e*}(\frac{C}{H}) = \frac{C}{H}$ implies that C = H. A non-zero R–module C is called an e*S–hollow module if each proper submodule of C is an e*S–small in C, and this is the definition of the e*S–hollow modules as generalizations of hollow modules. Give a description of e*S–hollow modules and prove under conditions in which the direct sum of an e*S–hollow module is an e*S–hollow, in addition to presenting its basic properties.
Let H ⊆ D ⊆ C, if $\frac{D}{H} \ll \frac{C}{H}$, then H is called a coessential submodule of D in C [14], [15]. Now, we introduce the e*S-coessential submodule, which is a generalization of the coessential submodule. Let C be an R–module, and let D, H ⊆ C, such that D ⊆ H ⊆ C, then D is called an e*S–coessential submodule of H in C (denoted by $D \subseteq_{e*S\_ce} H$ in C) if $\frac{H}{D} \ll_{e*S} \frac{C}{D}$. A submodule D of C is a coclosed submodule of C (denoted by $D \subseteq_{cc} C$) if whenever $\frac{D}{L} \ll \frac{C}{L}$ implies that D = L, see [16], [17]. Based on this idea, we provide the following concept. Let C be an R–module and H submodule of C. We say that H is called an e*S–coclosed submodule of C (denoted by $H \subseteq_{e*S\_cc} C$) if whenever $D \subseteq_{e*S\_ce} H$, (i.e., $\frac{H}{D} \ll_{e*S} \frac{C}{D}$) implies that D = H. The fundamental characteristics of these ideas are shown in this work.

2. e*S-hollow modules

We illustrate some of the features of an e*S-hollow modules. As a generalization of hollow modules, and present them in this section.

First, we need to list basic properties of the concept of e*S–small [6].

Lemma 2.1: [6]**.** Let C be any R–module then,

1) If D ⊆ W ⊆ C. Then $W \ll_{e*S} C$ if and only if $D \ll_{e*S} C$ and $\frac{W}{D} \ll_{e*S} \frac{C}{D}$.
2) Let D and W be submodules of C. Then $D + W \ll_{e*S} C$ if and only if $D \ll_{e*S} C$ and $W \ll_{e*S} C$.
3) Let $N_1, N_2, \dots, N_n \subseteq C$. Then $\sum_{i=1}^{n} N_i \ll_{e*S} C$ if and only if $N_i \ll_{e*S} C$, ∀ i = 1, 2, …, n.
4) Let D ⊆ W be a submodule of C. If $D \ll_{e*S} W$, then $D \ll_{e*S} C$.
5) Let $f: C \longrightarrow D$ be a homomorphisim. If $W \ll_{e*S} C$, then $f(W) \ll_{e*S} D$.
6) Let $C = M_1 \oplus M_2$ be an R–module and $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$. Then $N_1 \oplus N_2 \ll_{e*S} M_1 \oplus M_2$ if and only if $N_1 \ll_{e*S} M_1$ and $N_2 \ll_{e*S} M_2$.

Lemma 2.2: [6] For any R-module C, and W, L be two submodules of C. If $Z_{e*}(\frac{C}{W}) = \frac{C}{W}$ then $Z_{e*}(\frac{C}{L+W}) = \frac{C}{L+W}$.

The concept of an e*S–small submodule, lead to introduce the following:

Definition 2.3: A non-zero R–module C is called **e*_Singular–hollow module** (used for brief e*S–hollow) if each proper submodule of C is an e*S–small in C.

Examples and remarks 2.4:

1) Clearly, every hollow module is an e*S–hollow module. But the convers need not be accurate in general for example, let M = $Z_2 \oplus Z_2$ as Z_2–module, $Z_2 \oplus \{\bar{0}\}$ is a proper e*S–small submodule in M, but not small in M. See (Examples and remarks. 2) [6].

2) Every simple module is an e*S–hollow. For example, Z_p as Z–module (p is prime).

3) The Z_4 as Z–module is an e*S–hollow. By (1).

4) Consider M = $Z \oplus Z_{p^\infty}$ as Z–module is not an e*S–hollow. Since $0 \oplus Z_{p^\infty}$ proper submodule of M but $0 \oplus Z_{p^\infty}$ is not an e*S–small of M. Since $Z_{e*}(\frac{M}{Z}) \cong Z_{e*}(Z_{p^\infty}) = Z_{p^\infty} \cong \frac{M}{Z}$, but M ≠ Z. So, Z_{p^∞} dose not an e*S–small submodule of M.

5) Since $\langle\bar{2}\rangle$ and $\langle\bar{3}\rangle$ are not e*S–small in Z_6.Then Z_6 as Z–module is not an e*S–hollow.

6) In Z-module Z is not an e*S–hollow. See (Examples and remarks. 2) [6].

Under a certain condition the concept of hollow and e*S–hollow submodules coincide.

Theorem 2.5: Let C be an e*_Singular module. Then C is an e*S–hollow module if and only if each proper submodule D of C is small in C.

Proof:

⇒) Let D be a proper submodule of C such that $Z_{e*}(\frac{C}{D}) = \frac{C}{D}$, to show that D ≪ C. Assume that there exists K ⊊ C such that C = D + K. Since C is e*S–hollow, then $K \ll_{e*S} C$ and we have $Z_{e*}(\frac{C}{D}) = \frac{C}{D}$, then C = D, which is a contradiction. Thus D ≪ C.

⇐) To show that C is an e*S–hollow, let D be a proper submodule of C. Assume that D is not e*S–small in C, there exists a proper submodule K of C such that $Z_{e*}(\frac{C}{K}) = \frac{C}{K}$ and C = D + K. By our assumption K ≪ C, then D = C, which is a contradiction. Thus, C is e*S–hollow.

Proposition 2.6: A non-zero epimorphic image of an e*S–hollow module is an e*S–hollow.

Proof:

Let f: C ⟶ W be an epimorphism, and C be an e*S–hollow module, with K ⊊ W, to show $K \ll_{e*S} W$, since K ⊊ W, then $f^{-1}(K) \subsetneq C$. If $f^{-1}(K) = C$, then K = f(C) = W, hence K = W, this is a contradiction and since C is e*S–hollow, therefore $f^{-1}(K) \ll_{e*S} C$, and by Lemma 2.1, $f(f^{-1}(K)) \ll_{e*S} W$, then $K \ll_{e*S} W$.

Corollary 2.7: Let C be an R–module and N ⊆ C, if C is an e*S–hollow then $\frac{C}{N}$ is e*S–hollow.

Remember that a fully invariant submodule D of C is defined as follows: g (D) ⊆ D, for every g ∈ End (C) and C is called **duo module** if each submodule of C is fully invariant. See [18], [19].

Proposition 2.8: Let C be duo module and C = $C_1 \oplus C_2$, then C is an e*S–hollow if and only if C_1 and C_2 are e*S–hollow. Provided $N \cap C_i \neq C_i$ for all i = 1, 2, …N ⊆ C.

Proof:

⇒) Let C is e*S–hollow and $N_1 \oplus N_2 \subsetneq C_1 \oplus C_2$,with $N_1 \subsetneq C_1$ and $N_2 \subsetneq C_2$, and $N_1 \oplus N_2 \ll_{e*S} C_1 \oplus C_2$= C, to show C_1 is an e*S–hollow. Let $\pi_1: C_1 \oplus C_2 \longrightarrow C_1$ be the projection map, which is define as follows, $\pi_1(c_1 + c_2) = c_1$, for all $c_1 + c_2 \in C_1 \oplus C_2$, since $N_1 \oplus N_2 \ll_{e*S} C_1 \oplus C_2$, then by Lemma 2.1, $\pi_1(N_1 \oplus N_2) \ll_{e*S} \pi_1(C_1 \oplus C_2)$, then, $N_1 \ll_{e*S} C_1$, thus C_1 is an e*S–hollow, and similarly C_2 is an e*S–hollow.

⇐) Let C_1 and C_2 be e*S–hollow. To prove, $N_1 \oplus N_2 \ll_{e*S} C_1 \oplus C_2$, since $N_1 \ll_{e*S} C_1 \subseteq C$, and $N_2 \ll_{e*S} C_2 \subseteq C$, then by Lemma 2.1, $N_1 \ll_{e*S} C$ and $N_2 \ll_{e*S} C$. By Lemma 2.1 again, $N_1 \oplus N_2 \ll_{e*S} C = C_1 \oplus C_2$.

Proposition 2.9: Let C be an e*S–hollow module, if C has e*S–small proper submodule of D and $\frac{C}{D}$ is a finitely generated e*_Singular, then C is finitely generated.

Proof:

Since $\frac{C}{D}$ is finitely generated there are $y_1, y_2, \dots, y_n \in C$, such that $\frac{C}{D} = \langle y_1 + D, y_2 + D, \dots, y_n + D\rangle$. We claim that $C = \langle y_1, y_2, \dots, y_n\rangle$ let $c \in C$. Hence, $c + D \in \frac{C}{D}$ and $c + D = (r_1y_1 + r_2y_2 + \dots + r_ny_n) + D$, for some $r_1, r_2, \dots, r_n \in R$. So, $c - (r_1y_1 + r_2y_2 + \dots + r_ny_n) \in D$. Let $n = c - (r_1y_1 + r_2y_2 + \dots + r_ny_n)$ where $n \in D$. Hence, $c = (r_1y_1 + r_2y_2 + \dots + r_ny_n) + n$, thus $C = \langle y_1, y_2, \dots, y_n\rangle + D$. If $C \neq \langle y_1, y_2, \dots, y_n\rangle$, then $\langle y_1, y_2, \dots, y_n\rangle$ is e*S–small in C and since D is an e*S–small submodule. Hence, C = D which is a contradiction. Therefore, $C = \langle y_1, y_2, \dots, y_n\rangle$.

3. e*Singular–coessential submodules

This section defines the e*S–Coessential submodule and proves various features pertinent to our work. It is a generalization of the coessential submodule.

Definition 3.1: Let C be an R–module and D, H are submodules of C. Such that $D \subseteq H \subseteq C$, then D is called **e*_Singular–coessential** submodule of H in C (used for brief e*S–coessential submodule, denoted by $D \subseteq_{e*S\_ce} H$ in C) if $\frac{H}{D} \ll_{e*S} \frac{C}{D}$.

Examples and remarks 3.2:

1) Everyone can see that coessential submodule is e*S–coessential submodule. But the convers is not true in general for example: $\{\bar{0}\}$ is an e*S–coessential of $Z_3 \oplus \{\bar{0}\}$ in $M = Z_3 \oplus Z_3$ as Z_3–module, but not coessential in M.
2) Let C be an R–module and let D be a submodule of C. Then $D \ll_{e*S} C$ if and only if $\{0\} \subseteq_{e*S\_ce} D$ in C.
3) Z_6 as Z–module. Clear that $\langle\bar{0}\rangle$ is not e*S–coessential submodule of $\langle\bar{3}\rangle$ in $\mathbb{Z}_6$.
4) Z_8 as Z–module. As $\{\bar{0}, \bar{4}\} \subseteq_{e*S\_ce} \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ in Z_8.
5) In Z as Z–module. 4Z is e*S–coessential submodule of 2Z in Z.
6) Let $C = Z \oplus Z_{p^\infty}$ as Z–module. It is clear that $\langle\bar{0}\rangle$ is not e*S–coessential submodule of Z_{p^∞} in C. $\langle\bar{0}\rangle \subseteq Z_{p^\infty} \subseteq C$, since $\frac{Z_{p^\infty}}{\langle 0\rangle}$ not e*S–small in $\frac{C}{\langle 0\rangle}$.

The following proposition give a characterization of e*S–coessential submodule.

Theorem 3.3: Let C be an e*_singular module and $W \subseteq H \subseteq C$, then the following are equivalent.

1) $W \subseteq_{e*S\_ce} H$ in C;
2) For any submodule $X \subseteq C$, $H + X = C$ implies that $W + X = C$.

Proof:

(1)⇒(2) Let $C = H + X$, then $\frac{C}{W} = \frac{H}{W} + \frac{X+W}{W}$. Since $Z_{e*}(\frac{C}{W}) = \frac{C}{W}$, then by Lemma 2.2, $Z_{e*}(\frac{C}{X+W}) = \frac{C}{X+W}$. But $\frac{H}{W} \ll_{e*S} \frac{C}{W}$, therefore $\frac{C}{W} = \frac{X+W}{W}$. Thus $C = X + W$.
(2)⇒(1) Let $\frac{C}{W} = \frac{H}{W} + \frac{A}{W}$, where $W \subseteq A$ with $Z_{e*}(\frac{C}{A}) = \frac{C}{A}$. Then $C = H + A$, by (2) we get $C = W + A$. But $W \subseteq A$, therefore $C = A$. And hence $W \subseteq_{e*S\_ce} H$ in C.

The following proposition give some properties of e*S–coessential submodule which are needed later.

Proposition 3.4: Let C be an R–module and $L \subseteq K \subseteq N \subseteq C$. Then $K \subseteq_{e*S\_ce} N$ in C if and only if $\frac{K}{L} \subseteq_{e*S\_ce} \frac{N}{L}$ in $\frac{C}{L}$.

Proof:

⇒) Assume that $K \subseteq_{e*S\_ce} N$ in C, since $\frac{N/L}{K/L} \cong \frac{N}{K}$ and $\frac{C/L}{K/L} \cong \frac{C}{K}$, by the (Third Isomorphism Theorem) and $\frac{N}{K} \ll_{e*S} \frac{C}{K}$, we have $\frac{N/L}{K/L} \ll_{e*S} \frac{C/L}{K/L}$, thus $\frac{K}{L} \subseteq_{e*S\_ce} \frac{N}{L}$ in $\frac{C}{L}$.

⇐) Suppose that $\frac{K}{L} \subseteq_{e*S\_ce} \frac{N}{L}$ in $\frac{C}{L}$, since $\frac{N/L}{K/L} \cong \frac{N}{K}$, $\frac{C/L}{K/L} \cong \frac{C}{K}$, by the (Third Isomorphism Theorem) and since $\frac{N/L}{K/L} \ll_{e*S} \frac{C/L}{K/L}$, we have $\frac{N}{K} \ll_{e*S} \frac{C}{K}$, thus $K \subseteq_{e*S\_ce} N$ in C.

Proposition 3.5: For any R–module C, let $L \subseteq D \subseteq H \subseteq C$. Then $L \subseteq_{e*S\_ce} H$ in C if and only if $L \subseteq_{e*S\_ce} D$ in C and $D \subseteq_{e*S\_ce} H$ in C.

Proof:

⇒) Suppose that $L \subseteq_{e*S\_ce} H$ in C. Since $\frac{D}{L} \subseteq \frac{H}{L} \subseteq \frac{C}{L}$, and $\frac{H}{L} \ll_{e*S} \frac{C}{L}$, then $\frac{D}{L} \ll_{e*S} \frac{C}{L}$, by Lemma 2.1. So, $L \subseteq_{e*S\_ce} D$ in C. Now, define $f: \frac{C}{L} \longrightarrow \frac{C}{D}$ by $f(m + L) = m + D$ for all $m \in C$. Clearly f is an epimorphosis. Since $\frac{H}{L} \ll_{e*S} \frac{C}{L}$, hence $f(\frac{H}{L}) = \frac{H}{D} \ll_{e*S} \frac{C}{D}$, by Lemma 2.1. Thus $D \subseteq_{e*S\_ce} H$ in C.

⇐) Assume that $L \subseteq_{e*S\_ce} D$ in C and $D \subseteq_{e*S\_ce} H$ in C, to show $L \subseteq_{e*S\_ce} H$ in C. Let $\frac{H}{L} + \frac{X}{L} = \frac{C}{L}$, with $Z_{e*}(\frac{C}{X}) = \frac{C}{X}$, then $C = H + X$ and hence $\frac{C}{D} = \frac{H+X}{D} = \frac{H}{D} + \frac{X+D}{D}$. Since $Z_{e*}(\frac{C}{X}) = \frac{C}{X}$, by Lemma 2.2. $Z_{e*}(\frac{C}{X+D}) = \frac{C}{X+D}$. But $\frac{H}{D} \ll_{e*S} \frac{C}{D}$, therefore $\frac{X+D}{D} = \frac{C}{D}$, and hence $X + D = C$, therefore, $\frac{X}{L} + \frac{D}{L} = \frac{C}{L}$, since $\frac{D}{L} \ll_{e*S} \frac{C}{L}$, and $Z_{e*}(\frac{C}{X}) = \frac{C}{X}$, then $\frac{X}{L} = \frac{C}{L}$.

Proposition 3.6: For any R–module C. If $W \subseteq_{e*S\_ce} H$ in C and $L \subseteq C$, then $W + L \subseteq_{e*S\_ce} H + L$ in C. The converse is true if $L \ll_{e*S} C$.

Proof:

Assume that $W \subseteq_{e*S\_ce} H$ in C and $L \subseteq C$. To show that $W + L \subseteq_{e*S\_ce} H + L$ in C, let $\frac{H+L}{W+L} + \frac{Y}{W+L} = \frac{C}{W+L}$ with $Z_{e*}(\frac{C}{Y}) = \frac{C}{Y}$, then $C = H + L + Y$ since $L \subseteq W + L \subseteq Y$, then $C = H + Y$, $\frac{C}{W} = \frac{H+Y}{W} = \frac{H}{W} + \frac{Y}{W}$ and $Z_{e*}(\frac{C}{Y}) = \frac{C}{Y}$, and $\frac{H}{W} \ll_{e*S} \frac{C}{W}$, hence $\frac{C}{W} = \frac{Y}{W}$ and $\frac{Y}{W+L} = \frac{C}{W+L}$.

Conversely, suppose that $W + L \subseteq_{e*S\_ce} H + L$ in C and $L \ll_{e*S} C$. To show that $W \subseteq_{e*S\_ce} H$ in C. Let $\frac{C}{W} = \frac{H}{W} + \frac{Y}{W}$, with $Z_{e*}(\frac{C}{Y}) = \frac{C}{Y}$. Now, $C = H + Y$, hence $\frac{C}{W+L} = \frac{H+L}{W+L} + \frac{Y+L}{W+L}$. Since $Z_{e*}(\frac{C}{Y}) = \frac{C}{Y}$, then by Lemma 2.2, $Z_{e*}(\frac{C}{Y+L}) = \frac{C}{Y+L}$. But $\frac{H+L}{W+L} \ll_{e*S} \frac{C}{W+L}$, therefore $\frac{C}{W+L} = \frac{Y+L}{W+L}$ and hence $C = Y + L$. Since $L \ll_{e*S} C$ and $Z_{e*}(\frac{C}{Y}) = \frac{C}{Y}$, then $C = Y$. Therefore, $\frac{C}{W} = \frac{Y}{W}$ and $W \subseteq_{e*S\_ce} H$ in C.

Proposition 3.7: For any R–module C, let $W \ll_{e*S} C$. If $Y \subseteq_{e*S\_ce} H$ in C, then $Y \subseteq_{e*S\_ce} H + W$ in C.

Proof:

Suppose that $Y \subseteq_{e*S\_ce} H$ in C and $W \ll_{e*S} C$. To show that $Y \subseteq_{e*S\_ce} H + W$ in C. Let $\frac{C}{Y} = \frac{H+W}{Y} + \frac{X}{Y}$, with $Z_{e*}(\frac{C}{X}) = \frac{C}{X}$. Hence, C = H + W + X, since $Z_{e*}(\frac{C}{X}) = \frac{C}{X}$, then by Lemma 2.2, $Z_{e*}(\frac{C}{X+H}) = \frac{C}{X+H}$ and $W \ll_{e*S} C$, then C = H + X, and $\frac{C}{Y} = \frac{H}{Y} + \frac{X}{Y}$. But $\frac{H}{Y} \ll_{e*S} \frac{C}{Y}$ and $Z_{e*}(\frac{C}{X}) = \frac{C}{X}$, therefore $\frac{C}{Y} = \frac{X}{Y}$. Thus, $Y \subseteq_{e*S\_ce} H + W$ in C.

Proposition 3.8: Let C and W be an R–modules, let $f: C \longrightarrow W$ be an homomorphism if $D \subseteq_{e*S\_ce} H$ in C, then $f(D) \subseteq_{e*S\_ce} f(H)$ in $f(C)$.

Proof:

Suppose that $D \subseteq_{e*S\_ce} H$ in C. To show that $f(D) \subseteq_{e*S\_ce} f(H)$ in $f(C)$. Define $\varphi: \frac{C}{D} \longrightarrow \frac{f(C)}{f(D)}$ by $\varphi(m + D) = f(m) + f(D)$, for each $m \in C$, since $\frac{H}{D} \ll_{e*S} \frac{C}{D}$, then by Lemma 2.1. $\varphi(\frac{H}{D}) = \frac{f(H)}{f(D)} \ll_{e*S} \varphi(\frac{C}{D}) = \frac{f(C)}{f(D)}$. Thus, we get the result.

Proposition 3.9: For any R–module C, let $L \subseteq H \subseteq C$. If H = L + W and $W \ll_{e*S} C$, then $L \subseteq_{e*S\_ce} H$ in C.

Proof:

Suppose that H = L + W and $W \ll_{e*S} C$. Let $\frac{C}{L} = \frac{H}{L} + \frac{S}{L}$ with $Z_{e*}(\frac{C}{S}) = \frac{C}{S}$, for some $S \subseteq C$, then C = H + S, and hence C = L + W + S = S + W, since $W \ll_{e*S} C$ and $Z_{e*}(\frac{C}{S}) = \frac{C}{S}$, therefore C = S, and $\frac{C}{L} = \frac{S}{L}$. Thus, $L \subseteq_{e*S\_ce} H$ in C.

Proposition 3.10: For any R–module C, let $W \subseteq H \subseteq C$. If C = W + H, $W \subseteq X \subseteq C$ and $X \cap H \ll_{e*S} C$, then $W \subseteq_{e*S\_ce} X$ in C.

Proof:

Suppose that C = W + H, $W \subseteq X \subseteq C$ and $X \cap H \ll_{e*S} C$. Let $\frac{C}{W} = \frac{X}{W} + \frac{D}{W}$ with $Z_{e*}(\frac{C}{D}) = \frac{C}{D}$, where $D \subseteq C$, then C = X + D and $X = X \cap C = X \cap (W + H) = W + (X \cap H)$, by (Modular Law). Then $C = X + D = W + (X \cap H) + D$. So, $C = (X \cap H) + D$. But $X \cap H \ll_{e*S} C$ and $Z_{e*}(\frac{C}{D}) = \frac{C}{D}$, therefore C = D and $\frac{C}{W} = \frac{D}{W}$. Thus $W \subseteq_{e*S\_ce} X$ in C.

Proposition 3.11: Let C be an R–module. If $L \subseteq_{e*S\_ce} D$ in C and $X \subseteq_{e*S\_ce} H$ in C, then $L+X \subseteq_{e*S\_ce} D + H$ in C.

Proof:

Suppose that $L \subseteq_{e*S\_ce} D$ in C and $X \subseteq_{e*S\_ce} H$ in C. To show that $L + X \subseteq_{e*S\_ce} D + H$ in C, let $f: \frac{C}{L} \longrightarrow \frac{C}{L+X}$ be a map defined by $f(m + L) = m + (L + X)$ for each $m \in C$ and $g: \frac{C}{X} \longrightarrow \frac{C}{L+X}$ be a map defined by $g(m + X) = m + (L + X)$ for each $m \in C$. Clearly, each f and g are epimorphosis. Since $\frac{D}{L} \ll_{e*S} \frac{C}{L}$ and $\frac{H}{X} \ll_{e*S} \frac{C}{X}$, then $f(\frac{D}{L}) = \frac{(D+X)}{(L+X)} \ll_{e*S} \frac{C}{L+X}$ and $g(\frac{H}{X}) = \frac{(H+X)}{(L+X)} \ll_{e*S} \frac{H}{L+X}$, by Lemma 2.1. And hence $\frac{D+X}{L+X} + \frac{H+X}{L+X} = \frac{D+H}{L+X} \ll_{e*S} \frac{C}{L+X}$, by Lemma 2.1. Thus $L + X \subseteq_{e*S\_ce} D + H$ in C.

Proposition 3.12: Let B, D, H and X be submodules of an R–module C. The following statements are equivalent.

1) If $B \subseteq_{e*S\_ce} B + D$ in C, then $B \cap D \subseteq_{e*S\_ce} D$ in C;

2) If B $\subseteq_{e*S\_ce}$ D in C and Y ⊆ C, then B ∩ Y $\subseteq_{e*S\_ce}$ D ∩ Y in C;
3) If B $\subseteq_{e*S\_ce}$ D in C and X $\subseteq_{e*S\_ce}$ H in C, then B ∩ X $\subseteq_{e*S\_ce}$ D ∩ H in C;

Proof:
(1)⇒(2) Let B $\subseteq_{e*S\_ce}$ D in C and Y ⊆ C. Since B + (D ∩ Y) ⊆ D, then B $\subseteq_{e*S\_ce}$ B + (D ∩ Y) in C, by Proposition 3.5. Hence B ∩ (D ∩ Y) $\subseteq_{e*S\_ce}$ (D ∩ Y) in C, by (1). This implies that B ∩ Y $\subseteq_{e*S\_ce}$ D ∩ Y in C.
(2)⇒(3) Let B $\subseteq_{e*S\_ce}$ D in C and X $\subseteq_{e*S\_ce}$ H in C. By (2), B ∩ X $\subseteq_{e*S\_ce}$ D ∩ X in C. Also, X $\subseteq_{e*S\_ce}$ H in C and D ⊆ C, then D ∩ X $\subseteq_{e*S\_ce}$ D ∩ H in C. Thus B ∩ X $\subseteq_{e*S\_ce}$ D ∩ H in C, by Proposition 3.5.
(3)⇒(1) Let B $\subseteq_{e*S\_ce}$ B + D in C. Since D $\subseteq_{e*S\_ce}$ D in C, then by (3) B ∩ D $\subseteq_{e*S\_ce}$ (B + D) ∩ D in C. Thus B ∩ D $\subseteq_{e*S\_ce}$ D in C.
We use the following lemma in next theorem.

Lemma 3.13: Let C be a module such that C = H + D and C = (H ∩ D) + W for submodules H, D and W of C. Then C = (D ∩ W) + H = (H ∩ W) + D.
Proof: See [20], Lemma 1.2.

Theorem 3.14: Let C = W + X be an e*_singular module. Let X ⊆ H and X $\subseteq_{e*S\_ce}$ H in C. Then W ∩ X $\subseteq_{e*S\_ce}$ W ∩ H in C.
Proof:
Let $\frac{C}{(W\cap X)} = \frac{(W\cap H)}{(W\cap X)} + \frac{L}{(W\cap X)}$ with $Z_{e*}(\frac{C}{L}) = \frac{C}{L}$, to prove $\frac{C}{(W\cap X)} = \frac{L}{(W\cap X)}$, C = (W ∩ H) + L, implies that C = H + L. By Lemma 3.13, C = (W ∩ L) + H, $\frac{C}{X} = \frac{(W\cap L)+X}{X} + \frac{H}{X}$. Since $Z_{e*}(\frac{C}{X}) = \frac{C}{X}$, then $Z_{e*}(\frac{C}{(W\cap L)+X}) = \frac{C}{(W\cap L)+X}$, by Lemma 2.2. But $\frac{H}{X} \ll_{e*S} \frac{C}{X}$, therefore C = (W ∩ L) + X. Again, by Lemma 3.13, C = (W ∩ X) + L = L. Thus $\frac{C}{(W\cap X)} = \frac{L}{(W\cap X)}$, and $\frac{W\cap H}{W\cap X} \ll_{e*S} \frac{C}{W\cap X}$.

4. e*Singular–coclosed submodules
Here we define an e*S-coclosed submodule and go over a few of its characteristics.

Definition 4.1: Let C be an R–module and D be submodule of C. We say that D is called **e*_Singular–coclosed** submodule of C (used for brief e*S–coclosed submodule, denoted by D $\subseteq_{e*S\_cc}$ C) if whenever H $\subseteq_{e*S\_ce}$ D, (i.e., $\frac{D}{H} \ll_{e*S} \frac{C}{H}$) implies that D = H.

Examples and remarks 4.2:
1) Every e*S–coclosed submodule is coclosed submodule.
Let C be a module, let W be an e*S–coclosed submodule of C, and let A be submodule of W such that $\frac{W}{A} \ll \frac{C}{A}$. By (Example and remark. 2) [6], $\frac{W}{A} \ll_{e*S} \frac{C}{A}$, because W is e*S–coclosed in C. Thus, W = A and hence W is a coclosed submodule of C.
2) The convers of (1) need not be accurate in general for example, let M = $Z_2 \oplus Z_2$ as Z_2–module, $\{\bar{0}\} \subseteq Z_2 \oplus \{\bar{0}\}$ and $\frac{Z_2 \oplus \{\bar{0}\}}{\{\bar{0}\}} = Z_2 \oplus \{\bar{0}\} \ll_{e*S} \frac{M}{\{\bar{0}\}}$ = M, but $\{\bar{0}\} \neq Z_2 \oplus \{\bar{0}\}$. So $Z_2 \oplus \{\bar{0}\}$ is not e*S–coclosed, but is coclosed, since $Z_2 \oplus \{\bar{0}\}$ not small in M.

3) Let C = Z ⊕ Z_{p^∞} as Z–module. It is clear that $\langle\bar{0}\rangle$ is proper submodule of Z_{p^∞} in C, but $\frac{Z_{p^\infty}}{\langle\bar{0}\rangle} \cong Z_{p^\infty}$, $\frac{C}{\langle\bar{0}\rangle} \cong C$. So, Z_{p^∞} dose not e*S–small submodule of C. Also, $Z_{p^\infty} \neq \langle\bar{0}\rangle$. Therefore, Z_{p^∞} is e*S–coclosed.

4) In Z as Z–module. A 2Z dose not an e*S–coclosed submodule of Z. Since 4Z is proper submodule of 2Z, $\frac{2Z}{4Z} \cong \langle\bar{2}\rangle$ and $\frac{Z}{4Z} \cong Z_4$. By (Example and remark. 2) [6]. $\langle\bar{2}\rangle \ll_{e*S} Z_4$ and 2Z ≠ 4Z.

5) Let W be an e*S–hollow module. Then W has only one proper e*S–coclosed, which is the zero submodule. Let D be a proper submodule of W. Then $D \ll_{e*S} W$ and so $\frac{D}{\{0\}} \ll_{e*S} \frac{W}{\{0\}}$. Thus, if D is an e*S–coclosed in W, then D = {0}.

The following gives some basic properties of an e*S–coclosed submodules.

Proposition 4.3: Let C be an R–module and let W ⊆ B ⊆ M. Then:

1) If B is an e*S–coclosed in C, then $\frac{B}{W}$ is an e*S–coclosed in $\frac{C}{W}$.

2) If $W \ll_{e*S} B$ and $\frac{B}{W}$ is an e*S–coclosed in $\frac{C}{W}$, then B is an e*S–coclosed in C (provided C is an e*_singular module).

3) If W is an e*S–coclosed in C, then W is an e*S–coclosed in B.

Proof:

1) Assume that B is an e*S–coclosed in C, let $\frac{X}{W} \subsetneq \frac{B}{W}$, such that $\frac{\frac{B}{W}}{\frac{X}{W}} \ll_{e*S} \frac{\frac{C}{W}}{\frac{X}{W}}$ by (The Third Isomorphism Theorem), $\frac{\frac{B}{W}}{\frac{X}{W}} \cong \frac{B}{X}$ and $\frac{\frac{C}{W}}{\frac{X}{W}} \cong \frac{C}{X}$. As a result, $\frac{B}{X} \ll_{e*S} \frac{C}{X}$, since B is e*S–coclosed in C. Thus, B = X and $\frac{X}{W} = \frac{B}{W}$, therefore $\frac{B}{W}$ is an e*S–coclosed in $\frac{C}{W}$.

2) Suppose that L ⊆ B, such that $L \subseteq_{e*S\_ce} B$ (i.e., $\frac{B}{L} \ll_{e*S} \frac{C}{L}$). Let $\pi: C \longrightarrow \frac{C}{W}$ be the natural epimorphism, so by Proposition 3.4, $\frac{L+W}{W} \subseteq_{e*S\_ce} \frac{B}{W}$ (i.e., $\frac{\frac{B}{W}}{\frac{L+W}{W}} \ll_{e*S} \frac{\frac{C}{W}}{\frac{L+W}{W}}$). Since $\frac{B}{W}$ is an e*S–coclosed in $\frac{C}{W}$, so $\frac{L+W}{W} = \frac{B}{W}$ and B = L + W. Since $W \ll_{e*S} B$, thus B = L. Therefore, B is an e*S–coclosed in C.

3) Let L ⊊ W such that $\frac{W}{L} \ll_{e*S} \frac{B}{L} \subseteq \frac{C}{L}$. So, by Lemma 2.1, $\frac{W}{L} \ll_{e*S} \frac{C}{L}$. Since W is an e*S–coclosed in C, so L = W. Therefore, W is an e*S–coclosed in B.

Proposition 4.4: Let $C = M_1 \oplus M_2$ be a module, and $L \subseteq_{e*S\_cc} M_1$. Then $L \subseteq_{e*S\_cc} C$.

Proof:

Let W ⊊ L such that $\frac{L}{W} \ll_{e*S} \frac{C}{W} = \frac{M_1 \oplus M_2}{W}$. Hence, $\frac{L}{W} \ll_{e*S} \frac{M_1}{W} \oplus \frac{W \oplus M_2}{W}$. So, $\frac{L}{W} \ll_{e*S} \frac{M_1}{W}$, by Lemma 2.1. Since $L \subseteq_{e*S\_cc} M_1$, therefore W = L and $L \subseteq_{e*S\_cc} C$.

Proposition 4.5: Let C be a module and K be a non-zero submodule of C. If $K \subseteq_{e*S\_cc} C$, then K is not an e*S–small in C.

Proof:

Assume K is an e*S–small in C and $K \subseteq_{e*S\_cc} C$. Because {0} ⊆ K and $K \cong \frac{K}{\{0\}} \ll_{e*S} \frac{M}{\{0\}} \cong C$. Then K = {0} which is a contradiction. Therefore, K is not an e*S–small in C.

The following proposition shows that the e*S–coclosed submodule is a condition to be the submodule of an e*S–hollow module is an e*S–hollow.

Proposition 4.6: Every non-zero e*S–coclosed submodule of an e*S–hollow module is an e*S–hollow

Proof:

Suppose that C is an e*S–hollow module and W is e*S–coclosed in C. Let A be a proper submodule of W, such that W = A + H with $Z_{e*}(\frac{W}{H}) = \frac{W}{H}$. Since C is e*S–hollow by Corollary 2.7, $\frac{C}{H}$ is e*S–hollow. Now, if $\frac{W}{H}$ is a proper submodule of $\frac{C}{H}$, then $\frac{W}{H}$ is an e*S–small submodule of $\frac{C}{H}$, since W is e*S–coclosed. Thus, W = H and A is an e*S–small submodule of W. Hence, W is e*S–hollow.

Proposition 4.7: Le C be an R–module, and let K be a non-zero e*S–hollow submodule of C, then either $K \ll_{e*S} C$ or K is e*S–coclosed submodule of C but not both.

Proof:

Let K is a non-zero e*S–hollow submodule of C and K is not an e*S–coclosed. We have to show that $K \ll_{e*S} C$. Since K is not an e*S–coclosed in C, then there exists $L \subsetneq K$ such that $\frac{K}{L} \ll_{e*S} \frac{C}{L}$. To prove that $K \ll_{e*S} C$, let C = K + A with $Z_{e*}(\frac{C}{A}) = \frac{C}{A}$, then $\frac{C}{L} = \frac{K}{L} + \frac{L+A}{L}$ by Lemma 2.2. $Z_{e*}(\frac{C}{L+A}) = \frac{C}{L+A}$, but $\frac{K}{L} \ll_{e*S} \frac{C}{L}$, therefore C = L + A. Now, K = K ∩ C = K ∩ (L + A) = L + (K ∩ A), by (Modular Law). Note that by (Second Isomorphism Theorem) $\frac{K}{K \cap A} \cong \frac{K+A}{A} = \frac{C}{A}$, which is $Z_{e*}(\frac{K}{K \cap A}) = \frac{K}{K \cap A}$. But K is an e*S–hollow and L is a proper submodule of K, therefore $L \ll_{e*S} K$, hence K = K ∩ A, K ⊆ A, then C = A. Thus, $K \ll_{e*S} C$.

If $K \ll_{e*S} C$ and K is an e*S–coclosed, then $\frac{K}{\{0\}} \ll_{e*S} \frac{C}{\{0\}}$ by Lemma 2.1. Implies that K = 0, which is a contradiction.

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