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The Influence of Fear and Predator-Dependent Refuge on The Dynamics of A Harvested Prey-Predator Model with Disease in Predator

Hussein Sabah Abdullah, Dahlia Khaled Bahlool *
Department of Mathematics, College of science, University of Baghdad, Baghdad, Iraq

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Abstract

This paper discussed fear and predator-dependent refuges in a harvested prey-predator model with disease in predator with using the relationship between a predator's consumption of prey and the presence of fear in the prey in the existence of an infected predator in the environment. The solution was consistently limited in order to assure the availability of a solution due to the availability of conditions. The solution points for the system were also found, and the conditions for the approach of estimating the solution has been studied stability. We studied the bifurcation of the system and illustrate that it works. This research examined the effect of harvest factor on prey impact, as well as the combined influence of death in the unwell predator and the harvest of its available resources. It makes a major effect on the outcome of the system. The outcomes obtained by varying the parameters are displayed in numerical work. And theoretical solution was confirmed using Mathematica graphing.

Keywords: Fear, harvesting, refuge, stability, prey-predator model, Boundedness.

تأثير الخوف وملجأ المفترس المعتمد على ديناميكيات حصاد نموذج الفريسة –المفترسة مع المرض في المفترس

حسين صباح عبدالله , داليا خالد بهلول *
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة

ناقش هذا البحث الخوف والملاجئ المعتمدة على المفترس في نموذج الفريسة المفترسة المصابة بالمرض مع استخدام العلاقة بين استهلاك المفترس للفريسة ومدى توفر الخوف لدى الفريسة في وجود مفترس مصاب في البيئة. تم وضع قيد للنظام لضمان الحل بشكل مستمر للتأكد من توفر الحل نظرا لتوفر الشروط، كما تم العثور على نقاط الحل للنظام، وتمت دراسة شروط منهج تقدير استقرارية الحل. قمنا دراسة التشعب للنظام وضحنا طريقة عمله .وتناول هذا البحث تأثير عامل الحصاد على تأثير الفريسة، بالإضافة إلى تأثير المشترك للموت في المفترس المريض وحصاد الموارد المتاحة له لما له تأثير كبير على نتيجة النظام. تم عرض النتائج التي تم الحصول عليها عن طريق تغيير المعلمات في الشغل العددي. وتم تأكيد الحل النظري باستخدام الرسم البياني ماثيماتيكا.

* Email : dahlia.khaled@sc.uobaghdad.edu.iq

1. Introduction

Predator-prey relationships can be affected by many factors, such as supply of resources, infectious diseases, refuge and harvesting. These factors have a substantial impact on the dynamics of prey-predator interactions in the real world. In order to understand the prey-predator system, mathematical equations must be formulated for them and these factors must consider since they have a significant role in the system. One of the basic factors is the presence of the refuge. It is a place that gives prey safety from predators, and when it is added to the mathematical model, we will study its effect on the system, as increasing refuges lead to the prey growing more and more, for example the authors in [1] examined the food chain model that provides a prey refuge and an additional source of food to investigate the stability of the system. It is necessary to take into account the consequences of the presence of a refuge, and this is what has been seen in studies [2 - 5]. Also, other studies, in addition to the presence of the shelter, studied the effect of the harvested with it [6 - 10]. In addition, another study was conducted to find out the probability of local bifurcation happening near the equilibrium points [11].

In order to understand predator-prey dynamics, it is essential to recognize that the fear factor, which has a major impact on the system, is one of the most important factors. A number of researchers have investigated the influence of fear on the prey-predator system. For instance, see [12 - 15]. It can be seen from the studies in [16 - 18] that different functional responses were discussed in predator-prey models where the Alee effect is present. In [19 - 21], authors talked about Leslie-Gower predator–prey models using a different functional response incorporating refuges into the models.

Relationships between predators and prey can also be influenced by infectious diseases. Disease outbreaks can cause a rapid decline in the populations of prey species. Consequently, this may result in a decrease in predator numbers, as they depend on the prey for survival. This is what has been discussed and studied in [22-26]. Keeping the limitation of infectious diseases is necessary for keeping predator-prey dynamics. This can be achieved by carefully tracking and quickly noticing diseases in both the animals that are hunted and the animals that hunt them, adopting restrictions if necessary, and supporting healthy environments and ecosystems to reduce the propagation of diseases.

The rest of the arrangement of the research paper goes as follows: The next section relates to mathematical results relating to the model system (1) from Sections 1-4. We mainly investigate the importance of prey hunting and the cooperative fear factor. After that, we found the stability points and studied the conditions for their stability, existence, and globalization. After that, in Sections 5-6-7, we discussed the bifurcation and its occurrence possibility. Numerical simulations are performed in Section 8 to confirm the existing analytical findings. At last, we complete the paper with a brief conclusion in Section 9.

2. Model description

An ecological system consisting of three species is examined using a prey-predator model. A linear form of functional response is used by the prey to be consumed by the susceptible, infected predator. The modeling investigation intends to examine the dynamics of a prey-predator model that is affected by harvesting and the existence of disease in the predator population. The impact caused by fear and predator-dependent refuge on the dynamics of a prey-predator model was focused on in the present investigation. The following nonlinear ordinary differential equations are required for the description of this ecological model's

$$\frac{d\mathcal{X}}{d\mathcal{T}} = \frac{r\mathcal{X}}{1+\mu\mathcal{Y}}\left(1-\frac{\mathcal{X}}{k}\right) - \beta(1-\alpha\mathcal{Y})\mathcal{X}\mathcal{Y} - q_1E_1\mathcal{X},$$
$$\frac{d\mathcal{Y}}{d\mathcal{T}} = e\beta(1-\alpha\mathcal{Y})\mathcal{X}\mathcal{Y} - S\mathcal{Y}\mathcal{Z} - d_1\mathcal{Y} - q_2E_2\mathcal{Y}, \quad (1)$$
$$\frac{d\mathcal{Z}}{d\mathcal{T}} = S\mathcal{Y}\mathcal{Z} - d_2\mathcal{Z} - q_3E_3\mathcal{Z}.$$

The populations of prey, susceptible predator, and infected predator are denoted as $\mathcal{X}(\mathcal{T}), \mathcal{Y}(\mathcal{T})$ and $\mathcal{Z}(\mathcal{T})$, respectively in the given equation. All the parameters in the system (1) are required to be non-negative.

Table 1: Biological significance of each parameter.

Parameter	Description
r	Growth rate of the prey.
μ	Fear rate of prey species.
β	Attack rate of susceptible predator.
e	Conversion rate of prey biomass to predator biomass.
$(1-\alpha\mathcal{Y})$	Predator-dependent refuge rates, where α denotes the coefficient of prey refuge in (0, 1) with $\mathcal{Y} < \frac{1}{\alpha}$.
k	Carrying capacity.
d_1, d_2	The mortality rates of susceptible predator and infected predator, subsequently.
q_1, q_2, q_3	The catchability constant of prey, susceptible predator and infected predator, subsequently.
E_1, E_2, E_3	Harvesting effort of prey, susceptible predator and infected predator, subsequently.
S	Infection rate of the predator.

3. Dimensionless aspect of the model

To convert a model into a dimensionless form, dimensionless quantities must be used to express the system. This strategy has several benefits. First, it streamlines the analytical process by reducing the number of parameters. Second, it makes it possible to compare parameters more effectively in terms of their magnitudes, which facilitates deeper system insights. Furthermore, it makes comparing various systems easier. Therefore, the following formulation of equation (1) is possible:

$$\frac{dx}{dt} = xf_1(x,y,z),$$
$$\frac{dy}{dt} = yf_2(x,y,z), \quad (2)$$
$$\frac{dz}{dt} = zf_3(x,y,z)$$

where

$$f_1(x,y,z) = \frac{1}{1+w_0y}(1-x) - (1-w_1y)y - w_2,$$
$$f_2(x,y,z) = w_3(1-w_1y)x - w_4z - (w_5+w_6),$$
$$f_3(x,y,z) = w_4y - (w_7+w_8).$$

With the dimensionless variables and parameters given by:

$$t = r\mathcal{T}, \; x = \frac{1}{k}\mathcal{X}, \; y = \frac{\beta}{r}\mathcal{Y}, \; z = \frac{\beta}{r}\mathcal{Z},$$
$$w_0 = \frac{\mu r}{\beta}, \; w_1 = \frac{\alpha r}{\beta}, \; w_2 = \frac{q_1E_1}{r}, \; w_3 = \frac{ek\beta}{r},$$
$$w_4 = \frac{S}{\beta}, w_5 = \frac{d_1}{r}, w_6 = \frac{q_2E_2}{r}, w_7 = \frac{d_2}{r}, w_8 = \frac{q_3E_3}{r}.$$

Theorem 3.1. The solutions for the model (2) for all positive initial conditions $x(0), y(0), z(0)$ are uniformly bounded.

Proof. From the first equation of system (2)

$$\frac{dx}{dt} = x\left(\frac{(1-x)}{1+w_0 y} - y(1-w_1 y) - w_2\right),$$

$$\frac{dx}{dt} \le \left(\frac{1}{1+w_0 y} - x\right)x \le x(1-x).$$

Then solving the above differential inequality gives that $x(t) \le 1$ as $t \to \infty$.

Now, we define $N(t) = x(t) + \frac{1}{w_3}y(t) + \frac{1}{w_3}z(t)$, then

$$\frac{dN}{dt} \le x(1-x) - w_2 x - \frac{(w_5+w_6)}{w_3}y - \frac{(w_7+w_8)}{w_3}z.$$

$$\le 2 - \mu\left[x + \frac{1}{w_3}y + \frac{1}{w_3}z\right].$$

Where $\mu = \min\{(1+w_2),(w_5+w_6),(w_7+w_8)\}$. This implies tha

$$\frac{dN}{dt} + \mu N \le \frac{1}{4}.$$

Again, by solving the final differential inequality, we have $N \le \frac{1}{4\mu}$, for $t \to \infty$.

Hence, the uniformly boundedness of system (2) confirms that it is valid. □

4. The equilibrium points

By setting all the equations in the system (2) equal to zero and solving for the variables $x(t), y(t)$, and $z(t)$, we can determine the equilibrium points.

$$\begin{aligned} x f_1(x,y,z) &= 0, \\ y f_2(x,y,z) &= 0, \\ z f_3(x,y,z) &= 0. \end{aligned} \tag{3}$$

Consequently, the following solutions equilibrium points are obtained:

The evanescence equilibrium point, $Q_0 = (0,0,0)$, always exists.

The axial equilibrium point, $Q_x = (\hat{x}, 0,0)$, where:

$$\hat{x} = 1 - w_2. \tag{4}$$

Direct computation shows that the positive root exists uniquely, and hence there is a unique point, say Q_x inside the first quadrant of $x - plane$, if sufficient condition is fulfilled

$$1 > w_2, \tag{5}$$

the infected predator -free equilibrium point, $Q_{xy} = (\breve{x}, \breve{y}, 0)$, where

$$\breve{x} = \frac{w_5+w_6}{w_3(1-w_1\breve{y})}. \tag{6}$$

While $\breve{y}$ is a solution with a positive value for the $4th$ degree equation,

$$\varsigma_1 y^4 + \varsigma_2 y^3 + \varsigma_3 y^2 + \varsigma_4 y^1 + \varsigma_5 = 0,$$

where

$$\begin{aligned} \varsigma_1 &= -(w_0 w_1^2 w_3), \\ \varsigma_2 &= 2w_0 w_1 w_3 - w_1^2 w_3 = w_1 w_3(2w_0 - w_1), \\ \varsigma_3 &= w_0 w_1 w_2 w_3 + 2w_1 w_3 - w_0 w_3, \\ \varsigma_4 &= w_1 w_2 w_3 - w_0 w_2 w_3 - w_1 w_3 - w_3, \\ \varsigma_5 &= w_3 - w_2 w_3 - w_5 - w_6. \end{aligned}$$

Calculation explicitly finds that there are either three or one positive roots of the above fourth-order polynomial equation, depending on the given conditions. Therefore, there exist three or one equilibrium point within the first quadrant of $xy - plane$ given that:

$$\breve{y} < \frac{1}{w_1}, \tag{7a}$$

$$w_0 < \frac{w_1}{2}, \tag{7b}$$

$$0 < \frac{w_5+w_6}{1-w_2} < w_3. \tag{7c}$$

It is important to observe that condition (7a) ensures that $\breve{x}$ is positive, while condition (7b) leads to $\varsigma_2 < 0$ and $\varsigma_3 > 0$. Finally, condition (7c) guarantees that $\varsigma_4 < 0$ and $\varsigma_5 > 0$ at the same time.

The coexistence (positive) equilibrium point, $Q_{xyz} = (\bar{x}, \bar{y}, \bar{z})$, where

$$\left.\begin{aligned} \bar{x} &= 1 - w_2 - \frac{w_7+w_8}{w_4} - \frac{w_0 w_2 (w_7+w_8)}{w_4} - \frac{w_0 (w_7+w_8)^2}{w_4^2} \\ &\quad + \frac{w_1 (w_7+w_8)^2}{w_4^2} + \frac{w_0 w_1 (w_7+w_8)^3}{w_4^3} \\ \bar{y} &= \frac{w_7+w_8}{w_4} \\ \bar{z} &= \frac{1}{w_4}\left(w_3\left(1 - \frac{w_1(w_7+w_8)}{w_4}\right)\bar{x} - (w_5+w_6)\right) \end{aligned}\right\}. \tag{8}$$

Direct computation shows that the positive root exists uniquely, and hence there is a unique point, say Q_{xyz} inside the first quadrant of $xyz - plane$, if sufficient conditions are fulfilled

$$\frac{w_7+w_8}{w_4} + \frac{w_0 w_2 (w_7+w_8)}{w_4} + \frac{w_0 (w_7+w_8)^2}{w_4^2} < 1 + \frac{w_1 (w_7+w_8)^2}{w_4^2} + \frac{w_0 w_1 (w_7+w_8)^3}{w_4^3}, \tag{9a}$$

$$(w_5 + w_6) < w_3\left(1 - \frac{w_1(w_7+w_8)}{w_4}\right)\bar{x}. \tag{9b}$$

5. Local stability analysis

We examine the local stability of every equilibrium point by utilizing the Jacobian matrix and determining the eigenvalues near each point. The Jacobian matrix H for three-dimensional systems can be determined by:

$$H(x,y,z) = \begin{bmatrix} f_1 + x\frac{\partial f_1}{\partial x} & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & f_2 + y\frac{\partial f_2}{\partial y} & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & f_3 + z\frac{\partial f_3}{\partial z} \end{bmatrix}. \tag{10}$$

where

$$\frac{\partial f_1}{\partial x} = -\frac{1}{1+w_0 y}, \frac{\partial f_1}{\partial y} = 2\mathrm{w}_1 y - 1 - \frac{w_0(1-x)}{(1+w_0 y)^2},$$
$$\frac{\partial f_1}{\partial z} = 0, \frac{\partial f_2}{\partial x} = w_3(1 - w_1 y), \frac{\partial f_2}{\partial y} = -\mathrm{w}_1 w_3 x,$$
$$\frac{\partial f_2}{\partial z} = -w_4, \frac{\partial f_3}{\partial x} = 0, \frac{\partial f_3}{\partial y} = \mathrm{w}_4, \frac{\partial f_3}{\partial z} = 0.$$

By replacing the equilibrium points mentioned above individually in the Jacobian matrix $H(x,y,z)$ and subsequently calculating their eigenvalues, it can be noted that:
The eigenvalues of the Jacobian matrix (10) at the evanescence equilibrium point (Q_0) are $(1 - w_2, -(w_5+w_6), -(w_7+w_8))$, which means Q_0 will be local asymptotic stability if and only if the next condition is achieved:

$$1 < w_2. \tag{11}$$

The eigenvalues of the Jacobian matrix (10) at the axial equilibrium point (Q_x) are computed as:

$$\lambda_{11} = -\hat{x}\,,\ \lambda_{12} = w_3\hat{x} - (w_5+w_6)\,,\ \lambda_{13} = -(w_7+w_8). \tag{12}$$

Therefore, Q_x is local asymptotic stability if and only if the following conditions is met:

$$\hat{x} < \frac{w_5+w_6}{w_3}. \tag{13}$$

The Jacobian matrix can be expressed in terms of the equilibrium point where infected predators are absent:

$$H\left(Q_{xy}\right) = \begin{bmatrix} -\frac{\breve{x}}{1+w_0\breve{y}} & \breve{x}\left(2w_1\breve{y} - 1 - \frac{w_0(1-\breve{x})}{(1+w_0\breve{y})^2}\right) & 0 \\ w_3\breve{y}(1-w_1\breve{y}) & -w_1w_3\breve{x}\breve{y} & -w_4\breve{y} \\ 0 & 0 & w_4\breve{y} - (w_7+w_8) \end{bmatrix}. \quad (14)$$

The equation that describes the characteristics equation of $H(Q_{xy})$ can be represented in the following form:

$$\left(\lambda^2 - T_{xy}\lambda + D_{xy}\right)(w_4\breve{y} - (w_7+w_8) - \lambda) = 0, \quad (15)$$

where:

$$T_{xy} = -\frac{\breve{x}}{1+w_0\breve{y}} - w_1w_3\breve{x}\breve{y}\ ,$$

$$D_{xy} = \frac{\breve{x}}{1+w_0\breve{y}}(w_1w_3\breve{x}\breve{y}) - w_3\breve{x}\breve{y}\left(2w_1\breve{y} - 1 - \frac{w_0(1-\breve{x})}{(1+w_0\breve{y})^2}\right)(1-w_1\breve{y}).$$

Consequently, the eigenvalues of the matrix $H(Q_{xy})$ are identified as $\lambda_{2i} = \frac{T_{xy}}{2} \pm \frac{1}{2}\sqrt{T_{xy}{}^2 - 4D_{xy}}$, for $i = 1,2$ and $\lambda_{23} = w_4\breve{y} - (w_7+w_8)$. Hence, if the following conditions are met, all eigenvalues will possess negative real parts, indicating that Q_{xy} is locally asymptotically stable.

$$\breve{y} < \frac{(w_7+w_8)}{w_4}. \quad (16a)$$

$$\breve{y} < \frac{1}{2w_1}\left(1 + \frac{w_0(1-\breve{x})}{(1+w_0\breve{y})^2}\right). \quad (16b)$$

Theorem 5.1. The coexistence equilibrium point of system (2) will be locally asymptotically stable if the conditions listed below are satisfied.

$$\bar{y} < \frac{1}{2w_1}\left(1 + \frac{w_0(1-\bar{x})}{(1+w_0\bar{y})^2}\right). \quad (17)$$

Proof. The Jacobian matrix of the system (2) for $Q_{xyz} = (\bar{x}, \bar{y}, \bar{z})$, can be written as:

$$H\left(Q_{xyz}\right) = \left[a_{ij}\right]_{3\times 3}, \quad (18)$$

where:

$$a_{11} = -\frac{\bar{x}}{1+w_0\bar{y}},\ a_{12} = \bar{x}\left(2w_1\bar{y} - 1 - \frac{w_0(1-\bar{x})}{(1+w_0\bar{y})^2}\right),$$

$$a_{13} = 0\ ,\ a_{21} = w_3\bar{y}(1-w_1\bar{y})\ ,\ a_{22} = -w_1w_3\bar{x}\bar{y},$$

$$a_{23} = -w_4\bar{y}\ ,\ a_{31} = 0\ ,\ a_{32} = w_4z\ ,\ a_{33} = 0.$$

Hence, the characteristic equation of $H(Q_{xyz})$ can be expressed as:

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \quad (19)$$

where

$$A = -(a_{11} + a_{22})\ ,$$

$$B = a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32},$$

$$C = a_{32}a_{11}a_{23}.$$

with

$$\Delta = AB - C = -(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21}) + a_{22}a_{23}\,a_{32}\ .$$

It should be noted that applying the Routh-Hurwitz criterion [27] requires satisfying the conditions $C > 0$, $A > 0$ with $\Delta > 0$, which guarantees all the solutions to the equation (19) contain real that are negative parts. By performing direct calculations, it can be shown that condition (17) is sufficient to satisfy the conditions needed for the Routh-Hurwitz criterion. Consequently, Q_{xyz} achieves local asymptotic stability. □

6. Global stability

In this section, the method of the Lyapunov function is employed to determine the basin of attraction associated with each locally asymptotically stable point in the domain $\mathbb{R}^3_+$. If the basin of attraction for an equilibrium point encompasses the whole domain $\mathbb{R}^3_+$, it is considered globally asymptotically stable. As given in the following theorems.

Theorem 6.1. The local asymptotic stability of Q_0 implies its global asymptotic stability if the condition (11) is satisfied.

Proof. The real-valued function $\mathrm{W}_1 = \mathrm{k}_1\mathrm{x} + \mathrm{k}_2\mathrm{y} + \mathrm{k}_3\mathrm{z}$ is defined. By performing direct calculations, it can be shown that $\mathrm{W}_1: M_1 \to \mathbb{R}$, when $M_1 = \{(x,y,z) \in \mathbb{R}^3_+ : x \geq 0, y \geq 0, z \geq 0\}$.

Consequently, this implies $\mathrm{W}_1(Q_0) = 0$, and $\mathrm{W}_1(x,y,z) > 0$, for every $(x,y,z) \in M_1 - Q_0$. Furthermore, simple calculations yield that:

$$\frac{d\mathrm{W}_1}{dt} = \mathrm{k}_1\frac{dx}{dt} + \mathrm{k}_2\frac{dy}{dt} + \mathrm{k}_3\frac{dz}{dt},$$

$$\frac{dW_1}{dt} \leq -k_1(w_2 - 1)x - \frac{k_1x^2}{1+w_0y} - (k_1 - k_2w_3)(1 - w_1y)xy$$
$$- k_2(w_5 + w_6)y - k_3(w_7 + w_8)z.$$

By choosing positive constant values as $\mathrm{k}_1 = 1$, and $\mathrm{k}_2 = k_3 = \frac{1}{w_3}$, the following results are obtained:

$$\frac{d\mathrm{W}_1}{dt} \leq -(w_2 - 1)x - \left(\frac{w_5+w_6}{w_3}\right)y - \left(\frac{w_7+w_8}{w_3}\right)z.$$

Therefore, from condition (5) imply that $\frac{d\mathrm{W}_1}{dt} < 0$. As a result, Q_0 exhibits global asymptotic stability. □

Theorem 6.2. The local asymptotic stability of Q_x implies its global asymptotic stability if the condition outlined below is satisfied.

$$\hat{x} < \frac{w_5+w_6}{w_3(2w_0+1)}. \tag{20}$$

Proof. The real-valued function $\mathrm{W}_2 = \mathrm{q}_1\int_{\hat{x}}^{x}\frac{u-\hat{x}}{u}du + \mathrm{q}_2y + \mathrm{q}_3z$ is defined. By performing direct calculations, it can be shown that $\mathrm{W}_2: M_2 \to \mathbb{R}$, when $M_2 = \{(x,y,z) \in \mathbb{R}^3_+ : x > 0, y \geq 0, z \geq 0\}$. Consequently, this implies $\mathrm{W}_2(Q_x) = 0$, and $\mathrm{W}_2(x,y,z) > 0$ for every $(x,y,z) \in M_2 - Q_x$. Furthermore, simple calculations yield that:

$$\frac{d\mathrm{W}_2}{dt} = \mathrm{q}_1\left(\frac{x-\hat{x}}{x}\right)\frac{dx}{dt} + \mathrm{q}_2\frac{dy}{dt} + \mathrm{q}_3\frac{dz}{dt}.$$

Employing direct computation leads to the following outcome:

$$\frac{dw_2}{dt} \leq -q_1\frac{(x-\hat{x})^2}{1+w_0y} - q_1\frac{w_0(1-\hat{x})y(x-\hat{x})}{1+w_0y} - q_1(1 - w_1y)xy + q_1(1 - w_1y)\hat{x}y$$
$$+q_2w_3(1 - w_1y)xy - q_2w_4yz - q_2(w_5 + w_6)y + q_3w_4yz$$
$$-q_3(w_7 + w_8)z.$$

Now, by choosing positive constant values as $q_1 = 1$ and $q_2 = q_3 = \frac{1}{w_3}$, the following results are obtained

$$\frac{dW_2}{dt} \leq -\frac{(x-\hat{x})^2}{1+w_0y} - \left(\frac{(w_5+w_6)}{w_3} - (2w_0 + 1)\hat{x}\right)y - \frac{(w_7+w_8)}{w_3}.$$

Note that, $\frac{dW_2}{dt}$ is a negative definite function, and hence the proof is complete. □

Theorem 6.3. Assume that there is only one equilibrium point Q_{xy} that is locally stable, then it has a basin of attraction that satisfies the following conditions.

$$\frac{1}{1+w_0y} + \frac{L}{2} > 0, \tag{21a}$$

$$w_1\breve{x}+\frac{L}{2}>0. \qquad (21b)$$

Proof. The real-valued function $W_3=m_1\int_{\breve{x}}^{x}\frac{u-\breve{x}}{u}du+m_2\int_{\breve{y}}^{y}\frac{v-\breve{y}}{v}dv+m_3 z$ is defined. By performing direct calculations, it can be shown that $W_3: M_3\to\mathbb{R}$, when $M_3=\{(x,y,z)\in\mathbb{R}_+^3: x>0, y>0, z\geq 0\}$. Consequently, this implies $W_3(Q_{xy})=0$, and $W_3(x,y,z)>0$ for every $(x,y,z)\in M_3-Q_{xy}$. Furthermore, simple calculations yield that:

$$\frac{dW_3}{dt}=m_1\left(\frac{x-\breve{x}}{x}\right)\frac{dx}{dt}+m_2\left(\frac{y-\breve{y}}{y}\right)\frac{dy}{dt}+m_3\frac{dz}{dt}.$$

By employing direct computation incorporating the principle of maximizing, and using the upper bound constant of the variables x and y leads to the following outcome:

$$\frac{dW_3}{dt}=m_1(x-\breve{x})\left(\frac{(1-x)(1+w_0\breve{y})-(1-\breve{x})(1+w_0 y)}{(1+w_0\breve{y})(1+w_0 y)}-\left(1-w_1(y-\breve{y})\right)(y-\breve{y})\right)$$
$$+m_2(y-\breve{y})\left(w_3(1-w_1 y)(x-\breve{x})-w_1w_3\breve{x}(y-\breve{y})-w_4 z\right)$$
$$+m_3w_4yz-m_3(w_7+w_8)z.$$

Now, by choosing positive constant values as $m_1=1$ and $m_2=m_3=\frac{1}{w_3}$, the following results are obtained

$$\frac{dW_3}{dt}=-\frac{(x-\breve{x})^2}{1+w_0y}-w_1\breve{x}(y-\breve{y})^2-\left(\frac{w_0(1-\breve{x})}{(1+w_0\breve{y})(1+w_0y)}-w_1\breve{y}\right)(x-\breve{x})(y-\breve{y})$$
$$-\left(\frac{(w_7+w_8)}{w_3}-\frac{w_4}{w_3}\breve{y}\right)z.$$

Let $L=\frac{w_0(1-\breve{x})}{(1+w_0\breve{y})(1+w_0y)}-w_1\breve{y}$

$$\leq-\frac{(x-\breve{x})^2}{1+w_0y}-w_1\breve{x}(y-\breve{y})^2-\frac{L}{2}(x-\breve{x})^2-\frac{L}{2}(y-\breve{y})^2-\left(\frac{(w_7+w_8)}{w_3}-\frac{w_4}{w_3}\breve{y}\right)z$$
$$\leq-\left(\frac{1}{1+w_0y}+\frac{L}{2}\right)(x-\breve{x})^2-\left(w_1\breve{x}+\frac{L}{2}\right)(y-\breve{y})^2-\left(\frac{(w_7+w_8)}{w_3}-\frac{w_4}{w_3}\breve{y}\right)z.$$

Note that, $\frac{dW_3}{dt}$ is a negative definite function in the region that satisfies the given conditions, and hence the proof is complete. □

Theorem 6.4. If the conditions outlined below are satisfied, the local asymptotic stable point Q_{xyz} has the basin of attraction satisfies the following conditions.

$$\frac{1}{1+w_0y}+\frac{M}{2}>0. \qquad (22a)$$
$$w_1\bar{x}+\frac{M}{2}>0. \qquad (22b)$$

Proof. The real-valued function $W_4=p_1\int_{\bar{x}}^{x}\frac{u-\bar{x}}{u}du+p_2\int_{\bar{y}}^{y}\frac{v-\bar{y}}{v}dv+p_3\int_{\bar{z}}^{z}\frac{c-\bar{z}}{c}dc$ is defined. By performing direct calculations, it can be shown that $W_4: M_4\to\mathbb{R}$, when $M_4=\{(x,y,z)\in\mathbb{R}_+^3: x>0, y>0, z>0\}$, holds. Consequently, this implies $W_4(Q_{xyz})=0$, and $W_4(x,y,z)>0$, for every $(x,y,z)\in M_4-Q_{xyz}$. Furthermore, simple calculations yield that:

$$\frac{dW_4}{dt}=p_1\left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt}+p_2\left(\frac{y-\bar{y}}{y}\right)\frac{dy}{dt}+p_3\left(\frac{z-\bar{z}}{z}\right)\frac{dz}{dt}.$$

Likewise, a direct calculation that results in:

$$\frac{dW_4}{dt}=p_1(x-\bar{x})\left(\frac{(1-x)(1+w_0\bar{y})-(1-\bar{x})(1+w_0y)}{(1+w_0\bar{y})(1+w_0y)}-\left(1-w_1(y-\bar{y})\right)(y-\bar{y})\right)$$
$$+p_2(y-\bar{y})\left(\left(w_3(1-w_1y)(x-\bar{x})-w_1w_3\bar{x}(y-\bar{y})-w_4(z-\bar{z})\right)\right.$$
$$+p_3w_4(z-\bar{z})(y-\bar{y}).$$

By choosing positive constant values as $p_1 = 1$, and $p_2 = p_3 = \frac{1}{w_3}$, the following results are obtained:

$$\frac{dW_4}{dt} = -\frac{(x-\bar{x})^2}{1+w_0 y} - w_1\bar{x}(y-\bar{y})^2 - \left(\frac{w_0(1-\bar{x})}{(1+w_0\bar{y})(1+w_0 y)} - w_1\bar{y}\right)(x-\bar{x})(y-\bar{y}).$$

Let $M = \frac{w_0(1-\bar{x})}{(1+w_0\bar{y})(1+w_0 y)} - w_1\bar{y}$.

$$\le -\frac{(x-\bar{x})^2}{1+w_0 y} - w_1\bar{x}(y-\bar{y})^2 - \frac{M}{2}(x-\bar{x})^2 - \frac{M}{2}(y-\bar{y})^2.$$

$$\le -\left(\frac{1}{1+w_0 y} + \frac{M}{2}\right)(x-\bar{x})^2 - \left(w_1\bar{x} + \frac{M}{2}\right)(y-\bar{y})^2.$$

Note that the derivative $\frac{dW_4}{dt}$ is a negative semi-definite. Since the equilibrium point Q_{xyz} is a unique invariant set within the set of point that satisfies $\frac{dW_4}{dt} = 0$, hence, with the help of LaSalle's invariant set theorem [28] the proof is complete. □

7. Local bifurcation

System (2) can be reformulated using Sotomayor's theorem [27] to examine the local bifurcation that may arise near the non-hyperbolic equilibrium point. The objective is to understand the impact of variable variations on the dynamic behavior of the system. Now, let's rephrase system (2) in the following manner.

To rewrite System (2), it is necessary to express it as the derivative of X concerning t, denoted as $\frac{dX}{dt}$, which is equal to the function $G(X)$. Here, X is a column vector $(x,y,z)^T$, and $G(X)$ represents the column vector $(xf_1, yf_2, zf_3)^T$. Consequently, the second derivative of the Jacobian matrix can be expressed likewise with the following generic vector $S = (s_1, s_2, s_3)^T$:

$$D^2G(X).(S,S) = [e_{i1}]_{3\times 1}, \tag{23}$$

where

$$e_{11} = -\frac{2s_1^2}{1+yw_0} + 2xs_2^2\left(-\frac{(-1+x)w_0^2}{(1+yw_0)^3} + w_1\right) + 2s_1s_2\left(-1 + \frac{(-1+2x)w_0}{(1+yw_0)^2} + 2yw_1\right),$$
$$e_{21} = -2s_2(xs_2w_1w_3 + s_1(-1+2yw_1)w_3 + s_3w_4),$$
$$e_{31} = 2s_2s_3w_4.$$

Theorem 7.1. The system (2) will exhibit a Transcritical bifurcation near the Q_0 if w_2 passes the value $\breve{w}_2 = 1$.

Proof. The form of the Jacobian matrix for system (2) at Q_0, when $w_2 = \breve{w}_2$, can be represented as:

$$H(Q_0, \breve{w}_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(w_5+w_6) & 0 \\ 0 & 0 & -(w_7+w_8) \end{bmatrix}.$$

Therefore, the eigenvalues of $H(Q_0, \breve{w}_2)$, can be expressed as $\lambda_{32} = -(w_5+w_6) < 0$, $\lambda_{33} = -(w_7+w_8) < 0$ subject to condition (11), and $\lambda_{31} = 0$.

If we consider $\breve{S} = (s_{31}, s_{32}, s_{33})^T$ as the eigenvector of $H(Q_0, \breve{w}_2)$ corresponding to $\breve{\lambda}_{31} = 0$, we can derive that $\breve{S} = (s_{31}, 0, 0)^T$, where $s_{31} \neq 0$, $s_{31} \in \mathbb{R}$.

If we consider $\breve{\Psi} = (\Psi_{31}, \Psi_{32}, \Psi_{33})^T$, to be the eigenvector of $H(Q_0, \breve{w}_2)^T$ related to $\breve{\lambda}_{31} = 0$, we can deduce that $\breve{\Psi} = (\Psi_{31}, 0, 0)$, is obtained, where $\Psi_{31} \neq 0$ and $\Psi_{31} \in \mathbb{R}$.

Furthermore, by calculating $\frac{\partial G}{\partial w_2} = G_{w_2} = (-x, 0, 0)^T$, we find that $G_{w_2}(Q_0, \breve{w}_2) = (0,0,0)^T$.

Consequently, when evaluating $\breve{\Psi}^T[G_{w_2}(Q_0, \breve{w}_2)] = 0$. Additionally, direct computation demonstrates that:

$$DG_{w_2}(Q_0,\breve{w}_2)=\begin{pmatrix}-1&0&0\\0&0&0\\0&0&0\end{pmatrix}\Rightarrow DG_{w_5}(Q_0,\breve{w}_2)\check{S}=(-s_{31},0,0)^T,$$

Subsequently $\breve{\Psi}^T\left[DG_{w_2}(Q_0,\breve{w}_2)\check{S}\right]=-\Psi_{31}s_{31}\neq 0$.

Upon examining equation (23), it can be noted that

$$\left[D^2G(Q_0,\breve{w}_2)\left(\check{S},\check{S}\right)\right]=\begin{pmatrix}-2s_{31}^2\\0\\0\end{pmatrix}.$$

thus

$$\breve{\Psi}^T\left[D^2G(Q_0,\breve{w}_2)\left(\check{S},\check{S}\right)\right]=-2\Psi_{31}s_{31}^2\neq 0.$$

Therefore, it can be concluded that a Transcritical bifurcation takes place as per Sotomayor's theorem, so the proof has been completed.

Theorem 7.2. If the condition (13) is satisfied, the system (2) will exhibit a Transcritical bifurcation near the $Q_x=(\hat{x},0,0)$ if w_5 passes the value $\widehat{w}_5=w_3\hat{x}-w_6$.

Proof. The form of the Jacobian matrix for system (2) at Q_x, when $w_5=\widehat{w}_5$, can be represented as:

$$H(Q_x,\widehat{w}_5)=\begin{bmatrix}-\hat{x}&\hat{x}(-1-w_0(1-\hat{x}))&0\\0&0&0\\0&0&-w_7-w_8\end{bmatrix}=\left(d_{ij}\right).$$

Therefore, the eigenvalues of $H(Q_x,\widehat{w}_5)$, can be expressed as $\lambda_{11}=-\hat{x}$, $\lambda_{13}=-(w_7+w_8)<0$ subject to condition (13), and $\lambda_{12}=0$.

If we consider $\hat{S}=(s_{11},s_{12},s_{13})^T$ as the eigenvector of $H(Q_x,\widehat{w}_5)$ corresponding to $\hat{\lambda}_{12}=0$, we can derive that $\hat{S}=(\delta_{11}s_{12},s_{12},0)^T$, where $s_{12}\neq 0$, $s_{12}\in\mathbb{R}$, and $\delta_{11}=-\frac{c_{12}}{c_{11}}<0$, where c_{ij} are the elements of $H(Q_x,\widehat{w}_5)$.

If we consider $\widehat{\Psi}=(\Psi_{11},\Psi_{12},\Psi_{13})^T$, to be the eigenvector of $H(Q_x,\widehat{w}_5)^T$ related to $\hat{\lambda}_{12}=0$, we can deduce that $\widehat{\Psi}=(0,\Psi_{12},0)$, is obtained, where $\Psi_{12}\neq 0$ and $\Psi_{12}\in\mathbb{R}$.

Furthermore, by calculating $\frac{\partial G}{\partial w_5}=G_{w_5}=(0,-y,0)^T$, we find that $G_{w_5}(Q_x,\widehat{w}_5)=(0,0,0)^T$.

Consequently, when evaluating $\widehat{\Psi}^T[G_{w_5}(Q_x,\widehat{w}_5)]=0$. Additionally, direct computation demonstrates that:

$$DG_{w_5}(Q_x,\widehat{w}_5)=\begin{pmatrix}0&0&0\\0&-1&0\\0&0&0\end{pmatrix}\Rightarrow DG_{w_5}(Q_x,\widehat{w}_5)\hat{S}=(0,-s_{12},0)^T,$$

Subsequently $\widehat{\Psi}^T\left[DG_{w_5}(Q_x,\widehat{w}_5)\hat{S}\right]=-\Psi_{13}s_{13}\neq 0$.

Upon examining equation (23), it can be noted that

$$\left[D^2G(Q_x,\widehat{w}_5)\left(\hat{S},\hat{S}\right)\right]=\begin{pmatrix}-2s_{12}^2(\delta_{11}+\delta_{11}{}^2+(\delta_{11}-2\delta_{11}\hat{x})w_0+(-1+\hat{x})\hat{x}w_0^2-\hat{x}w_1)\\s_{12}^2(2\delta_{11}w_3-2\hat{x}w_1w_3)\\0\end{pmatrix}.$$

thus

$$\widehat{\Psi}^T\left[D^2G(Q_x,\widehat{w}_5)\left(\hat{S},\hat{S}\right)\right]=\Psi_{12}s_{12}^2(2\delta_{11}w_3-2\hat{x}w_1w_3)\neq 0.$$

Therefore, it can be concluded that a Transcritical bifurcation takes place as per Sotomayor's theorem, so the proof has been completed.

□

Theorem 7.3. In the system (2) will demonstrate a Transcritical bifurcation near the $Q_{xy}=(\breve{x},\breve{y},0)$, if w_7 passes the value $\breve{w}_7=w_4\breve{y}-w_8$.

Proof. The form of the Jacobian matrix for system (2) at Q_{xy}, when $w_7 = \breve{w}_7$, can be represented as:

$$H(Q_{xy}, \breve{w}_7) = \begin{bmatrix} -\frac{\breve{x}}{1+w_0\breve{y}} & \breve{x}\left(2w_1\breve{y} - 1 - \frac{w_0(1-\breve{x})}{(1+w_0\breve{y})^2}\right) & 0 \\ w_3\breve{y}(1 - w_1\breve{y}) & -w_1w_3\breve{x}\breve{y} & -w_4\breve{y} \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the eigenvalues of $H(Q_{xy}, \breve{w}_7)$, can be expressed as $\lambda_{2i} = \frac{T_{xy}}{2} \pm \frac{1}{2}\sqrt{T_{xy}{}^2 - 4D_{xy}}$, for $i = 1,2$, while the third eigenvalue is given by $\breve{\lambda}_{23} = 0$.
If we consider $\breve{S} = (s_{21}, s_{22}, s_{23})^T$ as the eigenvector of $H(Q_{xy}, \breve{w}_7)$ corresponding to $\breve{\lambda}_{23} = 0$, we can derive that $\breve{S} = (\delta_{21}s_{23}, \delta_{22}s_{23}, s_{23})^T$, where $s_{23} \neq 0$, $s_{23} \in \mathbb{R}$, and $\delta_{21} = \frac{b_{23}b_{12}}{b_{11}b_{22}-b_{12}b_{21}}$, $\delta_{22} = \frac{-b_{23}b_{11}}{b_{11}b_{22}-b_{12}b_{21}}$.
If we consider $\breve{\Psi} = (\Psi_{21}, \Psi_{22}, \Psi_{23})^T$, to be the eigenvector of $H(Q_{xy}, \breve{w}_7)^T$ associated with $\breve{\lambda}_{23} = 0$, we can deduce that $\breve{\Psi} = (0,0,\Psi_{23})$, is obtained, where $\Psi_{23} \neq 0$ and $\Psi_{23} \in \mathbb{R}$. Furthermore, by calculating $\frac{\partial G}{\partial w_8} = G_{w_8} = (0,0,-z)^T$, we find that $G_{w_7}(Q_{xy}, \breve{w}_7) = (0,0,0)^T$.Consequently, when evaluating $\breve{\Psi}^T[G_{w_7}(Q_x, \breve{w}_7)] = 0$. Additionally, direct computation demonstrates that:

$$DG_{w_7}(Q_{xy}, \breve{w}_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Longrightarrow DG_{w_7}(Q_{xy}, \breve{w}_7)\,\breve{S} = (0,0,-s_{23})^T,$$

subsequently, $\breve{\Psi}^T\left[DG_{w_7}(Q_{xy}, \breve{w}_7)\,\breve{S}\right] = -\Psi_{23}s_{23} \neq 0$.
Upon examining equation (23), it can be noted that

$$\left[D^2G(Q_{xy}, \breve{w}_7)(\breve{S}, \breve{S})\right] = [e_{i1}]_{3\times1},$$

where

$$e_{11} = -\frac{2(\delta_{21}s_{23})^2}{1+w_0\breve{y}} + 2\breve{x}(\delta_{22}s_{23})^2\left(-\frac{(-1+\breve{x})w_0^2}{(1+w_0\breve{y})^3} + w_1\right) + 2\delta_{21}\delta_{22}s_{23}{}^2\left(-1 + \frac{(-1+2\breve{x})w_0}{(1+w_0\breve{y})^2} + 2w_1\breve{y}\right),$$
$$e_{21} = -2\delta_{22}s_{23}(\breve{x}\delta_{22}s_{23}w_1w_3 + \delta_{21}s_{23}(-1 + 2w_1\breve{y})w_3 + s_{23}w_4),$$
$$e_{31} = 2\delta_{22}s_{23}{}^2w_4.$$

Thus $\breve{\Psi}^T\left[D^2G(Q_{xy}, \breve{w}_7)(\breve{S}, \breve{S})\right] = \Psi_{23}(2\delta_{22}s_{23}{}^2w_4) \neq 0$,

Therefore, it can be concluded that a Transcritical bifurcation occurs near Q_{xy} when $w_7 = \breve{w}_7$. According to the determinant of the positive point, its sign is always positive and unconditionally. Therefore, there is no zero eigenvalue, meaning that the system does not contain a bifurcation at the positive point. □

8. Numerical simulation

Now we have come to prove the validity of the theories that were discussed previously, to prove the existence of points of stability for system (2) and to study the stability of the points or not, and to ensure this by setting special conditions to guarantee what we want to obtain. Therefore, we will use Mathematica 13.2 for carefully selected data to obtain the influence of every variable on the system that we studied, and the data are shown in a Table 2.

Table 2: Data of parameter values.

w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
0.9	0.9	0.25	0.9	0.4	0.09	0.08	0.08	0.04

By replacing the data in Table 2, which consists of nine parameters in system (2), we obtained globally asymptotically stable positive points $Q_{xyz} = (0.404, 0.3, 0.239)$. Each of the cases was drawn individually in Figure 1, once for the prey, the other for susceptible predator, and the infected predator from several initial point. Here, for example, we chose five initial points and noted it reached an approximate solution in a stable state for each of the three species, and then we drew a combination of these three species together, and after that we drew a three-dimensional drawing to see it with better accuracy and clarification to reach the point $Q_{xyz} = (0.404, 0.3, 0.239)$. As for Figure 2, it is a drawing of the positive point with one condition, and it is also shown in two drawings of the $2\mathcal{D}$ and $3\mathcal{D}$.

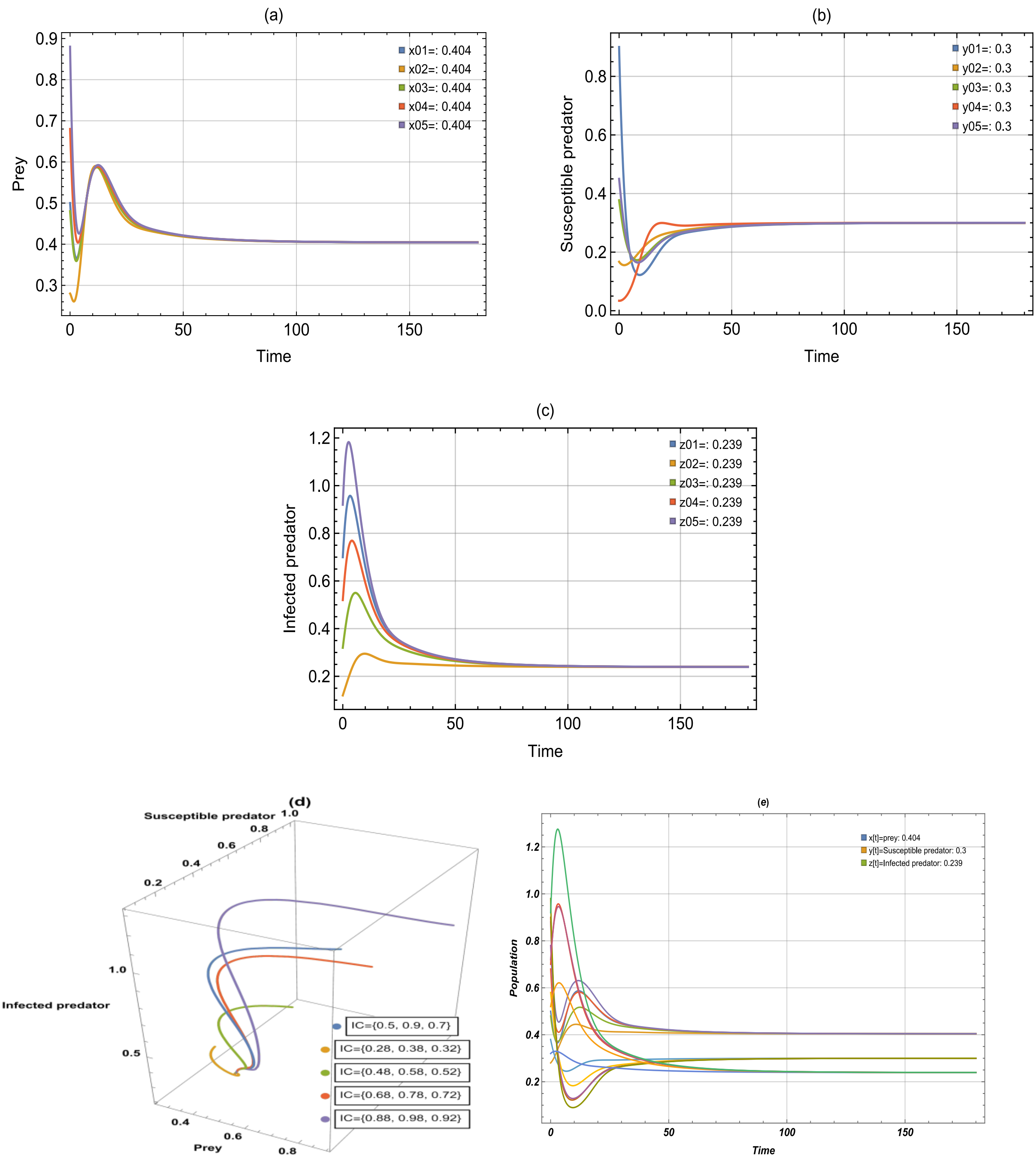

Figure 1: The trajectories of the system (2) by utilizing Table (2) and beginning from various

initial points. (a) The trajectories show case the motion exhibited by the prey with time. (b) The trajectories showcase the motion exhibited by a predator with time. (c) The trajectories show case the motion exhibited by scavengers versus time. (d) $3\mathcal{D}$ -Phase portrait of system (2). (e) Time series for the trajectories prey, susceptible predator, and infected of the system (2).

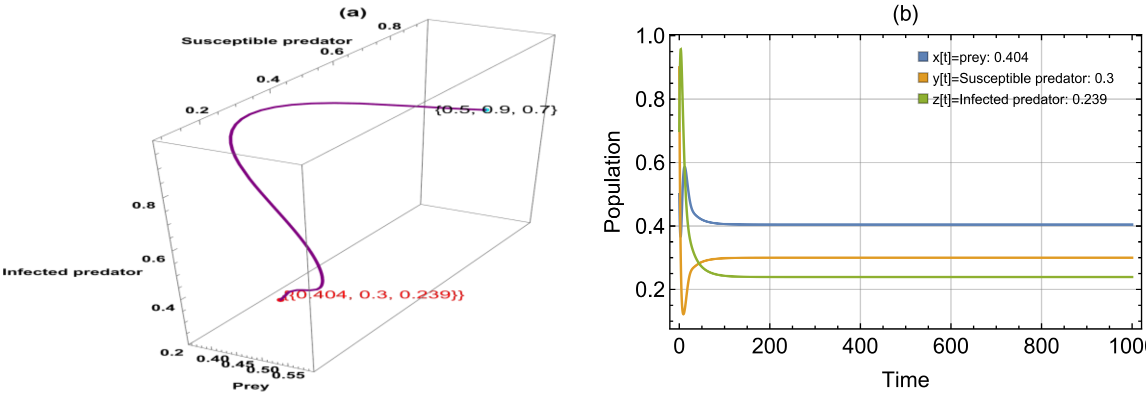

Figure 2: (a) $3\mathcal{D}$-Phase portrait of the system (2). (b)The time series exhibits the trajectories of the system (2) by utilizing Table (2), the trajectories of three species demonstrate an asymptotic positive convergence towards $Q_{xyz} = (0.404, 0.3, 0.239)$.

Now, we will begin to discuss each parameter and its impact on the system (2), as we noticed that there is only a quantitative effect on the second parameter w_1. As for parameter w_0, we notice two cases, the first is a case of approaching a globally asymptotically stable positive points $Q_{xyz} = (0.489, 0.3, 0.378)$, in the range $(0, 2.093)$ and the other is approaching a point where the extinction of the infected predator appears $Q_{xy} = (0.253, 0.282, 0)$, in the interval $[2.093, 5)$, which is shown in Figure 3.

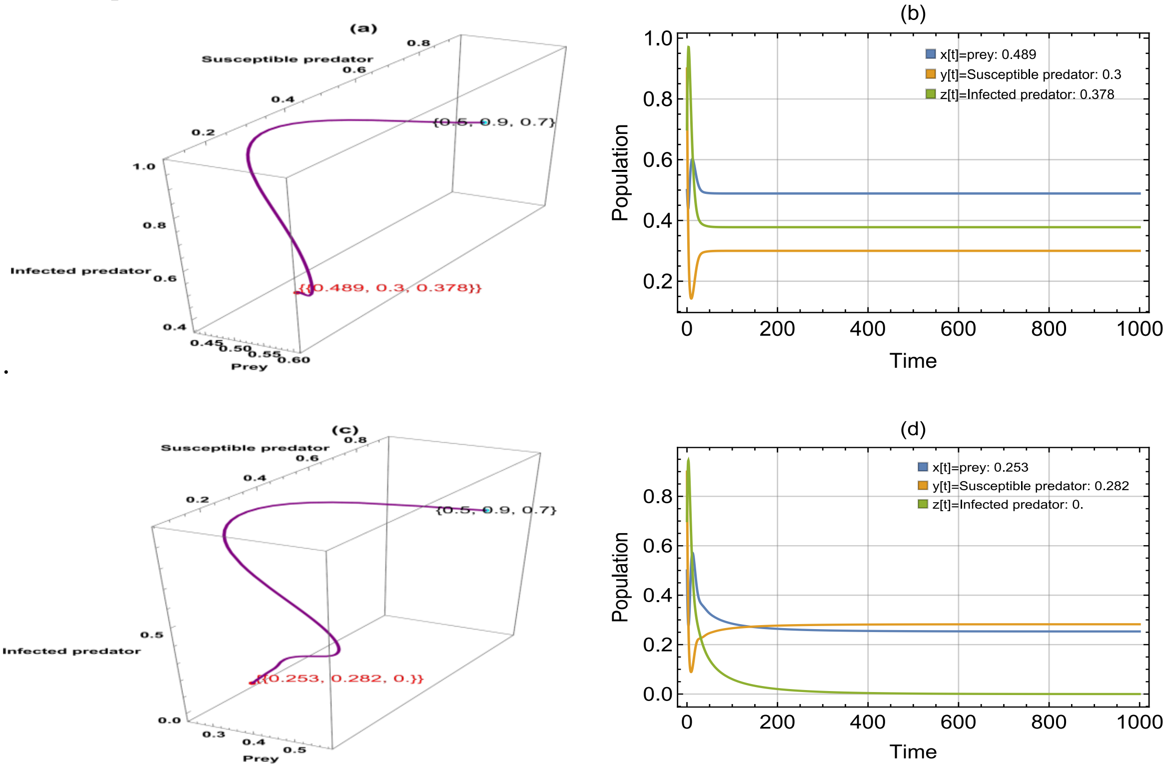

Figure 3: The trajectories of the system (2) by utilizing Table (2) with values of w_0. (a) $3\mathcal{D}$-Phase portrait when $w_0 = 0.3$. (b) Time series for the trajectories when $w_0 = 0.3$. (c) $3\mathcal{D}$-Phase portrait when $w_0 = 2.2$. (d) Time series for the trajectories when $w_0 = 2.2$.

When a parameter w_2 was substituted into multiple periods of range taken, we obtained three different approaches to globally asymptotically stable points of system (2), which are as follows, $Q_{xyz} = (0.341, 0.3, 0.135)$ in the period $(0, 0.376)$, $Q_{xy} = (0.228, 0.191, 0)$, in the interval $(0.376, 0.811]$, and $Q_x = (0.1, 0, 0)$ in the range $[0.811, 1)$ which is shown respectively in the Figure (4), and parameters are given by Table (2).

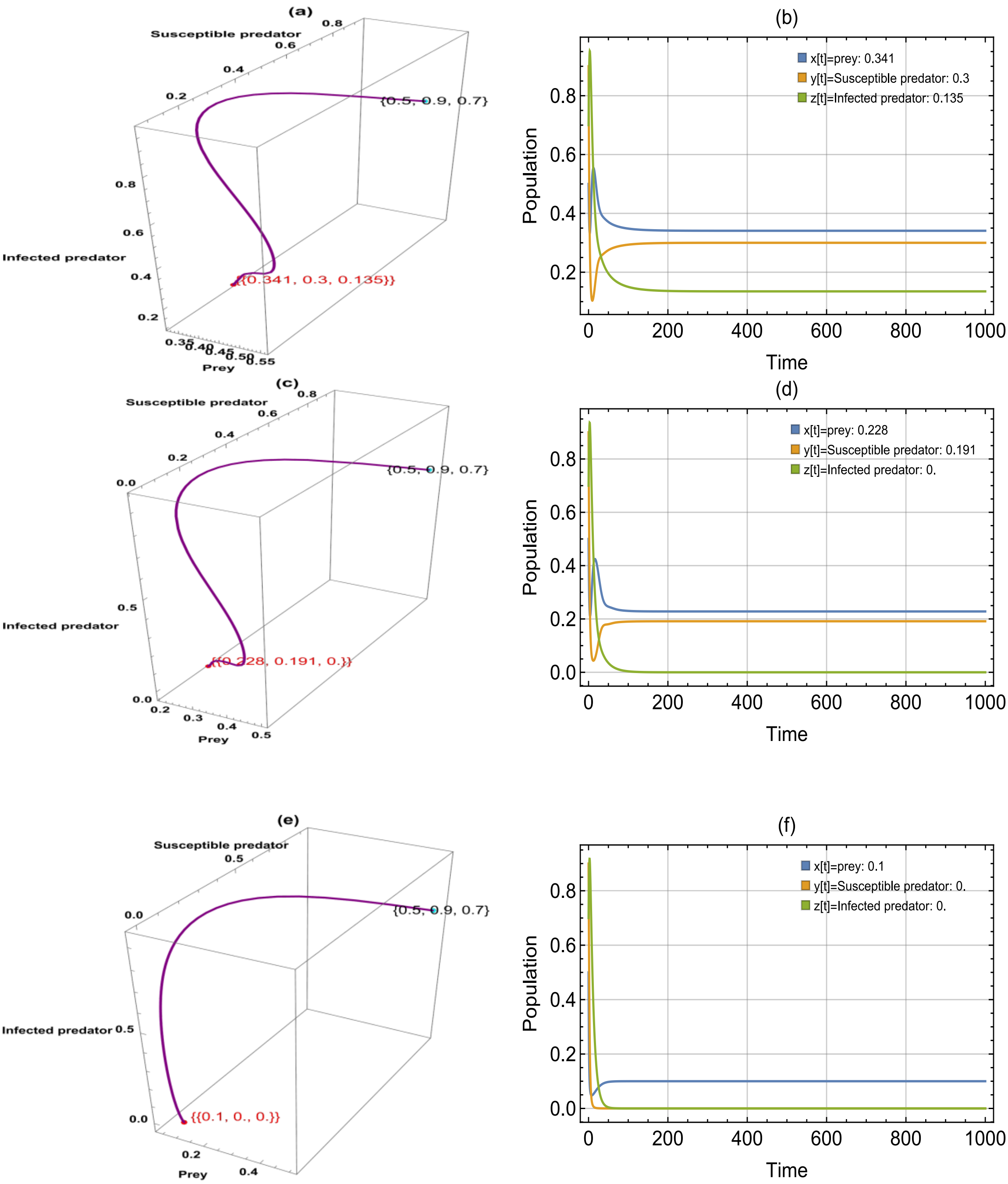

Figure 4: The trajectories of the system (2) by utilizing Table (2) with different values of w_1. (a) $3\mathcal{D}$-Phase portrait when $w_2 = 0.3$. (b) Time series for the trajectories when $w_2 = 0.3$. (c) $3\mathcal{D}$-Phase portrait when $w_2 = 0.5$. (d) Time series for the trajectories when $w_2 = 0.50$. (e) $3\mathcal{D}$-Phase portrait at $w_2 = 0.9$. (f) Time series for the trajectories when $w_2 = 0.9$.

Whenever a parameter w_3 was substituted into multiple periods of range taken, we obtained three different approaches to globally asymptotically stable points of system (2), which are as follows, $Q_{xyz} = (0.404, 0.3, 0.128)$ in the period $(0.558, 1)$, $Q_{xy} = (0.477, 0.231, 0)$, in the interval $(0.224, 0.558]$, and $Q_x = (0.75, 0, 0)$ in the range $(0, 0.224]$ which is shown respectively in the Figure (5), and parameters are given by Table (2).

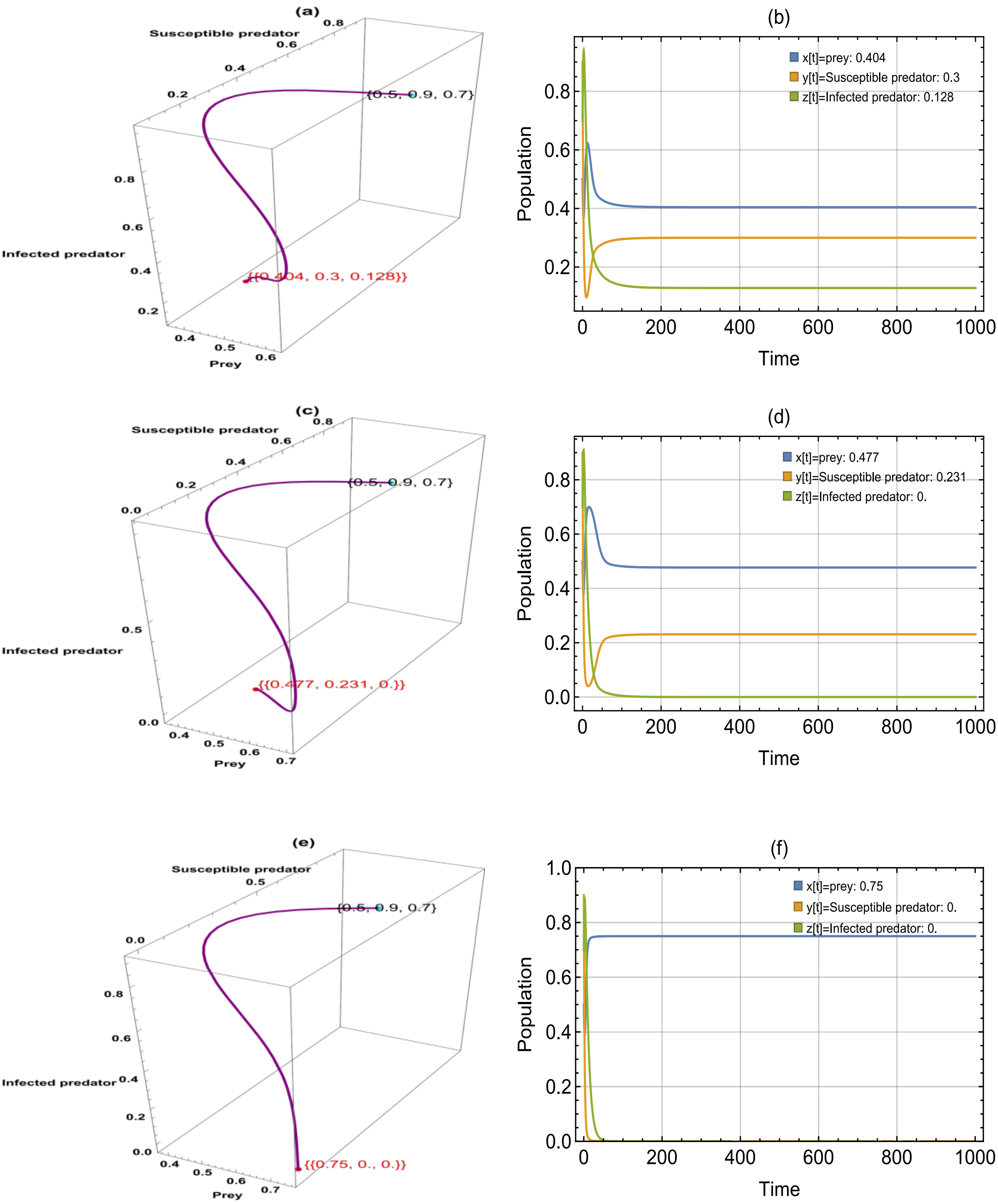

Figure 5: The trajectories of the system (2) by utilizing Table (2) with different values of w_3. (a) $3\mathcal{D}$-Phase portrait when $w_3 = 0.75$. (b) The trajectories of the time series at $w_3 = 0.75$. (c) $3\mathcal{D}$-Phase portrait when $w_3 = 0.45$. (d) The trajectories of the time series at $w_3 = 0.45$. (e) $3\mathcal{D}$-Phase portrait when $w_3 = 0.15$. (f) The trajectories of the time series when $w_3 = 0.15$.

Furthermore, a parameter w_4 was substituted into multiple periods of range taken, we obtained two different approaches to globally asymptotically stable points of system (2), which are as follows, $Q_{xyz} = (0.573, 0.146, 0.339)$ in the period $(0.279, 1)$ and $Q_{xy} = (0.301, 0.413, 0)$, in the interval $(0, 0.279]$, which is shown respectively in the Figure (6), and parameters are given by Table (2) .

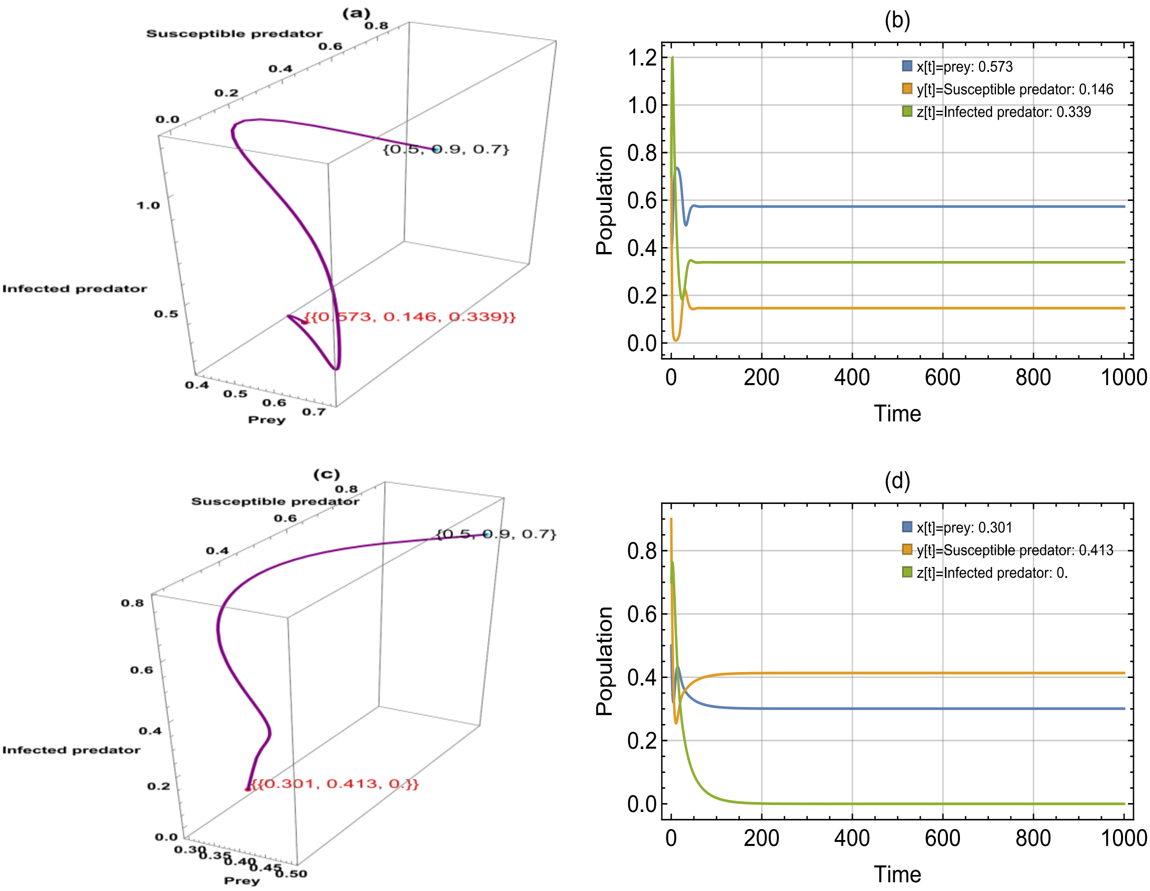

Figure 6: The trajectories of the system (2) by utilizing Table (2) with various values of w_4. (a) $3\mathcal{D}$-Phase portrait when $w_4 = 0.82$. (b) The trajectories of the time series at $w_4 = 0.82$. (c) $3\mathcal{D}$-Phase portrait when $w_4 = 0.22$. (d) The trajectories of the time series at $w_3 = 0.22$.

Alternatively a parameter w_5 was substituted into multiple periods of range taken, we obtained three different approaches to globally asymptotically stable points of system (2), which are as follows, $Q_{xyz} = (0.404, 0.3, 0.139)$ in the period $(0, 0.196)$, $Q_{xy} = (0.54, 0.175, 0)$, in the interval $[0.196, 0.595)$, and $Q_x = (0.75, 0, 0)$ in the range $[0.595, 1)$ which is shown respectively in the Figure (7), and parameters are given by Table (2) .

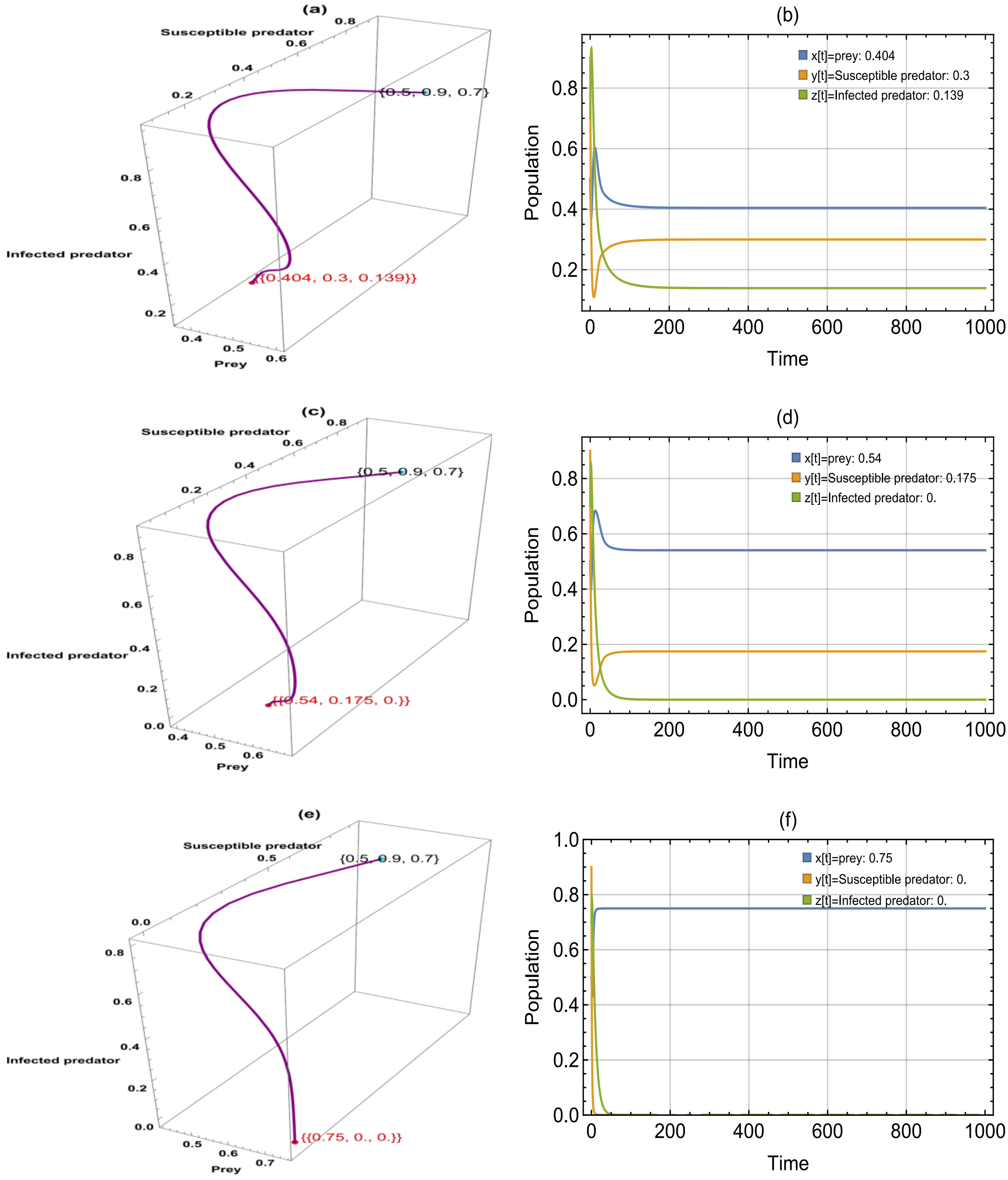

Figure 7: The trajectories of the system (2) by utilizing Table (2) with various values of w_5. (a) $3\mathcal{D}$-Phase portrait when $w_5 = 0.13$. (b) The time series of the trajectories when $w_5 = 0.13$. (c) $3\mathcal{D}$-Phase portrait when $w_5 = 0.33$. (d) The trajectories of the time series at $w_5 = 0.33$. (e) $3\mathcal{D}$-Phase portrait when $w_5 = 0.73$. (f) The trajectories of the time series at $w_5 = 0.73$.

However, a parameter w_6 was substituted into multiple periods of range taken, we obtained three different approaches to globally asymptotically stable points of system (2), which are as follows, $Q_{xyz} = (0.404, 0.3, 0.089)$ in the period $(0 , 0.186)$, $Q_{xy} = (0.641, 0.09, 0)$, in the interval $[0.186 , 0.585)$, and $Q_x = (0.75, 0, 0)$ in the range $[0.585 , 1)$ which is shown respectively in the Figure (8), and parameters are given by Table (2).

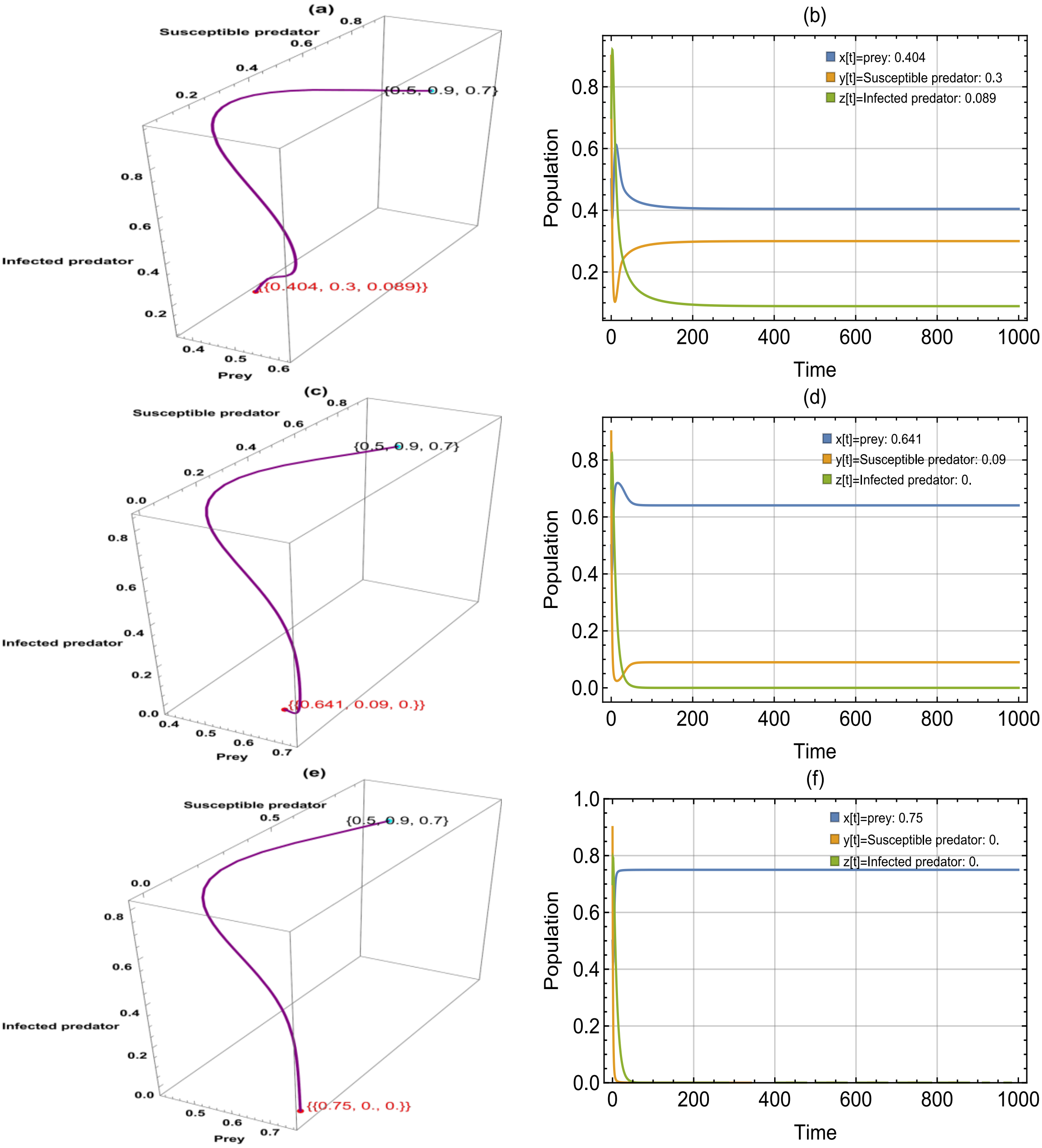

Figure 8: The trajectories of the system (2) by utilizing Table (2) with various values of w_6. (a) $3\mathcal{D}$-Phase portrait when $w_6 = 0.14$. (b) The trajectories of the time series at $w_6 = 0.14$. (c) $3\mathcal{D}$-Phase portrait when $w_6 = 0.44$. (d) The trajectories of the time series at $w_6 = 0.44$. (e) $3\mathcal{D}$-Phase portrait when $w_6 = 0.64$. (f) The trajectories of the time series at $w_6 = 0.64$.

When a parameter w_7 was substituted into multiple periods of range taken, we obtained two different approaches to globally asymptotically stable points of system (2), which are as follows, $Q_{xyz} = (0.569, 0.15, 0.682)$ in the period $(0, 0.129)$ and $Q_{xy} = (0.301, 0.413, 0)$, in the interval $[0.129, 1)$, which is shown respectively in the Figure (9), and parameters are given by Table (2).

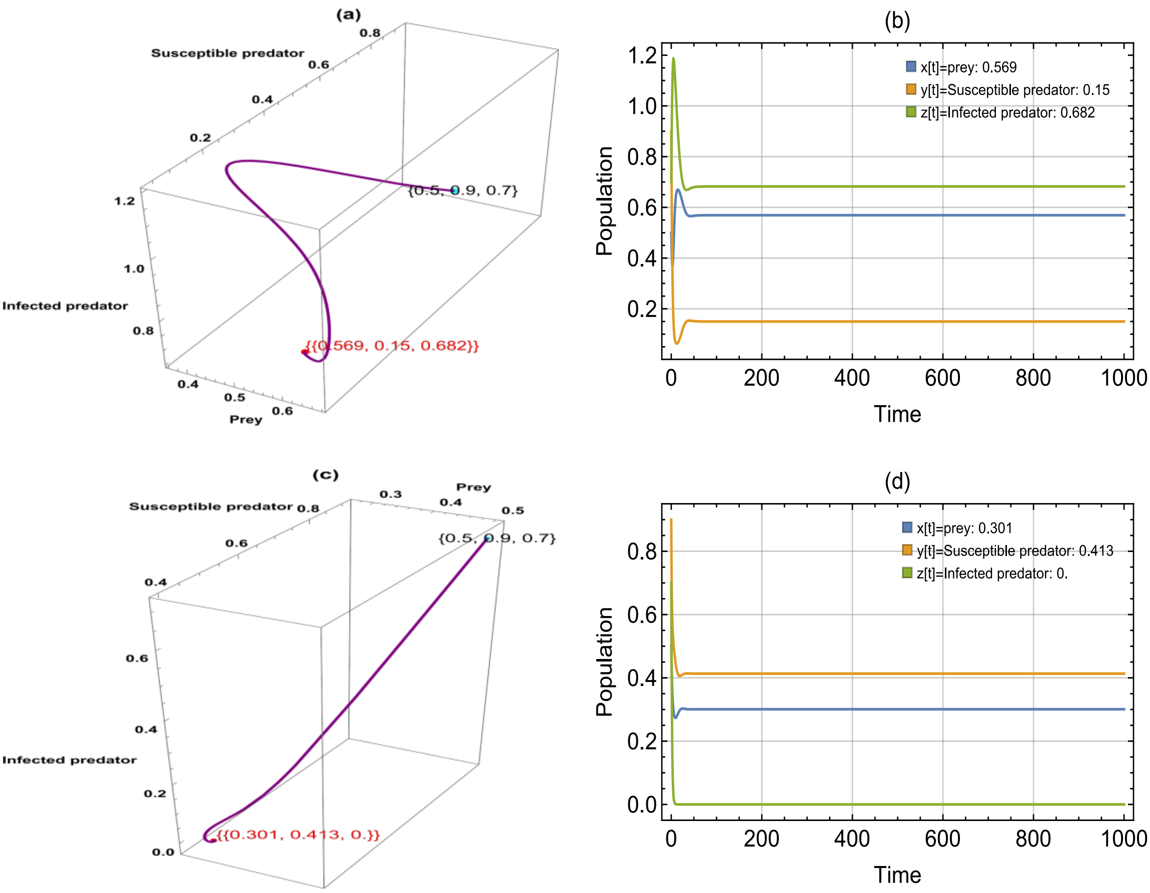

Figure 9: The trajectories of the system (2) by utilizing Table (2) with different values of w_7. (a) $3\mathcal{D}$-Phase portrait when $w_7 = 0.02$. (b) The trajectories of the time series at $w_7 = 0.02$. (c) $3\mathcal{D}$-Phase portrait when $w_7 = 0.86$. (d) The trajectories of the time series at $w_7 = 0.86$.

Finally, a parameter w_8 was substituted into multiple periods of range taken, we obtained two different approaches to globally asymptotically stable points of system (2), which are as follows, $Q_{xyz} = (0.363, 0.343, 0.14)$ in the period $(0, 0.089)$ and $Q_{xy} = (0.301, 0.413, 0)$, in the interval $[0.089, 1)$, which is shown respectively in the Figure (10), and parameters are given by Table (2) .

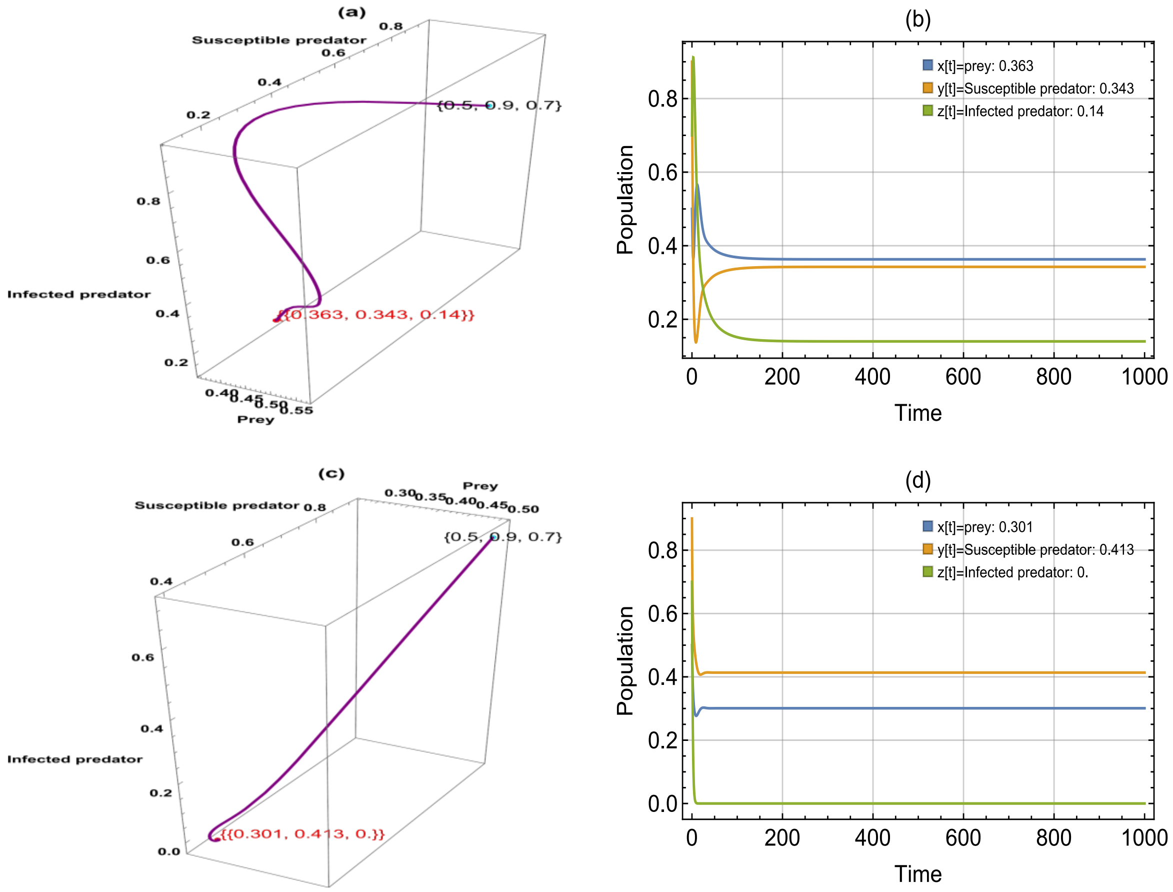

Figure 10: The trajectories of the system (2) by utilizing Table (2) with various values of w_8. (a) $3\mathcal{D}$-Phase portrait when $w_8 = 0.057$. (b) The trajectories of the time series at $w_8 = 0.057$. (c) $3\mathcal{D}$-Phase portrait when $w_8 = 0.77$. (d) The trajectories of the time series at $w_8 = 0.77$.

Through our investigation of the system (2), we have found that at one of its points, which is the point $Q_{xy} = (\breve{x}, \breve{y}, 0)$, when applying the conditions (7a,7b,7c) a bi-statble case appeared at the value of $w_3 = 0.9$. This outcome proves there is no unique global point. In order to confirm the theoretical findings, a numerical representation has been shown in the Figure (11)

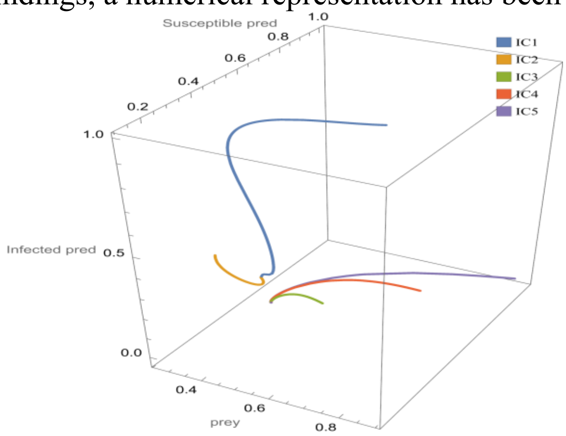

Figure 11: $3\mathcal{D}$-Phase portrait when $w_3 = 0.9$

9. Conclusions

Relationships between predators and their prey are an important aspect of the natural world, having consequences for the equilibrium of ecosystems and the survival of species. These interactions are intricate and can be affected by different factors like existence of refuge and the prevalence of infectious diseases. A model was taken for the prey and the predator, and there was a presence of an infected predator in one of the predators. The existence which has four solution points has been proven, and stability conditions were set for them. During the investigation of solution points in a bi-stable, it has been discovered that there is no unique global point at the value of Q_{xz}.

After that, each point was created for its local stability. Each of these was done by developing theories to prove them theoretically, as well as their numerical analysis has been studied to confirm their validity. The results reached were consistent, as we noticed that when changing the parameters in Table 2, the effect of some parameters become more than others, and only one parameter had a quantitative effect. Fear of the prey was one of the parameters that had a clear effect on changing the behavior of the system. The presence of the harvest in the system in the three equations had a great impact, and this is what we saw through the drawing, as well as the effect of both death and births was clear.

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