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Certain Properties of The Result Involution Graph Related to The O'Nan Group

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Abstract
The behavior of the result involution graph associated with the finite simple group O'N was investigated in this article. This includes determining particular properties of the graph $\Gamma_{O'N}^{RI}$ and showing their effect on the algebraic properties of the group O'N.

Mathematics Subject Classification: 05C25 20D60 20D08.
Keywords: Pariahs groups, Result Involution Graph, Connectedness, Girth.

خصائص معينة لبيان ناتج الالتفاف المتعلقة بزمرة أونان

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الخلاصة
تمت سلوك بيان ناتج الالتفاف المرتبط في الزمرة البسيطة المنتهية O'N في هذا المقال. وهذا تضمن تحديد خواص معينة للبيان $\Gamma_{O'N}^{RI}$ وإظهار تأثيرها على الخصائص الجبرية للزمرة O'N.

1. Introduction

The method of creating graphs with a vertex set consisting of specific group elements has been successfully applied in several applications. Approaches from recently have proven useful in this field of studies see [1-6]. Involution elements of a group are those with order 2. In graph theory there are many graphs base on the involution elements for example commuting involution graphs and S3-involution graphs see [7-12]. Another type of graph that primarily depends on the involution components is the results involution graph. The result involution graph on a finite group G denoted by Γ_G^{RI} with $V(\Gamma_G^{RI})=G$ (elements of G) and $(\alpha,\beta)\in E(\Gamma_G^{RI})$ if and only if $\alpha\neq\beta$ and $\alpha\beta\in I(G)$, where $I(G)$ stands for the involution elements of G. In [13], Jund and Salih were the first to construct this graph, also determinate several features for Γ_G^{RI}, for example, provided the connectivity for Γ_{Sn}^{RI} and Γ_{An}^{RI} when n>4, where S_n and A_n stand for the symmetric and alternating group respectively. Furthermore, estimate the girth , the diameter and the radius of these graphs. In addition, for the Mathieu groups M$_i$ for $i\in\{11,12,22,23,24\}$, the foundation of $\Gamma_{M_i}^{RI}$ are analyzed with full information can be seen in [14]. For the notations and general

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terminology of this paper, assume that Γ is a finite graph, $V(\Gamma)$ and $E(\Gamma)$ are the vertex set and edges set of Γ, respectively. For a random a clique in Γ we use the symbol C_q. Let $d(\alpha, \beta)$ be the distance between the vertices α, β in the graph Γ, then the $Diam(\Gamma) = max\{d(\alpha,\beta) : \alpha,\beta \in V(\Gamma)\}$ and if the diameter exists the radius can be define as $R(\Gamma) = min\{d(\alpha,\beta) : \alpha,\beta \in V(\Gamma)\}$. For a vertex α, the neighbor $N(\alpha)=\{\beta \in V(\Gamma): d(\alpha,\beta)=1\}$ and the close neighbor $N[\alpha] = N(\alpha)\cup\{\alpha\}$. Moreover, local clustering coefficient for α is define as $C(\alpha) = \frac{2*\xi}{|N(\alpha)|*(|N(\alpha)|-1)}$, where ξ represents the number of edges connected with vertices of $N(\alpha)$. Furthermore, the chromatic number is define to be the minimum amount of characters required to cover the vertex set, ensuring that any two neighboring vertices have different characters. Finally, if some vertices in the graph with degree $\mathfrak{r}$ and the other with dgree $\mathfrak{s}$ then the graph is said to be $(\mathfrak{r}, \mathfrak{s})$-birgular graph. Additional details on this setting can be found in [15] and [16]. However, as a novel simple group, Michael O'Nan established the existing of the O'N group and presented it as a permutation group on a 122760 points in 1976. For a deep study of the properties of O'N group see [17]. The work aims at establishing the connectedness of $\Gamma_{O'N}^{RI}$ together with specific graph characteristics and examining how these features reflect on the structure of the group O'N.

2. Preliminary

Consider a finite group G, in this section particular terminology and result associate with Γ_G^{RI} are offered. Let we assume that the set of order 4 elements in G is denoted by $F(G)$. The proof of the next results can be found in [8]:

Proposition 2.1: [13] The size of the graph Γ_G^{RI} equal to half of the amount $(|I(G)||G| - |F(G)|)$.

Definition 2.2: [13] Given a finite group G, a simple graph whose vertex set are the conjugacy classes of G and two vertices L, K are adjacent if $(\alpha,\beta) \in E(\Gamma_G^{RI})$ where α and β are in L and K, respectively. This graph is called the resize graph of Γ_G^{RI} and expressed as Γ_G^{RS}.

A crucial finding about the impact of connectivity of Γ_G^{RS} on the structure of Γ_G^{RI} is presented in the following:

Proposition 2.3: [13] Given a finite group G, then the connectedness of Γ_G^{RS} is a necessary and sufficient condition for the graph Γ_G^{RS} to being connected.

With aid of the programming GAP [18] ant its graph package Yags [19], the following example can be held. We will replace the vertex by its position in the vertex set in all figures of this paper.

Example 2.4: Take into consideration the group G with order 24 and ID 8 inside the Small Group library of GAP. Thus, $G=<\alpha,\beta,\mu: [\alpha^2, \beta^2, \mu^3, (\alpha*\mu)^2, \mu^{-1}*\beta*\mu*\beta, (\beta*\alpha)^4]>$ and thus the elements of G presented as : $V(\Gamma_G^{RI})=\{e, \beta, \alpha, (\alpha*\beta)^2, \mu, \beta*\alpha, \alpha*\beta*\alpha, \beta*\mu, \beta*\alpha*\beta, \alpha*\mu, (\alpha*\beta)^2*\mu, \mu^{-1}, \alpha*\beta, \beta*\alpha*\mu, \alpha*\beta*\alpha*\mu, \beta*\mu^{-1}, \beta*\alpha*\beta*\mu, \alpha*\mu^{-1}, (\alpha*\beta)^2*\mu^{-1}, \alpha*\beta*\mu, \beta*\alpha*\mu^{-1}, \alpha*\beta*\alpha*\mu^{-1}, \beta*\alpha*\beta*\mu^{-1}, \alpha*\beta*\mu^{-1}\}$

Furthermore, Γ_G^{RI} is connected of size 105 with diameter, radius and girth equal to 3, and clique number 4. Moreover, the vertex degree for each vertex is respectively, equal:

9, 9, 9, 9, 9, 8, 9, 9, 9, 9, 9, 9, 8, 8, 9, 9, 9, 9, 9, 8, 8, 9, 9, 8.

On the other hand, $V(\Gamma_G^{RS})$={ 1∨A, 2∨A, 2∨B, 2∨C, 3∨A, 4∨A, 6∨A, 6∨B, 6∨C}, where the vertex n∨Alphabetical is the class n Alphabetical that mentioned in the online Atlas [20]. Thus, the graph Γ_G^{RS} is connected of size 15 with 105 with diameter, clique number and girth equal to 3, and radius 2. Moreover, the vertex degree for each vertex is respectively, equal:

3, 5, 3, 3, 4, 3, 3, 4,3.

In the next figures we present both Γ_G^{RI} and Γ_G^{RS}:

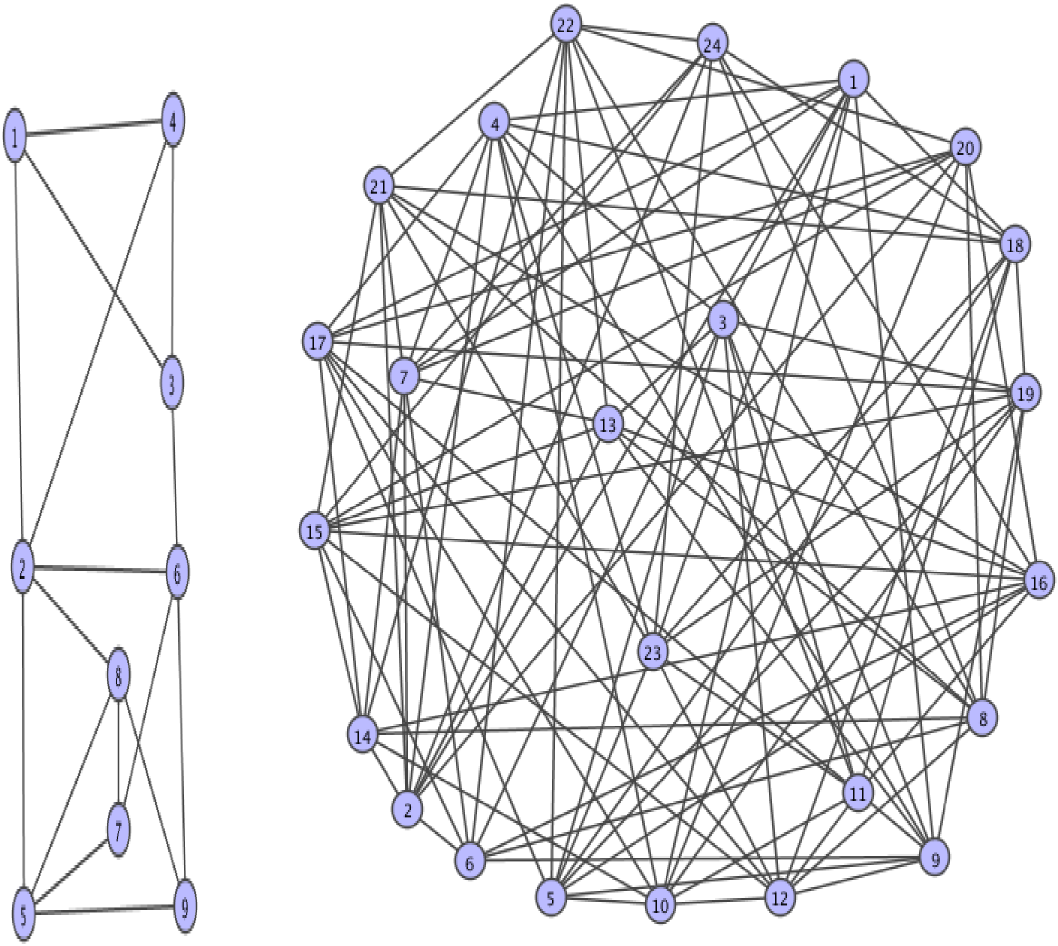


Figure 1: The Result Involution Graph and Resize graph of G.

3 .Main results

The online Atlas provide permutation representation on 122760 points for the O'Nan group which can be implemented within GAP. Furthermore, the order of this group is 460815505920 with one class of involution. This means we need to deal with the result involution graph has order about 5×10^{11} which is very complicated even computationally. Thus, we first work on the resize graph of this group to obtain some properties of the graph $\Gamma_{O'N}^{RI}$.

3.1. *Observations about $\Gamma_{O'N}^{RS}$*

We will first make some observations on the graph $\Gamma_{O'N}^{RS}$, the graph has an order of 30 along with a vertex set determined by $V(\Gamma_{O'N}^{RS})$={ 1∨A, 2∨A, 3∨A, 4∨A, 4∨B, 5∨A,

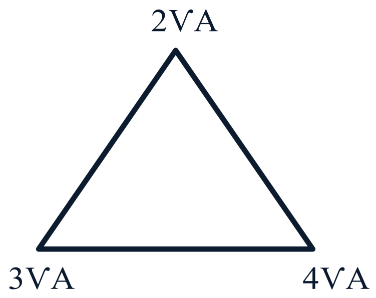
6VA, 7VA, 7VB, 8VA, 8VB, 10VA, 11VA, 12VA, 14VA, 15VA, 15VB, 16VA, 16VB, 16VC, 16VD, 19VA, 19VB, 19VC, 20VA, 20VB, 28VA, 28VB,31VA, 31VB}.
The neighbor of $V(\Gamma_{O'N}^{RS})$ along with their local clustering coefficient presented in the following:

i-N(1VA)={2VA} with C(1VA)=0.
iiN(2VA)={1VA,3VA,4VA,4VB,5VA,6VA,7VA,7VB,8VA,8VB,10VA,11VA,12VA, 14VA,15VA,15VB,16VA,16VB,16VC,16VD,19VA,19VB,19VC,28VA,28VB}
then we have C(2VA)=0.455.
iii-N(3VA)={2VA,4VA,4VB,5VA,6VA,7VA,7VB,8VA,8VB,10VA,11VA,12VA, 14VA,15VA,15VB,16VA,16VB,16VC,16VD,19VA,19VB,19VC,20VA,20VB,28VA, 28VB,31VA,31VB}. Also for L ∈ {4VB,5VA, 6VA, 7VA, 7VB, 8VA, 8VB, 10VA, 11VA,14VA,16VA,16VB,16VC,16VD,19VA,19VB,19VC,28VA,28VB}, N(L)= N[3VA]\{ L } and , for K=3VA or L we have $C(K)$=0.4907407407407408.
iv-N(4VA)={2VA,3VA,4VB,5VA,6VA,7VA,7VB,8VA,8VB,10VA,11VA,14VA,16VA, 16VB,16VC,16VD,19VA,19VB,19VC,20VA,20VB,28VA,28VB,31VA,31VB},
in this case we have C(4A)=0.4933333333333333.
vN(12VA)={2VA,3VA,4VB,5VA,6VA,7VA,7VB,8VA,8VB,10VA,11VA,14VA,15VA,15VB,16VA,16VB,16VC,16VD,19VA,19VB,19VC,20VA,20VB,28VA,28VB,31VA,31VB}. Also, for L ∈ {15VA,15VB}, N(L)= N[12VA]\{ L} and $C(K)$=0.4943019943019943, for K=12VA or L.
viN(20VA)={3VA,4VA,4VB,5VA,6VA,7VA,7VB,8VA,8VB,10VA,11VA,12VA,14VA,15VA,15VB,16VA,16VB,16VC,16VD,19VA,19VB,19VC,20VB,28VA,28VB,31VA,31VB}.Also for L ∈ {20VB, 31VA,31VB}, N(L)= N[20VA]\{ L } and $C(K)$=0.4957264957264957, for K=20VA or L.

As a result, the set $V(\Gamma_{O'N}^{RS})$ have degrees respectively equal to { 1, 25, 28, 25, 28, 28, 28, 28, 28, 28, 28, 28, 28, 27, 28, 27, 27, 28, 28, 28, 28, 28, 28, 28, 27, 27, 28, 28, 27, 27}.

Finally, by the above observation we have the size of $\Gamma_{O'N}^{RS}$ equal to 400.

Lemma 3.1: The graph $\Gamma_{O'N}^{RS}$ is connected such that Ʀ($\Gamma_{O'N}^{RS}$)=2, Diam($\Gamma_{O'N}^{RS}$)=3, girth of size 3, clique number 27 and chromatic number 27.
Proof: We have N(3VA) = $V(\Gamma_{O'N}^{RS})$ \{ 3VA,1VA}, but since 1VA linked with 2VA thus we obtain the connectivity of $\Gamma_{O'N}^{RS}$. Furthermore, the graph $\Gamma_{O'N}^{RS}$ contains the following cycle:



Therefore, we get the girth of $\Gamma_{O'N}^{RS}$ is 3. Moreover, have 1≤{ȡ(L, K) : $L, K \in V(\Gamma_{O'N}^{RS})$} ≤3, in fact, the minimum vertex eccentricity is 2VA such that {ȡ(2VA, K) : $K \in V(\Gamma_{O'N}^{RS})$ }={1,2}, while the maximum vertex eccentricity is 31VA where {ȡ(31VA, K) : $K \in V(\Gamma_{O'N}^{RS})$ }={1,2,3}, thus we conclude that $Ʀ(\Gamma_{O'N}^{RS})$=2 and $Diam(\Gamma_{O'N}^{RS})$=3. Also, one can see that the graph $\Gamma^{RS}_{O'N}$ has 5 cliques are given as follows:

C_q1={1VA,2VA} with order 2,
C_q2={2VA,3VA,4VA,4VB,5VA,6VA,7VA,7VB,8VA,8VB,10VA,11VA,14VA, 16VA,16VB,16VC,16VD,19VA,19VB,19VC,28VA,28VB} of order 22,
C_q3={2VA,3VA,4VB,5VA,6VA,7VA,7VB,8VA,8VB,10VA,11VA,12VA,14VA,15VA,15VB,16VA,16VB,16VC,16VD,19VA,19VB,19VC,28VA,28VB} of order 24,
C_q4={3VA,4VA,4VB,5VA,6VA,7VA,7VB,8VA,8VB,10VA,11VA,14VA,16VA,16VB,16VC,16VD,19VA,19VB,19VC,20VA,20VB,28VA,28VB,31VA,31VB} of order 25, and
C_q5={3VA,4VB,5VA,6VA,7VA,7VB,8VA,8VB,10VA,11VA,12VA,14VA,15VA,15VB,16VA,16VB,16VC,16VD,19VA,19VB,19VC,20VA,20VB,28VA,28VB,31VA,31VB} of order 27.

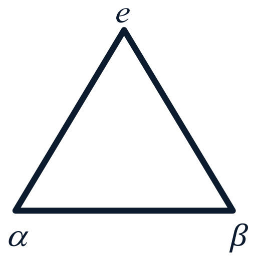
Therefore, the clique number of $\Gamma^{RS}_{O'N}$ is 27. Then the chromatic number of $\Gamma^{RS}_{O'N}$ is greater than or equal to 27. Thus, we need to deal with the vertices {1VA,2VA,4VA} which are not in C_q5. Since 31VA ∉N(2VA) then 2VA and 31VA could have the same color. Similarly, 4VA, 15VA have the same color, also 1VA and 31VB. Consequently, we have chromatic number is 27.

3.2. The structure of $\mathbf{\Gamma^{RI}_{O'N}}$

First, we apply the online Atlas and Gap to find the number of the edges between any two O'N-conjugacy classes. For example the number of the edges for the classes 3A and the other O'N-conjugacy classes given as follows:
{3VA(115203876480),4VA(12800430720),4VB(1497650394240),5VA(1984066761600),6VA(8633890520640),7VA(115203876480),7VB(8755494612480),8VA(12211610906880), 8VB(12211610906880),10VA(12326814783360),11VA(36404424967680),12VA(12544422105600),14VA(21312717148800),15VA(8397082552320),15VB(8397082552320),16VA(25805668331520),16VB(25805668331520),16VC(25805668331520),16VD(25805668331520),19VA(21197513272320),19VB(21197513272320),19VC(21197513272320),20VA(20736697766400),20VB(20736697766400),28VA(14054872930560),28VB(14054872930560), 31VA(12442018659840),31VB(12442018659840)}.

Lemma 3.2: For all $\alpha \in V(\Gamma^{RI}_{O'N})$ we have $d(e,\alpha) \leq 3$.
Proof: Since N(e)=2VA, thus $d(e,\alpha) = 1$, for all $\alpha \in$ 2VA. Also, for $\alpha \in L$, where $L \in$ N(2VA) there is an edge linked α with certain involution in 2VA, hence, $d(e,\alpha) = 2$ for all $\alpha \in L$. Finally, take $\alpha \in V(\Gamma^{RI}_{O'N}) \setminus L$, such that $L \in$ N[2VA]∪ $\{e\}$. Then again there is an edge between α and particular vertex in the classes of N(2VA), so that $d(e,\alpha) = 3$, in this case. Then we conclude that $d(e,\alpha) \leq 3$ for all $\alpha \in V(\Gamma^{RI}_{O'N})$.

Theorem 3.3: The result involution graph Γ^{RI}_{G} for G≅ O'N is connected (2857239, 2857238)-biregular graph with girth equal *3*.
Proof: By using Lemma 3.1 we have $\Gamma^{RS}_{O'N}$ is connected thus Proposition 2.3 implies the connectivity of the graph $\Gamma^{RI}_{O'N}$. Also, the graph Γ^{RI}_{G} is (2857239, 2857238)-biregular because we have *degree*(α)=2857239 for all $\alpha \in V(\Gamma^{RI}_{O'N})$ such that α not of order 4 and when $\alpha \in F$(G), we get *degree*(α)=2857238 this because α^2 will be an involution element. Now, let α be an involution elements in G then | N(α)∩2VA|=1750 , thus since N(e)=2VA, there is $\beta \in$ N(α) such that we have the following cycle in Γ^{RI}_{G}:



Thus, the girth is 3. From the structure of the result involution graph of O'N, we can see that the eccentricity of the non-identity vertex is 2 this because the graph $\Gamma^{RS}_{O'N\backslash\{e\}}$ has diameter 2. Furthermore, by Lemma 3.2 the eccentricity of e is 3, as a result we obtain $R(\Gamma^{RI}_{O'N})$=2, and $Diam(\Gamma^{RI}_{O'N})$=3.

The structure of the graph $\Gamma^{RI}_{O'N}$ can be utilized to obtain the following results.

Proposition 3.4: Let $G\cong$ O'N, then the number of D_{2n} in G are given in the following table:

Table 3.1: The number for D_{2n} in O'N for certain n

n, Number of D_{2n}	n=2, 2500084125	n=3, 58881981312	n=4, 121604091840
n=5, 128004307200	n=6, 345611629440	n=7, 131661573120	n=8, 691223258880
n=10, 230407752960	n=11, 460815505920	n=12, 460815505920	n=14, 460815505920
n=15, 921631011840	n=16, 1843262023680	n=19, 1382446517760	n=28, 921631011840

Proof: The dihedral group D_{2n} generate by two elements $<\alpha,\beta>$ such that α is an involution element and β elements of order n such that $d(\alpha,\beta)$=1 in Γ^{RI}_{G}. We have the number of the edges between the elements of the class 2VA and the elements of the class of order n are given in the below:
2VA(2500084125),3VA(58881981312),4VA(6400215360),4VB(115203876480), 5VA(128004307200),6VA(345611629440),7VA(65830786560),7VB(65830786560),8VA(345611629440),8VB(345611629440),10VA(230407752960),11VA(460815505920),12VA(460815505920),14VA(460815505920),15VA(460815505920),15B(460815505920),16VA(4 60815505920),16VB(460815505920),16VC(460815505920),16VD(460815505920),19VA(460815505920),19VB(460815505920),19VC(460815505920),28VA(460815505920),28VB (460815505920).
Then we conclude our result.

Proposition 3.5: Let $\alpha\in$ 3VA, then the set $\{\beta\in$ 5VA: $<\alpha,\beta>>\cong A_5\}$ has size 13950.
Proof: Given $\alpha\in$ 3VA, then from the presentation of A_5 (see The Online Atlas) we have $<\alpha,\beta>>\cong A_5$ is true only if $\beta\in$ N(α). However, between 3VA and 5VA there are 1984066761600 edges. Therefore, the result follow by divided this number by |3VA|=142227008 as we look at certain vertex in 3VA.

4. Conclusions

During this work, the behavior of the elements of the O'N group and their relationship with the involution elements was studied. The result involution graph was used to obtain a number of important results. For example, the number of D_{2n} in O'N for certain n, and the construction of A_5 in O'N are explored. This work significantly paves the way for further investigations into Γ^{RI}_{G} on various related types of simple groups, including the exceptional lie type groups, the linear groups, and classical groups.

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