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Inequality of Fuzzy Soft κ Quasi-Hyponormal Operator Formulated Along with Several Analytic Features

Salim Dawood Mohsen*

Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq

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Abstract

Fuzzy soft operator theory (FS-OT) is the connection between operator theory and fuzzy theory that has recently been raised and discussed, which has a central significant role in the evolution of functional analysis. In this sequel, a new general class of fuzzy soft operators on fuzzy soft Hilbert space (FS-H) are introduced, called fuzzy soft κ quasi-hyponormal operators. These operators are generalized fuzzy soft hyponormal operators. Then, a sufficiently equivalent major condition for these imposed operators called feature characterization is achieved. Besides, several interesting features of quasi-hyponormal operators are yielded along with other outcomes. Therefore, conditions concerning the achieving of those analytical features are also investigated.

Keywords: Soft set, fuzzy soft Hilbert space, fuzzy soft operator, fuzzy soft quasi-hyponormal operator.

متراجحة المؤثر شبه فوق السوي *k* الضبابي الناعم المعرف مع صفاته التحليلية

سالم داود محسن*

[1]قسم الرياضيات, كلية التربية, جامعة المستنصرية, بغداد, العراق

الخلاصة

نظرية المؤثر الضبابي الناعم (FS–OT) هي العلاقة بين نظرية المؤثر والنظرية الضبابية التي تم طرحها ومناقشتها مؤخرًا، والتي لها دور مركزي مهم في تطور التحليل الدالي. في هذالبحث، تم تقديم فئة عامة جديدة من المؤثرات الضبابية الناعمة في فضاءات هيلبرت الضبابية الناعم (FS–H)، وهي المؤثرات شبه فوق السوية k الضبابية الناعمة ، وهي مؤثرات معممة للمؤثرات فوق السوية الضبتبية الناعمة. بعد ذلك، يتم تحقيق شرط رئيسي مكافئ بدرجة كافية لهذا المؤثر المطروح يسمى توصيف الميزات. الى جانب ذلك، يتم الحصول على العديد من الصفات المثيرة للاهتمام لهذا المؤثر جنبا إلى جنب مع نتائج أخرى. علاوة على ذلك، هناك بعض الصفات الذي لايمكن تحقيقها. ولذلك، يتم أيضًا دراسة الشروط المتعلقة بتحقيق تلك الصفات التحليلية.

1. Introduction

Functional analysis (FA) is a remarkable classical mathematical discipline, that closely correlates with operator theory, integral equation theory, and operational calculus. This discipline's origin was constructed independently in 1933. Since then, diverse renowned scholars have recently discussed interesting functional analysis aspects having fruitful for the

*Email: dr_salim2015@yahoo.com

development of a variety of mathematical domains, for instance, Dowari and Goswami [1], Rashid [2], Mecheri [3], Yuwaningsih and Muhammad [4], Hoxha et al. [5], Mortad [6-7], Dehimi and Mortad [8], Dharmarha and Ram [9-11], Ramesh and Osaka [12-13], Ram and Dharmarha [14], Mohsen [15], Enose et al. [16], Ganesh and Reddy [17], and others. On the other hands, the theory of soft sets (Ss-T) is principally interested in mathematical modeling and is an extended theory of sets. It is employed as a tool for modeling uncertainties by

correlating a set along with a parameterized set to resolve numerous convoluted problems that cannot dealt with through classical ways in various disciplines, for instance, medical science, engineering, and economics. This theory was first put forward by Molodtsov [18] in 1999. For the last two decades, the study of Ss-T has received significant and rising attention. In 2002, Maji et al. [19] utilized Ss-T for decision-making, modeling problems based on rough mathematics. Subsequently, in 2003, Maji et al. [20] conducted rigorous and interesting studies on Ss-T. Later, a lot of investigators imposed new extended notions in terms of soft sets and discussed their features, such as soft metric space [21], soft normed space [22], soft inner product spaces [23], and soft Hilbert spaces [24]. In 2002, Maji et.al [25] first studied fuzzy soft sets (FSs) that include fuzzy sets terms and soft sets terms. This original study drove the birth of fuzzy soft set theory (FSs-T), which has a gorgeous role in the study of functional analysis. Since then, by utilizing FSs-T concepts, several renowned scientists have developed diverse, captivating implementations of Ss-T. In 2007, Roy and Maji [26] yielded some fruitful implementations of FSs-T. In the same year, Yang et al. [27] presented operations on FSs based on the logic of fuzzy operators. Next, in 2009, Yang et al. [28] investigated a combination of an interval-valued fuzzy set along with a soft set. Afterward, in 2014, Beaula and Gunaseeli [29] provided spaces called fuzzy soft metric spaces (FS-M). The following year, in 2015, Beaula and Priyanga [30] offered more general of FSs-M, namely fuzzy soft normed spaces (FS-N). Moreover, diverse scholars have examined a variety of fabulous implementations of FSs-T in diverse disciplines and contributed to the development and highlighting of significant new aspects of this theme, referring to some of their pivotal contributions, for instance, [31-38]. In the same period, in 2020, Nashat et al. [39], provided several magnificent and effective outcomes in FSs-T. Significant generalizations of FS-N named fuzzy soft inner product (FS-IP) posed on fuzzy soft vector spaces (FS-V) and fuzzy soft Hilbert spaces (FS-H) were originated. They also attempted to evolve Fuzzy soft operator theory (FS-OT) by studying certain fuzzy soft linear operators on FS-H, such as fuzzy soft right and left shift operators, and to investigate several concerning features of fuzzy soft spectral theory (FS-ST). Later, in 2023, S. D. Mohsen [40] presented a special class of quasi normal sort operators, namely fuzzy soft normal operators. They examined the connection between the proposed operator and fuzzy soft self-adjoint operator. The present study is principally related to FS-OT. This effort investigates and presents a new class of fuzzy soft κ quasi hyponormal operators. It is a generalized fuzzy soft hyponormal operator. Several features of this considered operator are examined, including the characterization condition, inverse, and power, Then, addition and multiplication of fuzzy soft κ quasi hyponormal operators are also analysed.

2. Preliminaries

This section reviews significant terms and several relevant results in the sense of FS-OT, which will be employed for our discussions of certain analytical features involving characterization feature, inverse, and power, Then, addition and multiplication of new proposed fuzzy soft κ quasi hyponormal operators.

Definition 2.1. [13] The set of order pair $\hat{\mathcal{A}} = \{(\varkappa, \mu_{\hat{\mathcal{A}}}(\varkappa)) | \varkappa \in \mathcal{X},\ \mu_{\hat{\mathcal{A}}}(\varkappa) \in \mathcal{I}\}$ named fuzzy set on $\mathcal{X}$ with a membership function $\mu_{\hat{\mathcal{A}}}: \mathcal{X} \longrightarrow \mathcal{I}$, where $\mathcal{I} = [0,1]$.

Definition 2.2. [3] The set $\mathcal{Q}_{\mathcal{A}} = \{\mathcal{Q}(\omega) \in \mathcal{P}(\mathcal{X}):\ \omega \in \mathcal{A}\}$ named soft set over $\mathcal{X}$, with a set of parameters Σ and $\mathcal{P}(\mathcal{X})$ the set of all subsets $\mathcal{X}$ such that $\mathcal{A} \subseteq \Sigma, \mathcal{Q}: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{X})$, represented by $\mathcal{Q}_{\mathcal{A}}$.

Definition 2.3. [4] The soft set $\mathcal{Q}_{\mathcal{A}}$ is named fuzzy soft set (FSs) over $\mathcal{X}$, where $\mathcal{Q}: \mathcal{A} \rightarrow \mathcal{I}^{\mathcal{X}}$, has the range $\{\mathcal{Q}(\omega) \in \mathcal{I}^{\mathcal{X}}: \omega \in \mathcal{A}\}$, and the class of all (FSs), indicated by $\mathcal{FSs}(\widetilde{\mathcal{X}})$.

Definition 2.4 [13] Let $\widetilde{\mathcal{X}}$ be FS-V and $\mathcal{R}(\mathcal{A})$ be fuzzy soft real set. A mapping $\widetilde{\|.\|}: \widetilde{\mathcal{X}} \rightarrow \mathcal{R}(\mathcal{A})$, namely fuzzy soft norm (FS-N) on $\widetilde{\mathcal{X}}$ if $\widetilde{\|.\|}$ achieves the following:

i. for all $\tilde{\varkappa}_{\mu_{\mathcal{Q}(\omega)}} \in \widetilde{\mathcal{X}}$, gains $\left\| \widetilde{\tilde{\varkappa}_{\mu_{\mathcal{Q}(\omega)}}} \right\| \geqq \tilde{0}$,

ii. $\left\| \widetilde{\tilde{\varkappa}_{\mu_{\mathcal{Q}(\omega)}}} \right\| \cong \tilde{0}$ if and only if $\tilde{\varkappa}_{\mu_{\mathcal{Q}(\omega)}} \cong \tilde{0}$,

iii. for all $\tilde{\varkappa}_{\mu_{\mathcal{Q}(\omega)}} \in \widetilde{\mathcal{X}}$, and $\tilde{\wp} \in \mathcal{C}(\mathcal{A})$, yields $\left\| \widetilde{\tilde{\wp} \cdot \tilde{\varkappa}_{\mu_{\mathcal{Q}(\omega)}}} \right\| \cong |\tilde{\wp}| \left\| \widetilde{\tilde{\varkappa}_{\mu_{\mathcal{Q}(\omega)}}} \right\|$,

iv. for all $\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}} \in \widetilde{\mathcal{X}}$, obtains $\left\| \widetilde{\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}} + \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}}} \right\| \leqq \left\| \widetilde{\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}} \right\| + \left\| \widetilde{\tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}}} \right\|$.

The FS-V $\widetilde{\mathcal{X}}$ with $\widetilde{\|.\|}$ indicated by $\left(\widetilde{\mathcal{X}}, \widetilde{\|\ .\|}\right)$, namely fuzzy soft normed space (FS-N space).

Definition2.5. [8] Let $\widetilde{\mathcal{X}}$ be FS-V and $\mathcal{C}(\mathcal{A})$ be fuzzy soft complex set. Then, the mapping $\widetilde{\langle .,. \rangle}: \widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}} \rightarrow \mathcal{C}(\mathcal{A})$, called fuzzy soft inner product (FS-IP) on $\widetilde{\mathcal{X}}$ if $\widetilde{\langle .,. \rangle}$ achieves the following:

i. for all $\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}} \in \widetilde{\mathcal{X}}$, yields $\widetilde{\langle \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}} \rangle} \geqq \tilde{0}$,

ii. $\widetilde{\langle \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}} \rangle} \cong \tilde{0}$ if and only if $\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}} \cong \tilde{0}$,

iii. for all $\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}} \in \widetilde{\mathcal{X}}$, gains $\widetilde{\langle \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}} \rangle} \cong \overline{\widetilde{\langle \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}}, \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}} \rangle}}$,

iv. for all $\tilde{x}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}} \in \widetilde{\mathcal{X}}$ and $\tilde{\wp} \in \mathcal{C}(\mathcal{A})$, attains $\widetilde{\langle \tilde{\wp}\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}} \rangle} \cong \tilde{\wp}\, \widetilde{\langle \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}} \rangle}$

v. for all $\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}}, \tilde{z}_{\mu_{3\mathcal{Q}(\omega_3)}} \in \widetilde{\mathcal{X}}$, we yield $\widetilde{\langle \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}} + \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}}, \tilde{z}_{\mu_{3\mathcal{Q}(\omega_3)}} \rangle} \cong \widetilde{\langle \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{z}_{\mu_{3\mathcal{Q}(\omega_3)}} \rangle} + \widetilde{\langle \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}}, \tilde{z}_{\mu_{3\mathcal{Q}(\omega_3)}} \rangle}$.

The FS-V $\widetilde{\mathcal{X}}$ with $\widetilde{\langle .,. \rangle}$ indicated by $\left(\widetilde{\mathcal{X}}, \widetilde{\langle .,. \rangle}\right)$, namely fuzzy soft inner product space (FS-IP space).

Definition 2.6. [11] The complete of FS-IP space$\left(\widetilde{\mathcal{X}}, \widetilde{\langle .,. \rangle}\right)$ is named a fuzzy soft Hilbert space (FS-H space) and indicated by $\left(\widetilde{\mathcal{H}}, \widetilde{\langle , \rangle}\right)$ or $\widetilde{\mathcal{H}}$.

Definition 2.7. [7] Let $\widetilde{\mathcal{H}}$ be FS-H space, the operator $\tilde{\vartheta}: \widetilde{\mathcal{H}} \rightarrow \widetilde{\mathcal{H}}$, called fuzzy soft linear operator (FS-operator), if for all $\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{\tilde{y}}_{\mu_{2\mathcal{Q}(\omega_2)}} \in \widetilde{\mathcal{H}}$ and $\tilde{\sigma}, \tilde{\rho} \in \mathcal{C}(\mathcal{A})$, then

$$\tilde{\vartheta}\left(\tilde{\sigma}\, \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}} + \tilde{\rho}\, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}}\right) \cong \tilde{\sigma}\, \tilde{\vartheta}\left(\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}\right) + \tilde{\rho}\, \tilde{\vartheta}\left(\tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}}\right).$$

As well as, called fuzzy soft bounded operator (FSB operator) if $\exists\ \tilde{0} \gtrsim \tilde{\xi} \in \mathcal{R}(\mathcal{A})$ provided that

$\left\| \tilde{\vartheta}\left(\widetilde{\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}}\right) \right\| \leqq \tilde{\xi} \left\| \widetilde{\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}} \right\|$, for all $\varkappa_{\mu_{1\mathcal{Q}(\omega_1)}} \in \widetilde{\mathcal{H}}$.

The term fuzzy soft adjoint of FSB operator is offered as:

Definition 2.8. [9] Let $\widetilde{\mathcal{H}}$ be FS-H space and $\tilde{\vartheta}: \widetilde{\mathcal{H}} \rightarrow \widetilde{\mathcal{H}}$ be FS-operator named the fuzzy soft adjoint operator $\tilde{\vartheta}^*$ is given by $\widetilde{\langle \tilde{\vartheta}\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}} \rangle} \cong \widetilde{\langle \tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{\vartheta}^* \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}} \rangle}$, for all $\tilde{\varkappa}_{\mu_{1\mathcal{Q}(\omega_1)}}, \tilde{y}_{\mu_{2\mathcal{Q}(\omega_2)}} \in \widetilde{\mathcal{H}}$.

3. Analytical outcomes

This section states a special class of fuzzy soft operators that involves a fuzzy soft hyponormal operator in FS-H space.

Definition 3.1. Let $\widetilde{\mathcal{H}}$ be FS-H space, and $\tilde{\vartheta}: \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ be FS-operator, then $\tilde{\vartheta}$ is named fuzzy soft κ quasi hyponormal operator (FS κ quasi hyponormal operator) if there is $0 < \eta$, such that

$$\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa} \lesseqqgtr \tilde{\vartheta}^{*\kappa}\left(\eta^{2}\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{\kappa}.$$

Equivalently, $\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa} \lesseqqgtr \eta^{2}\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{\kappa}$.

In order the above definition to be clear, we consider fuzzy soft operator $\tilde{\vartheta}_1 = \begin{bmatrix}(0.1,0) & (0.5,0\\ (0.2,0) & (0.1,0)\end{bmatrix}$ is an FS κ quasi hyponormal operator, although $\tilde{\vartheta}_2 = \begin{bmatrix}(0.2,0) & (0.3,0\\ (0.6,0) & (0.7,0)\end{bmatrix}$ is not an FS κ quasi hyponormal operator.

Remarks 3.2.

1) It is clearly that, every fuzzy soft self adjoint (fuzzy soft hyponormal, fuzzy soft $\mathcal{M}$ -hyponormal, fuzzy soft quasi $\mathcal{M}$ -hyponormal) is an FS κ quasi hyponormal operator.
2) FS κ quasi hyponormal operator becomes fuzzy soft quasi-$\mathcal{M}$-hyponormal when $\eta = I, k = 1$.

Theorem 3.3. Let $\tilde{\vartheta}: \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$, then $\tilde{\vartheta}$ is FS κ quasi hyponormal operator if and only if there is $\eta > 0$, such that for all $\tilde{\gamma}_{\mu_{Q(\omega)}} \in \widetilde{\mathcal{H}}$,

$$\| \left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\tilde{\vartheta}^{k}\widetilde{\gamma_{\mu_{Q(\omega)}}} \| \lesseqqgtr \eta^{2} \| \left(\tilde{\vartheta}-\tilde{\delta}\right)\tilde{\vartheta}^{k}\widetilde{\gamma_{\mu_{Q(\omega)}}} \|.$$

Proof. $\| (\tilde{\vartheta}-\tilde{\delta})^{*}\tilde{\vartheta}^{k}\widetilde{\gamma_{\mu_{Q(\omega)}}} \|^{2} \cong < (\tilde{\vartheta}-\tilde{\eta})^{*}\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), (\tilde{\vartheta}-\tilde{\eta})^{*}\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$

$\cong < \tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), (\tilde{\vartheta}-\tilde{\delta})(\tilde{\vartheta}-\tilde{\delta})^{*}\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$

$\cong < \tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), \tilde{\vartheta}^{*k}((\tilde{\vartheta}-\tilde{\delta})(\tilde{\vartheta}-\tilde{\delta})^{*})\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$

$\lesseqqgtr \eta^{2} < \tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), \tilde{\vartheta}^{*k}((\tilde{\vartheta}-\tilde{\delta})^{*}(\tilde{\vartheta}-\tilde{\delta}))\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$

$\lesseqqgtr \eta^{2} < \tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), (\tilde{\vartheta}-\tilde{\delta})^{*}(\tilde{\vartheta}-\tilde{\delta})\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$

$\lesseqqgtr \eta^{2} < (\tilde{\vartheta}-\tilde{\delta})\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), (\tilde{\vartheta}-\tilde{\delta})\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$

$\lesseqqgtr \eta^{2} \| (\tilde{\vartheta}-\tilde{\delta})\tilde{\vartheta}^{k}\widetilde{\gamma_{\mu_{Q(\omega)}}} \|^{2}$.

Conversely, for $\eta^{2} \| (\tilde{\vartheta}-\tilde{\delta})\tilde{\vartheta}^{k}\widetilde{\gamma_{\mu_{Q(\omega)}}} \|^{2} \gtreqqless \| (\tilde{\vartheta}-\tilde{\delta})^{*}\tilde{\vartheta}^{k}\widetilde{\gamma_{\mu_{Q(\omega)}}} \|^{2}$, then

$< (\tilde{\vartheta}-\tilde{\delta})^{*}\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), (\tilde{\vartheta}-\tilde{\delta})^{*}\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$

$\cong < \tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), (\tilde{\vartheta}-\tilde{\delta})(\tilde{\vartheta}-\tilde{\delta})^{*}\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$

$\cong < \tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), \tilde{\vartheta}^{*k}((\tilde{\vartheta}-\tilde{\delta})(\tilde{\vartheta}-\tilde{\delta})^{*})\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$

$\lesseqqgtr \eta^{2} < \tilde{\gamma}_{\mu_{Q(\omega)}}(\omega), \tilde{\vartheta}^{*k}((\tilde{\vartheta}-\tilde{\delta})^{*}(\tilde{\vartheta}-\tilde{\delta}))\tilde{\vartheta}^{k}\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) >$.

Thus,

$< (\eta^{2}\tilde{\vartheta}^{*k}\left((\tilde{\vartheta}-\tilde{\delta})^{*}(\tilde{\vartheta}-\tilde{\delta})\right)\tilde{\vartheta}^{k} - \tilde{\vartheta}^{*k}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{k})\tilde{v}_{\mu_{Q(\omega)}}(\omega)), \tilde{\gamma}_{\mu_{Q(\omega)}}(\omega) > \gtreqqless 0$,

which yields $(\eta^{2}\tilde{\vartheta}^{*k}\left((\tilde{\vartheta}-\tilde{\delta})^{*}(\tilde{\vartheta}-\tilde{\delta})\right)\tilde{\vartheta}^{k} - (\tilde{\vartheta}^{*k}\left((\tilde{\vartheta}-\tilde{\delta})(\tilde{\vartheta}-\tilde{\delta})^{*}\right)\tilde{\vartheta}^{k})\tilde{\gamma}_{\mu_{Q(\omega)}}(\omega)) \gtreqqless 0$.

Then, $\eta^{2}\tilde{\vartheta}^{*k}\left((\tilde{\vartheta}-\tilde{\delta})^{*}(\tilde{\vartheta}-\tilde{\delta})\right)\tilde{\vartheta}^{k} - (\tilde{\vartheta}^{*k}\left((\tilde{\vartheta}-\tilde{\delta})(\tilde{\vartheta}-\tilde{\delta})^{*}\right)\tilde{\vartheta}^{k} \gtreqqless 0$.

Hence, $\eta^{2}\tilde{\vartheta}^{*k}\left((\tilde{\vartheta}-\tilde{\delta})^{*}(\tilde{\vartheta}-\tilde{\delta})\right)\tilde{\vartheta}^{k} \geq (\tilde{\vartheta}^{*k}\left((\tilde{\vartheta}-\tilde{\delta})(\tilde{\vartheta}-\tilde{\delta})^{*}\right)\tilde{\vartheta}^{k}$.

Therefore, FS κ quasi hyponormal operator.

The iteration of (multi-product of itself) of FS κ quasi hyponormal operator does not achieve FS κ quasi hyponormal operator. The following outcome presents and determines the adequate conditions to attain this feature.

Proposition 3.4. If $\tilde{\vartheta}$ is an FS κ quasi hyponormal operator on $\widetilde{\mathcal{H}}$ into itself, and $(\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa})^{n} \lesseqqgtr \eta^{2}(\tilde{\vartheta}^{*\kappa}\left(\eta^{2}\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{\kappa})^{n}$, then $\tilde{\vartheta}^{n}$ is an FS κ quasi hyponormal operator.

Proof. By utilizing mathematical induction, since $\tilde{\vartheta}$ is an FS κ quasi hyponormal operator, then $(\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa})^{n} \lesseqqgtr \eta^{2}(\tilde{\vartheta}^{*\kappa}\left(\eta^{2}\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{\kappa})^{n}$, is an FS κ quasi hyponormal operator. For $n=1$, then

$$(\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa})^{1} \lesseqqgtr \eta^{2}(\tilde{\vartheta}^{*\kappa}\left(\eta^{2}\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{\kappa})^{1}. \quad (1)$$

Given this outcome is attained for $n=m$, consequently

$$(\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa})^{m} \lesseqqgtr \eta^{2}(\tilde{\vartheta}^{*\kappa}\left(\eta^{2}\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{\kappa})^{m}. \quad (2)$$

Then, to demonstrate the validity of the outcome for $n = m+1$,

$$(\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa})^{m+1} \lesseqqgtr \eta^{2}(\tilde{\vartheta}^{*\kappa}\left(\eta^{2}\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{\kappa})^{m+1},$$

From Eq. (1) and Eq. (2), we yield

$$\left(\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa}\right)^{m}\left(\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa}\right)^{1} \lesseqqgtr \eta^{2}(\tilde{\vartheta}^{*\kappa}\left(\eta^{2}\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{\kappa})^{m+1}.$$

Therefore, $(\tilde{\vartheta}^{*\kappa}\left(\left(\tilde{\vartheta}-\tilde{\delta}\right)\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\right)\tilde{\vartheta}^{\kappa})^{n} \lesseqqgtr \eta^{2}(\tilde{\vartheta}^{*\kappa}\left(\eta^{2}\left(\tilde{\vartheta}-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}-\tilde{\delta}\right)\right)\tilde{\vartheta}^{\kappa})^{n}$ is an FS κ quasi hyponormal operator.

The subsequent outcomes present appropriate stipulations to attain the addition feature and multiplication feature of an FS κ quasi-hyponormal operators.

Theorem 3.5. If $\tilde{\vartheta}_1$ and $\tilde{\vartheta}_2$ are FS κ quasi hyponormal operators on $\widetilde{\mathcal{H}}$, and $\tilde{\vartheta}_1^{*\kappa}\,(\tilde{\vartheta}_2-\tilde{\delta})^{*} \cong \tilde{\vartheta}_2^{*\kappa}\left(\tilde{\vartheta}_1-\tilde{\delta}\right)^{*} \cong \left(\tilde{\vartheta}_1-\tilde{\delta}\right)\tilde{\vartheta}_1^{*\kappa} \cong \left(\tilde{\vartheta}_2-\tilde{\delta}\right)\tilde{\vartheta}_1^{\kappa} \cong \left(\tilde{\vartheta}_1-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}_1-\tilde{\delta}\right) \cong 0$. Then, $\tilde{\vartheta}_1+\tilde{\vartheta}_2$ is an FS κ quasi hyponormal operators. And there is $0<\eta$ such that $\eta_1^2, \eta_2^2. \leq \eta^2$

Proof. Since $\tilde{\vartheta}_1$ and $\tilde{\vartheta}_2$ are FS κ quasi hyponormal operators, and $\tilde{\vartheta}_1^{*\kappa}\,(\tilde{\vartheta}_2-\tilde{\delta})^{*} \cong \tilde{\vartheta}_2^{*\kappa}\,(\tilde{\vartheta}_1-\tilde{\lambda})^{*} \cong (\tilde{\vartheta}_1-\tilde{\delta})\,\tilde{\vartheta}_2^{\kappa} \cong (\tilde{\vartheta}_2-\tilde{\delta})\,\tilde{\vartheta}_1^{\kappa} \cong (\tilde{\vartheta}_2-\tilde{\delta})^{*}(\tilde{\vartheta}_1-\tilde{\delta}) \cong (\tilde{\vartheta}_1-\tilde{\delta})^{*}\,(\tilde{\vartheta}_1-\tilde{\delta}) \cong 0$.

$$\eta^2\left(\left(\tilde{\vartheta}_1+\tilde{\vartheta}_2\right)^{*\kappa}\left((\tilde{\vartheta}_1-\tilde{\delta})+(\tilde{\vartheta}_2-\tilde{\delta})\right)^{*}\left((\tilde{\vartheta}_1-\tilde{\delta})+(\tilde{\vartheta}_2-\tilde{\delta})\right)\left(\tilde{\vartheta}_1+\tilde{\vartheta}_2\right)^{\kappa}\right)$$

$$\cong \eta^2\left(\tilde{\vartheta}_1^{*\kappa}+\tilde{\vartheta}_2^{*\kappa}\right)\left((\tilde{\vartheta}_1-\tilde{\delta})^{*}+(\tilde{\vartheta}_2-\tilde{\delta})^{*}\right)\left((\tilde{\vartheta}_1-\tilde{\delta})+(\tilde{\vartheta}_2-\tilde{\delta})\right)\left(\tilde{\vartheta}_1^{\kappa}+\tilde{\vartheta}_2^{\kappa}\right)$$

$$\cong \eta^2\left(\tilde{\vartheta}_1^{*\kappa}\,(\tilde{\vartheta}_1-\tilde{\delta})^{*}+\tilde{\vartheta}_1^{*\kappa}\,(\tilde{\vartheta}_2-\tilde{\delta})^{*}+\tilde{\vartheta}_2^{*\kappa}\left(\tilde{\vartheta}_1-\tilde{\delta}\right)^{*}+\tilde{\vartheta}_2^{*\kappa}\left(\tilde{\vartheta}_2-\tilde{\delta}\right)^{*}\right)\left(\left(\tilde{\vartheta}_1-\tilde{\delta}\right)\tilde{\vartheta}_1^{\kappa}+\left(\tilde{\vartheta}_1-\tilde{\delta}\right)\tilde{\vartheta}_2^{\kappa}+\left(\tilde{\vartheta}_2-\tilde{\delta}\right)\tilde{\vartheta}_1^{\kappa}+\left(\tilde{\vartheta}_2-\tilde{\delta}\right)\tilde{\vartheta}_2^{\kappa}\right)$$

$$\cong \eta^2\left(\tilde{\vartheta}_1^{*\kappa}\left(\tilde{\vartheta}_1-\tilde{\delta}\right)^{*}+\tilde{\vartheta}_2^{*\kappa}\left(\tilde{\vartheta}_2-\tilde{\delta}\right)^{*}\right)\left(\left(\tilde{\vartheta}_1-\tilde{\delta}\right)\tilde{\vartheta}_1^{\kappa}+\left(\tilde{\vartheta}_2-\tilde{\delta}\right)\tilde{\vartheta}_2^{\kappa}\right)$$

$$\cong \eta^2\left(\tilde{\vartheta}_1^{*\kappa}\,(\tilde{\vartheta}_1-\tilde{\delta})^{*}\right)\left(\left(\tilde{\vartheta}_1-\tilde{\delta}\right)\tilde{\vartheta}_1^{\kappa}\right)+\mathcal{M}^2\left(\tilde{\vartheta}_1^{*\kappa}\,(\tilde{\vartheta}_1-\tilde{\delta})^{*}\right)\left(\left(\tilde{\vartheta}_2-\tilde{\delta}\right)\tilde{\vartheta}_2^{\kappa}\right)+\eta^2\left(\tilde{\vartheta}_2^{*\kappa}\,(\tilde{\vartheta}_2-\tilde{\delta})^{*}\right)\left(\left(\tilde{\vartheta}_1-\tilde{\delta}\right)\tilde{\vartheta}_1^{\kappa}\right)+\mathcal{M}^2\left(\tilde{\vartheta}_2^{*\kappa}\,(\tilde{\vartheta}_2-\tilde{\delta})^{*}\right)\left(\left(\tilde{\vartheta}_2-\tilde{\delta}\right)\tilde{\vartheta}_2^{\kappa}\right)$$

$$\cong \eta_1^2\left(\tilde{\vartheta}_1^{*\kappa}\,(\tilde{\vartheta}_1-\tilde{\delta})^{*}\right)\left((\tilde{\vartheta}_1-\tilde{\delta})\;\tilde{\vartheta}_1^{\kappa}\right)+\eta_2^2\left(\tilde{\vartheta}_2^{*\kappa}\,\tilde{\vartheta}_2^{*}\right)\left(\left(\tilde{\vartheta}_2-\tilde{\delta}\right)\tilde{\vartheta}_2^{\kappa}\right)$$

$$\cong \eta_1^2\left(\tilde{\vartheta}_1^{*\kappa}\left(\tilde{\vartheta}_1-\tilde{\delta}\right)^{*}\left(\tilde{\vartheta}_1-\tilde{\delta}\right)\tilde{\vartheta}_1^{\kappa}\right)+\eta_2^2\left(\tilde{\vartheta}_2^{*\kappa}\,(\tilde{\vartheta}_2-\tilde{\delta})^{*}\left(\tilde{\vartheta}_2-\tilde{\delta}\right)\tilde{\vartheta}_2^{\kappa}\right).$$

Since $\tilde{\vartheta}_1$ and $\tilde{\vartheta}_2$ are FS κ quasi hyponormal operators, then

$\cong \eta_1^2\left(\tilde{\vartheta}_1^{*\kappa}\left(\tilde{\vartheta}_1-\tilde{\delta}\right)^*\left(\tilde{\vartheta}_1-\tilde{\delta}\right)\tilde{\vartheta}_1^{\kappa}\right)+\eta_2^2\left(\tilde{\vartheta}_2^{*\kappa}\left(\tilde{\vartheta}_2-\tilde{\delta}\right)^*\left(\tilde{\vartheta}_2-\tilde{\delta}\right)\tilde{\vartheta}_2^{\kappa}\right) \geqq \left(\tilde{\vartheta}_1^{*\kappa}\left(\tilde{\vartheta}_1-\tilde{\delta}\right)\left(\tilde{\vartheta}_1-\tilde{\delta}\right)^*\tilde{\vartheta}_1^{\kappa}\right)+\left(\tilde{\vartheta}_2^{*\kappa}(\tilde{\vartheta}_2-\tilde{\delta})(\tilde{\vartheta}_2-\tilde{\delta})^*\tilde{\vartheta}_2^{\kappa}\right)$

$\cong \left(\tilde{\vartheta}_1-\tilde{\delta}\right)+\left(\tilde{\vartheta}_2-\tilde{\delta}\right)$.

Therefore, $\tilde{\vartheta}_1+\tilde{\vartheta}_2$ is an FS κ quasi hyponormal operator.

Theorem 3.6. Let $\tilde{\vartheta}_1$ and $\tilde{\vartheta}_2$ be FS κ quasi hyponormal operators on $\tilde{\mathcal{H}}$. Then $\tilde{\vartheta}_1\tilde{\vartheta}_2$ is an FS κ quasi hyponormal operators if

i. $(\tilde{\vartheta}_1-\tilde{\delta})^*(\tilde{\vartheta}_1-\tilde{\delta}) \cong (\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_1-\tilde{\delta})^*$

ii. $(\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_2-\tilde{\delta})^* \cong \left(\tilde{\vartheta}_2-\tilde{\delta}\right)^*\left(\tilde{\vartheta}_1-\tilde{\delta}\right)$ are achieved.

Proof. Assume that $\tilde{\vartheta}_1$ and $\tilde{\vartheta}_2$ be FS κ quasi hyponormal operators, and

$(\tilde{\vartheta}_1-\tilde{\delta})^*(\tilde{\vartheta}_1-\tilde{\delta}) \cong (\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_1-\tilde{\delta})^*$,$(\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_2-\tilde{\delta})^* \cong \left(\tilde{\vartheta}_2-\tilde{\delta}\right)^*(\tilde{\vartheta}_1-\tilde{\delta})$

$\eta^2\left(\left(\tilde{\vartheta}_1\tilde{\vartheta}_2\right)^{*\kappa}\left((\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_2-\tilde{\delta})\right)^*\left((\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_2-\tilde{\delta})\right)\left(\tilde{\vartheta}_1\tilde{\vartheta}_2\right)^{\kappa}\right)$

$\geqq \eta^2\left(\tilde{\vartheta}_2^{*\kappa}\,\tilde{\vartheta}_1^{*\kappa}\right)\tilde{\vartheta}_2^{\,*}(\tilde{\vartheta}_1-\tilde{\delta})^*(\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_2-\tilde{\delta})\left(\tilde{\vartheta}_1^{\kappa}\,\tilde{\vartheta}_2^{\kappa}\right)$

$\geqq \left(\tilde{\vartheta}_2^{*\kappa}\,\tilde{\vartheta}_1^{*\kappa}\right)\left(\tilde{\vartheta}_2-\tilde{\delta}\right)^*(\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_1-\tilde{\delta})^*(\tilde{\vartheta}_2-\tilde{\delta})\left(\tilde{\vartheta}_1^{\kappa}\,\tilde{\vartheta}_2^{\kappa}\right)$

$\geqq \left(\tilde{\vartheta}_2^{*\kappa}\,\tilde{\vartheta}_1^{*\kappa}\right)(\tilde{\vartheta}_1-\tilde{\delta})(\left(\tilde{\vartheta}_2-\tilde{\delta}\right)^*(\tilde{\vartheta}_1-\tilde{\delta})^*)(\tilde{\vartheta}_2-\tilde{\delta})\left(\tilde{\vartheta}_1^{\kappa}\,\tilde{\vartheta}_2^{\kappa}\right)$

$\geqq \left(\tilde{\vartheta}_2^{*\kappa}\,\tilde{\vartheta}_1^{*\kappa}\right)(\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_2-\tilde{\delta})(\left(\tilde{\vartheta}_2-\tilde{\delta}\right)^*(\tilde{\vartheta}_1-\tilde{\delta})^*)\left(\tilde{\vartheta}_1^{\kappa}\,\tilde{\vartheta}_2^{\kappa}\right)$

$\geqq \left(\left(\tilde{\vartheta}_1\tilde{\vartheta}_2\right)^{*\kappa}\left((\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_2-\tilde{\delta})\right)\left((\tilde{\vartheta}_1-\tilde{\delta})(\tilde{\vartheta}_2-\tilde{\delta})\right)^*\left(\tilde{\vartheta}_1\tilde{\vartheta}_2\right)^{\kappa}\right)$.

Therefore, $\tilde{\vartheta}_1\tilde{\vartheta}_2$ is an FS κ quasi hyponormal operator.

4. Conclusions

Recently, determining the bounded operator in terms of fuzzy soft is essential in functional analysis. Operator theory (OP) evaluation is a dynamic for the prosperity multidisciplinary in mathematics. The term such as fuzzy soft Hilbert space has considerably been utilized to represent the classification of fuzzy soft operators. This study concentrates on the FS-OT realm, the class of fuzzy soft κ quasi-hyponormal operators has been presented, as a generalized class of fuzzy soft hyponormal operators. The significant characteristic of this operator has also been posed, rising to the yielding of various other features. Furthermore, the prominence of discussing the interaction of conditions and analytical attributes involving power, addition, and multiplication of fuzzy soft κ quasi hyponormal operators, is also studied.

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