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RESEARCH ARTICLE

Mixed Galerkin- Implicit Differences Methods for Solving Couple Nonlinear Parabolic System with Constant Coefficients

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ABSTRACT

In this paper, the mixed Galerkin- implicit difference method (MGIFDM) is used to solve a couple of nonlinear systems of parabolic partial differential equations with constant coefficients which are abbreviated by (CNPSCC). At first the weak form of the CNPSCC is formulated and then it is discretized using the proposed method, the method mixes the Galerkin finite element method (GFEM) in space variables with the implicit finite difference method (IFDM) in time variables, so the method was named by MGIFDM. At any discrete time tj the method transforms the CNPSCC into a couple Galerkin nonlinear algebraic system (CGNAS), which is solved by applying the predictor-corrector techniques, this technique is used to reduce the "nonlinear" CGNAS into a couple of linear Galerkin system algebraic, of course at any discrete time tj. Then the Cholesky decomposition method is utilized to solve it (at any time tj). The convergence theorem is given and demonstrated, to show the convergence of the solutions to the proposed problem. Two examples are given to illustrate and examine the method, and the results are given by tables and by figures and show the efficiency and accuracy of the proposed method.

Keywords: Coupled nonlinear parabolic system, Cholesky decomposition method, Galerkin method, Implicit difference method, Predictor and corrector techniques

Introduction

Many phenomena in life can be expressed with mathematical models which are described in general by partial differential equations (PDEs).[1–3] A parabolic PDE is a particular type of PDF which is utilized to characterize a large diversity of time –dependent phenomena such a conduction of the heat,[4] control problems,[5] acoustic diffusion[6] and many others. Due to the particular significance of nonlinear parabolic PDEs, numerous investigators are interested in finding the numerical or approximate solution for this kind of differential equations. Such as; a hybrid method, the Crank-Nicolson with the GFEM, and the homotopy perturbation with transform method which were suggested by the authors[7–9] in 2019 to solve nonlinear PPDEs. Whilst in 2020 the mixed Haar wavelet collocation method with finite difference[10] was utilized to solve a PPDEs. Later and in 2022 others investigators[11–13] were solved such of these problems through implementing the GFEM with Bernstein Polynomial bases, the GFEM, finite difference with finite element methods, and the Haar wavelets method respectively. Recently; mathematical problems which involving PPDEs were solved by utilizing; a hybrid numerical technique, spline collocation methods, orthogonal cubic splines[14–16] and an artificial neural network.[17]

In this paper the approximate solution for a new type of PPDEs is studied, more precisely for a couple of nonlinear PPDEs with constant coefficients (CNPSCC) is proposed. In fact, the approximate

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solution of this type of differential equation has never been investigated before this time, especially using the mixed Galerkin- implicit difference methods (the MGIFDM). For this reason, our attention was focused on studying the solution of this type of differential equation.

The paper begins with a description of the continuous type of the CNPSCC, then the weak formulation of the problem is found and then the approximation problem obtained from the discretization of the CCNPSCC by using the GFEM for the space variable and the IFDM for the time variable. The mix of these two methods converted the problem into a couple Galerkin nonlinear algebraic systems (CGNLAS). Then the predictor- corrector technique is used to reduce this "nonlinear" CGNLAS into a couple Galerkin linear algebraic system (CGLAS). Then the CGLAS is solved by applying the Cholesky decomposition method which is considered one of the more accurate and fast method than the other methods like the Haar wavelets method.[13] The convergent of the proposed method is proved. Finally, illustrative examples are given to illustrate and examine the method $M = 9, 19, 29$, $NT = 20, 50, 100$ and $T = 1, 2$; the results are given by tables and figures which show the efficiency and the accuracy of the MGIFDM.

Problem description

Let $\Omega = \{\vec{x} = (x_1, x_2) \in R^2 : \ a < x_1, x_2 < b\}$, be a region with boundary $\partial\Omega$, and $I = [0, T < \infty]$, $D = \Omega \times I$, then the CNPSCC are represented as:

$$u_{1t} - \Delta u_1 + u_1 - u_2 = w_1(\vec{x}, t, u_1), \text{ in } D, \text{ in } D \tag{1}$$

$$u_{2t} - \Delta u_2 + u_1 + u_2 = w_2(\vec{x}, t, u_2), \text{ in } D \tag{2}$$

with

$$u_r(\vec{x}, 0) = u_r^0(\vec{x}), \text{ in } \Omega, \ (r = 1, 2) \tag{3}$$

and the Drichlet boundary conditions

$$u_r(\vec{x}, t) = 0, \text{ on } \partial\Omega \times I, (r = 1, 2) \tag{4}$$

The classical solution of Eqs. (1) to (4) is $\vec{u} = (u_1,\ u_2) \in (H^2(D))^2$, s.t $\Delta\vec{u} = \sum_{i=1}^{2} \frac{\partial^2 \vec{u}}{\partial x_i^2}$ and $\vec{w}(\vec{u}) = ((w_1(\vec{x}, t, u_1), w_2(\vec{x}, t, u_2)) \in (L^2(D))^2$ is given. Let $(.,.)$ and $\|.\|_0$ be referred to the inner product and norm in $L^2(D)$ and $\|.\|_1$ be referred to the norm in Sobelev space $V = H_0^1(\Omega)$.

The weak formula

Let $H_0^1(\Omega) = \{v : v \in H^1(\Omega),\ v = 0 \text{ on } \partial\Omega\}$, then the weak formula of the CNPSCC (Eqs. (1) to (4)) is

$$(u_{1t}, v_1) + a_1(t, u_1, v_1) - (u_2, v_1)$$
$$= (w_1(u_1), v_1) \,\forall\, v_1 \in H_0^1(\Omega), \vec{u} \in \left(H_0^2(D)\right)^2, \tag{5}$$

$$(u_1(0), v_1) = \left(u_1^0, v_1\right), \text{ in } H_0^1(\Omega), \tag{6}$$

$$(u_{2t}, v_2) + a_2(t, u_2, v_2) + (u_1, v_2)$$
$$= (w_2(u_2), v_2), \forall\, v_2 \in H_0^1(\Omega), \vec{u} \in \left(H_0^2(D)\right)^2, \tag{7}$$

$$(u_2(0), v_2) = \left(u_2^0, v_2\right), \text{ in } H_0^1(\Omega) \tag{8}$$

where

$a_r(t, u_r, v_r) = \sum_{s,t=1}^{2} (\frac{\partial u_r}{\partial x_t}, \frac{\partial v_r}{\partial x_s}) + (u_r, v_r)$, is a bilinear form for $r = 1, 2$.

Assumptions

i) For γ, $\overline{\gamma}, \gamma_r, \overline{\gamma_r} \in R^+$, $\forall\, r = 1, 2$, the following conditions are held:
   a) $|a_r(t, u_r, v_r)| \le \gamma_r\|u_r\|_1\|v_r\|_1$, $\forall\, u_r,\ v_r \in V$
   b) $a_r(t, u_r, v_r) \ge \bar{\gamma}_r\|u_r\|_1^2$, $\forall u_r \in V$
   c) $|a(t, \vec{u}, \vec{v})| \le \gamma\|\vec{u}\|_1\|\vec{v}\|_1$, $a(t,\ \vec{u}, \vec{u}) \ge \overline{\gamma_r}\|\vec{u}\|_1^2$, when $a(t,\ \vec{u}, \vec{u}) = \sum_{r=1}^{2} a_r(t, u_r, u_r)$

ii) W_r $(\forall r = 1, 2)$ is of the Caratheodary type on $D \times R$ and it holds the following conditions:
   a) $|w_r(\vec{x}, t, u_i)| \le \psi_r(\vec{x}, t) + d_r|u_r|$,
   b) $|w_r(\vec{x}, t, u_1) - w_r(\vec{x}, t, u_2)| \le L_r|u_1 - u_2|$,

where $(\vec{x}, t) \in D$, $d_r > 0$, $u_r \in R$, $\psi_r \in L^2(D, R)$, and L_r is a Lipischitz constant, $\forall r = 1, 2$.

Discretization of the weak formula

The discretization of the weak formula (Eqs. (5) to (8)) is obtained by employing the GFEM[18] for the space variable. So, let $M_1 \in Z^+$, $\bar{\Omega} = \cup_{i=1}^{N} O_i$, $O_i = O_i^n$, $i = 1, 2, \ldots, N$, $N = M^2$, $M = M_1 - 1$ be a regular "admissible" triangulation of $\bar{\Omega}$,[19] $h = 1/M_1$, $x_{i1} = ih$, $x_{i2} = ih$, $i = 0, 1, \ldots, M_1$ be points in $\bar{\Omega}$, s.t $0 = x_{01} < x_{11} < \cdots < x_{i1} < \cdots < x_{(M_1)1} = 1$ and $0 = x_{02} < x_{12} < \cdots < x_{i2} < \cdots < x_{(M_1)2} = 1$.

The interval of the time variable $I = [0, 1]$, is spilt to the subintervals $I_j = I_j^n := [t_j^n, t_{j+1}^n]$, where $t_j = j\Delta t$, $j = 0, 1, \ldots, NT - 1$, $NT \in Z^+$ with $\Delta t = T/NT$.

The approximate vector solution

To find the approximate vector solution $\vec{u}^n = (u_1^n, u_2^n)$ of Eqs. (5) to (8), using the GFEM, let $V_N \subset H_0^1(\Omega)$ (be piecewise affine functions of dimension N), with $V_N = \{v_i,\ i = 1, 2, \ldots, N,\ v_i(\vec{x}) = 0,\ \text{on } \partial\Omega\}$ and $\vec{V}_N = V_N \times V_N$, then the following proceedings can be applied:

Step 1: Eqs. (5) to (8) can be rewritten as

$$(u_{1t}^n, v_1) + a_1\left(t, u_1^n, v_1\right) - (u_2^n, v_1) = \left(w_1\left(t_{j+1}^n, {u_1}_{j+1}^n\right), v_1\right),\ \forall\, v_1 \in V_N, \vec{u}^n \in \vec{V}_N, \quad (9)$$

$$(u_1(0), v_1) = (u_1^0, v_1),\ \text{ in } V_N \quad (10)$$

$$(u_{2t}^n, v_2) + a_2\left(t, u_2^n, v_2\right) + (u_1^n, v_2) = \left(w_2\left(t_{j+1}^n, {u_2}_{j+1}^n\right), v_2\right),\ \forall\, v_2 \in V_N, \vec{u}^n \in \vec{V}_N, \quad (11)$$

$$(u_2(0), v_2) = (u_2^0, v_2),\ \text{ in } V_N, \quad (12)$$

Step 2: Applying the GFEM, the approximate vector solution $\vec{u}^n = (u_1^n, u_2^n)$ is approximated using the basis $(v_1, v_2, \ldots, v_N)$ of V_N, i.e. $u_1^n(\vec{x}, t) = \sum_{k=1}^N c_k(t) v_k(\vec{x})$, $u_2^n(\vec{x}, t) = \sum_{k=1}^N c_{k+N}(t) v_k(\vec{x})$, $u_1^n(\vec{x}, 0) = \sum_{k=1}^N c_k(0) v_k(\vec{x})$, and $u_2^n(\vec{x}, 0) = \sum_{k=1}^N c_{k+N}(0) v_k(\vec{x})$.
where $c_k(t)$ and $c_{k+N}(t)$ are unknown coefficients to be found

Step 3: Substitute $\vec{u}^n$ in Eqs. (9) to (12) with $v_1 = v_2 = v_m$, to get the following CGNAS with their ICs

$$(D + \Delta t E)\, C_k^{j+1} - \Delta t F C_{k+N}^{j+1} = D C_k^j + \Delta t \vec{b}_1\left(t_{j+1}\right), \quad (13)$$

$$D C_k(0) = \vec{b}_1^0, \quad (14)$$

$$(D + \Delta t G)\, C_{k+N}^{j+1} + \Delta t F C_k^{j+1} = D C_{k+N}^j + \Delta t \vec{b}_2\left(t_{j+1}\right), \quad (15)$$

$$D C_{k+N}(0) = \vec{b}_2^0, \quad (16)$$

where,
$D = (d_{mk})_{N\times N}$, $d_{mk} = (v_k, v_m)$, $E = (e_{mk})_{N\times N}$, $e_{mk} = a_1(v_k, v_m)$, $a_1(v_k, v_m) = \sum_{s,t=1}^2 (\frac{\partial v_k}{\partial x_t}, \frac{\partial v_m}{\partial x_s}) + (v_k, v_m)$, $F = (f_{mk})_{N\times N}$, $f_{mk} = (v_k, v_m)$, $G = (g_{mk})_{N\times N}$, $g_{mk} = a_2(v_k, v_m)$, $a_2(v_k, v_m) = \sum_{s,t=1}^2 (\frac{\partial v_k}{\partial x_t}, \frac{\partial v_m}{\partial x_s}) + (v_k, v_m)$, $C_k(t_j) = (c_k(t_j))_{N\times 1}$, $C_k(t_{j+1}) = (c_k(t_{j+1}))_{N\times 1}$, $C_{k+N}(t_j) = (c_{k+N}(t_j))_{N\times 1}$, $C_{k+N}(t_{j+1}) = (c_{k+N}(t_{j+1}))_{N\times 1}$, $\vec{b}_1 = (b_{1i})_{N\times 1}$, $\vec{b}_2 = (b_{2i})_{N\times 1} b_{1i} = (w_1(\vec{x}, t_{j+1}, C_k^j \vec{V}^T), v_m)$, $b_{2i} = (w_2(\vec{x}, t_{j+1}, C_{k+N}^j \vec{V}^T), v_m)$, $\vec{b}_1^0 = (b_{1i}^0)_{N\times 1}$, $b_{1i}^0 = (u_1^0, v_i)$, $\vec{b}_2^0 = (b_{2i}^0)_{N\times 1}$, $b_{2i}^0 = (u_2^0, v_i)$, $\forall\ m, k = 1, 2, \ldots, N$.

System Eqs. (13) to (16) has a unique solution. In fact the GLAS Eqs. (15) and (16) are solved at first to get C_k^0 and C_{k+N}^0 resp., then the CGNAS Eqs. (13) and (14) are solved using the predictor and the corrector technique as follows.

In the predictor technique, set $C^{j+1} = C^j$ $(\forall j = 0, 1, 2 \ldots, NT - 1)$ in the RHS of Eqs. (13) and (14), which transform the GNLAS into GLAS, solving it to get the predictor solution C^{j+1}. Next, in the corrector technique substitute $\bar{C}^{j+1} = C^{j+1}$ in the RHS of (Eqs. (13) and (14)) (i.e. in $\vec{b}_1$ and $\vec{b}_2$), then solving them to get the corrector solution C^{j+1} (this technique can be repeated for more than one time) by solving Eqs. (14) and (15) after setting the corrector solution $\bar{C}^{j+1} = C^{j+1}$ in the RHS of them, to get a new solution. Hence the corrector technique can be expressed as follows:

$$\left(\vec{u}_{j+1}^{(l+1)} - \vec{u}_j, \vec{v}_i\right) + \Delta t a\left(\vec{u}_{j+1}^{(l+1)}, \vec{v}_i\right) - \Delta t\left({u_2}_{j+1}^{(l+1)}, v_{1i}\right) + \Delta t\left({u_1}_{j+1}^{(l+1)}, v_{2i}\right) = \Delta t\left[\left(w_1\left({u_1}_{j+1}^{(l)}\right), v_{1i}\right) + \left(w_2\left({u_2}_{j+1}^{(l)}\right), v_{2i}\right)\right], \quad (17)$$

where $\vec{v}_i = (v_i, v_i)$ for $(i = 1, 2, \ldots, N)$, $\vec{u}_{j+1}^{(l)} = \vec{u}_{j+1}^n$ represents the (l) iteration of the predictor solution, $\vec{u}_{j+1}^{(l+1)} = \vec{u}_{j+1}^n$ and $\vec{u}_j = \vec{u}_j^n$ represent the $(l+1)$ iteration of the corrector solutions at the $(l+1)$ iteration and at the previous step j respectively.

Eq. (17) can be written as:

$$\vec{u}^{(l+1)} = g\left(\vec{u}^{(l)}\right). \quad (18)$$

In the following theorem the existence for a unique vector solution for the weak formula Eqs. (9) to (12) is stated and proved, the proof is a generalization to the proof of the theorem that appeared in,[19] since the weak formula here consists of a couple of nonlinear parabolic equations instead of "single" nonlinear parabolic equation.

Theorem 1: *The discrete weak form Eqs. (9) to (12) for fixed point j $(0 \le j \le NT - 1)$ and for Δt sufficiently small has a unique vector solution $\vec{u}^n = (u_1^n, u_2^n) = (u_{10}^n, u_{11}^n, \ldots, u_{1N}^n, u_{20}^n, u_{21}^n, \ldots, u_{2N}^n)$ and the $\{\vec{u}^{(l)}\}$ is convergence in R^2.*

Proof: let $\vec{u}^{(l+1)} = (u_1^{(l+1)}, u_2^{(l+1)}) = (u_{10}^{(l+1)}, u_{11}^{(l+1)}, \ldots, u_{1N}^{(l+1)}, u_{20}^{(l+1)}, u_{21}^{(l+1)}, \ldots, u_{2N}^{(l+1)})$, and $\vec{\bar{u}}^{(l+1)} =$

$(\vec{u}_1^{(l+1)}, \vec{u}_2^{(l+1)}) = (\vec{u}_{10}^{(l+1)}, \vec{u}_{11}^{(l+1)}, \ldots, \vec{u}_{1N}^{(l+1)}, \vec{u}_{20}^{(l+1)}, \vec{u}_{21}^{(l+1)}, \ldots, \vec{u}_{2N}^{(l+1)})$ are two classical vector solutions of Eq. (17), i.e.

$$\left(\vec{u}_{j+1}^{(l+1)} - \vec{u}_j, \vec{v}_i\right) + \Delta t a\left(\vec{u}_{j+1}^{(l+1)}, \vec{v}_i\right) - \Delta t\left(u_{2j+1}^{(l+1)}, v_{1i}\right) + \Delta t\left(u_{1j+1}^{(l+1)}, v_{2i}\right) = \Delta t\left[\left(w_1\left(u_{1\,j+1}^{(l)}\right), v_{1i}\right) + \left(w_2\left(u_{2\,j+1}^{(l)}\right), v_{2i}\right)\right], \quad (19)$$

$$\left(\vec{\bar{u}}_{j+1}^{(l+1)} - \vec{\bar{u}}_j, \vec{v}_i\right) + \Delta t a\left(\vec{\bar{u}}_{j+1}^{(l+1)}, \vec{v}_i\right) - \Delta t\left(\bar{u}_{2j+1}^{(l+1)}, v_{1i}\right) + \Delta t\left(\bar{u}_{1j+1}^{(l+1)}, v_{2i}\right) = \Delta t\left[\left(w_1\left(\bar{u}_{1j+1}^{(l)}\right), v_{1i}\right) + \left(w_2\left(\bar{u}_{2j+1}^{(l)}\right), v_{2i}\right)\right], \quad (20)$$

where $\vec{v}_i = (v_{1i}, v_{2i}) = (v_i, v_i),\ i = 1, 2, \ldots, N$.

By subtracting Eq. (20) from Eq. (19), then setting $\vec{v}_i = \vec{u}_{j+1}^{(l+1)} - \vec{\bar{u}}_{j+1}^{(l+1)}$ in the resulting equation and then using assumption (ii-b), it yields

$$\left\|\vec{u}_{j+1}^{(l+1)} - \vec{\bar{u}}_{j+1}^{(l+1)}\right\|_0^2 + \Delta t a\left(\vec{u}_{j+1}^{(l+1)} - \vec{\bar{u}}_{j+1}^{(l+1)}, \vec{u}_{j+1}^{(l+1)} - \vec{\bar{u}}_{j+1}^{(l+1)}\right)$$

$$\leq \Delta t L_1\left(\left|u_{1j+1}^{(l)} - \bar{u}_{1j+1}^{(l)}\right|, \left|u_{1j+1}^{(l+1)} - \bar{u}_{1j+1}^{(l+1)}\right|\right) + \Delta t L_2\left(\left|u_{2j+1}{}^{(l)} - \bar{u}_{2j+1}^{(l)}\right|, \left|u_{2j+1}^{(l+1)} - \bar{u}_{2j+1}^{(l+1)}\right|\right)$$

From assumption (ii-c), the 2^{nd} term in the LHS is nonnegative and then by applying the Cauchy Schwarz inequality on the RHS of above inequality, it becomes

$$\left\|\vec{u}_{j+1}^{(l+1)} - \vec{\bar{u}}_{j+1}^{(l+1)}\right\| \leq \Delta t L\left\|\vec{u}_{j+1}^{(l)} - \vec{\bar{u}}_{j+1}^{(l)}\right\|_0 = \alpha\left\|\vec{u}_{j+1}^{(l)} - \vec{\bar{u}}_{j+1}^{(l)}\right\|,$$

where $\alpha = \Delta t L$, and $L = L_1 + L_2$.

Applying Eq. (18), to get that

$$\left\|g\left(\vec{u}_{j+1}^{(l+1)}\right) - g\left(\vec{\bar{u}}_{j+1}^{(l+1)}\right)\right\|_0 \leq \alpha\left\|\vec{u}_{j+1}^{(l)} - \vec{\bar{u}}_{j+1}^{(l)}\right\|_0$$

Then for sufficiently small value of Δt, the value of $\alpha < 1$, thus g is contractive.

Hence Eqs. (9) to (12) has a unique vector solution. On the other hand since $\vec{u}^{(l)} \in R^2,\ \forall\ l$ and $g(\vec{u}^{(l)}) = \vec{u}^{(l+1)},\ \forall\ l$, then by Theorem 3,[20] once get that $\{\vec{u}^{(l)}\}$ is convergent to a point $\vec{u}$ in R^2.

Cholesky decomposition

This method is utilized here to solve the CGLAS with condition, every positive definite matrix $A = L.L^T$ can be decomposed into a product of a unique lower triangular matrix L and its transpose.[8] The method can be represented in the following steps:

Step 1: $L_{11} = (a_{11})^{1/2}$.

Step 2: $L_{pp} = \left(a_{pp} - \sum_{z=1}^{p-1} L_{pz}^2\right)^{1/2},\ p = 2, \ldots, N$.

Step 3: $L_{pq} = \frac{a_{pq} - \sum_{z=1}^{q-1} L_{qz}.L_{pz}}{L_{qq}},\ q = p+1, \ldots, N$.

Results and discussion

In this section, to examine the proposed method, two numerical examples are carried out. In the following examples the region $\Omega = (0,1) \times (0,1)$, $I = [0,1]$, $\Gamma = \Omega \times I$.

Example 1: *Let the CNPSCC are given as*

$$u_{1t} - \frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_2^2} + u_1 - u_2 = w_1(\vec{x}, t, u_1),$$

$$u_{2t} - \frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial^2 u_2}{\partial x_2^2} + u_1 + u_2 = w_2(\vec{x}, t, u_2),$$

with

$$u_1^0(\vec{x}) = 0.3\tan(1 - x_2)\sin(2\pi x_1),\ in\ \Omega,$$

$$u_2^0(\vec{x}) = 0,\ in\ \Omega$$

and the Dirichlet boundary conditions

$$u_1(\vec{x}, t) = 0,\ on\ \partial\Omega \times I$$

$$u_2(\vec{x}, t) = 0,\ on\ \partial\Omega \times I$$

where

$$w_1(\vec{x}, t, u_1) = x_1 x_2 t(x_1 - 1)(x_2 - 1) + 3\sin(2\pi x_1)\left[0.2\left(tan(x_2 - 1)^2 + 1\right) - x_2 tan(x_2 - 1) \times \left[0.1 + 0.4\pi^2 + 0.1\sin(0.3x_2 tan(x_2 - 1)\sin(2\pi x_1) + 0.2\left(tan(x_2 - 1)^2 + 1\right)\right]\right]$$

$$w_2(\vec{x}, t, u_2) = x_1 x_2(x_1 x_2 - x_1 - x_2 + 1) \times [t[\cos(t(x_1 x_2(x_1 x_2 - x_1 - x_2 + 1)) - 1)] - 1] + 2t\left[x_1^2 + x_2^2 - x_1 - x_2\right] - [3x_2\sin(2\pi x_1) tan(x_2 - 1)]/10$$

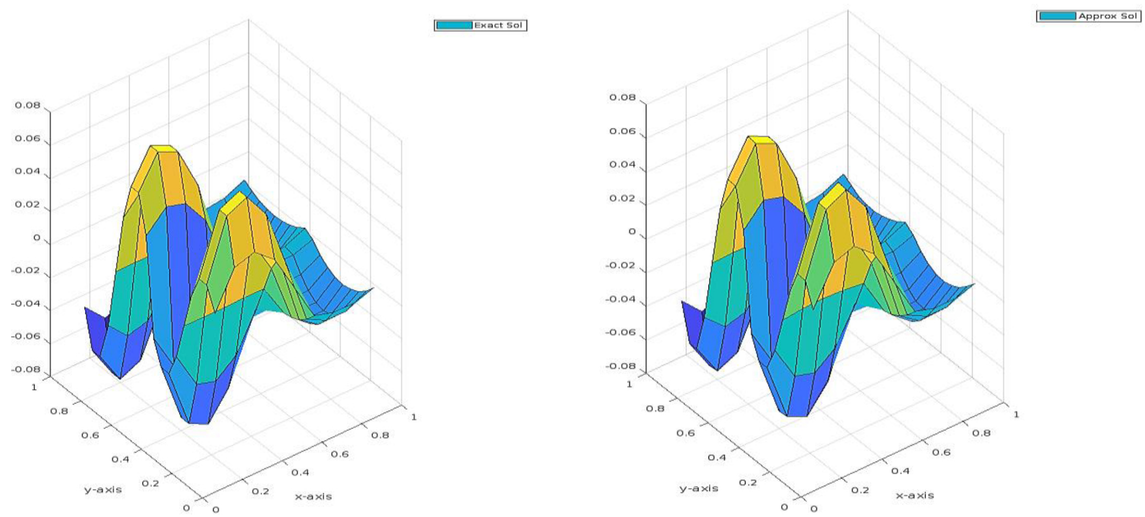

Fig. 1. Shows the exact (LHS) and the approximate vector solutions(RHS) at $\mathbf{t} = 0.5$.

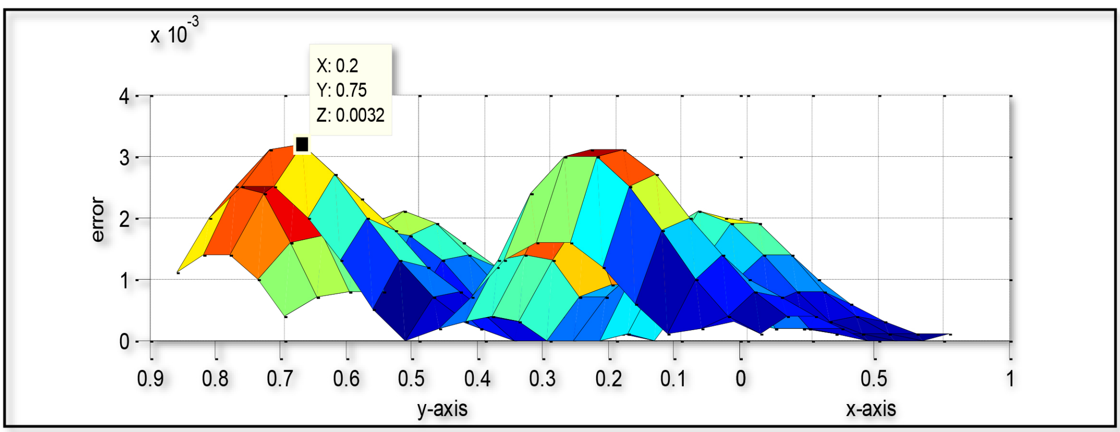

Fig. 2. Shows the absolute error between the exact and the approximate solutions at $\mathbf{t} = 0.5$.

The exact solutions are $u_1(\vec{x}, t) = 0.3x_2 \tan(1 - x_2)\sin(2\pi x_1)$, and $u_2(\vec{x}, t) = x_1 x_2 t(1 - x_1)(1 - x_2)$.

This problem is solved using the MGIFDM for $M = 9$, $NT = 20$ and $T = 1$, then the approximate vector solution $\vec{U}^n$ and its exact vector solution $\vec{U}$ at x_1 and x_2 are given at the time $\hat{t} = 0.5$ in the Table 1 and are shown in Fig. 1, the absolute maximum error is (0.0032) and is shown in Fig. 2.

Example 2: *The CNPSCC are given as*

$$u_{1t} - \frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_2^2} + u_1 - u_2 = w_1(\vec{x}, t, u_1)$$

$$u_{2t} - \frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial^2 u_2}{\partial x_2^2} + u_1 + u_2 = w_2(\vec{x}, t, u_2),$$

with

$$u_1^0(\vec{x}) = 0.2(1 - x_1)\left(1 - e^{0.9x_2}\right)\sin(2\pi x_2), \ \ in\ \Omega$$

$$u_2^0(\vec{x}) = (1 - x_1)(1 - x_2)\sin(x_1 x_2)/5, \ \ in\ \Omega$$

and the Dirichlet boundary conditions

$$u_1(\vec{x}, t) = 0, \text{ and } u_2(\vec{x}, t) = 0, \ \ on\ \partial\Omega \times I$$

where

$$w_1(\vec{x}, t, u_1) = 0.2(x_1 - 1)(x_2 - 1)\sin(x_1 x_2)\sqrt{e^{-0.11t}}$$
$$- \sec(t)\sin(2\pi x_2)\left[e^{0.9x_1}(0.162(x_1 - 1) + 0.36)\right.$$
$$-0.2(x_1 - 1)\left(e^{0.9x_1} - 1\right)\left[1 + 4\pi^2 \sec(t) + \sec(t)\sin(t)\right.$$
$$\left.\left. - \sin\left(0.2(x_1 - 1)\left(e^{0.9x_1} - 1\right)\right)\sec(t)\sin(2\pi x_2)\right]\right],$$

and

$$w_2(\vec{x}, t, u_2) = 0.2(x_1 - 1)\left(e^{0.9x_1} - 1\right)\sec(t)\sin(2\pi x_2)$$

Table 1. Comparison between the exact and the approximate vector solutions.

x_1	x_2	EXS	APPS	Absolute error	x_1	x_2	EXS	APPS	Absolute error
0.1	0.1	0.0222	0.0219	0.0003	0.1	0.1	–0.0040	–0.0041	0.0000
0.3	0.1	0.0360	0.0360	0.0001	0.3	0.1	–0.0095	–0.0096	0.0002
0.5	0.1	0	0.0012	0.0012	0.5	0.1	–0.0112	–0.0118	0.0006
0.7	0.1	–0.0360	–0.0343	0.0016	0.7	0.1	–0.0095	–0.0103	0.0008
0.9	0.1	–0.0222	–0.0216	0.0007	0.9	0.1	–0.0040	–0.0045	0.0004
0.1	0.3	0.0446	0.0446	0.0001	0.1	0.3	–0.0095	–0.0095	0.0000
0.3	0.3	0.0721	0.0731	0.0010	0.3	0.3	–0.0221	–0.0223	0.0003
0.5	0.3	0	0.0025	0.0025	0.5	0.3	–0.0262	–0.0275	0.0013
0.7	0.3	–0.0721	–0.0691	0.0030	0.7	0.3	–0.0221	–0.0239	0.0018
0.9	0.3	–0.0446	–0.0433	0.0013	0.9	0.3	–0.0095	–0.0104	0.0010
0.1	0.5	0.0482	0.0486	0.0004	0.1	0.5	–0.0112	–0.0113	0.0000
0.3	0.5	0.0779	0.0793	0.0014	0.3	0.5	–0.0262	–0.0266	0.0004
0.5	0.5	0	0.0027	0.0027	0.5	0.5	–0.0312	–0.0327	0.0014
0.7	0.5	–0.0779	–0.0749	0.0031	0.7	0.5	–0.0262	–0.0284	0.0021
0.9	0.5	–0.0482	–0.0467	0.0014	0.9	0.5	–0.0112	–0.0124	0.0011
0.1	0.7	0.0382	0.0386	0.0004	0.1	0.7	–0.0095	–0.0095	0.0001
0.3	0.7	0.0618	0.0627	0.0010	0.3	0.7	–0.0221	–0.0224	0.0004
0.5	0.7	0	0.0018	0.0018	0.5	0.7	–0.0262	–0.0275	0.0012
0.7	0.7	–0.0618	–0.0594	0.0023	0.7	0.7	–0.0221	–0.0238	0.0018
0.9	0.7	–0.0382	–0.0369	0.0013	0.9	0.7	–0.0095	–0.0104	0.0009
0.1	0.9	0.0159	0.0161	0.0002	0.1	0.9	–0.0040	–0.0041	0.0000
0.3	0.9	0.0258	0.0259	0.0002	0.3	0.9	–0.0095	–0.0097	0.0002
0.5	0.9	0	0.0005	0.0005	0.5	0.9	–0.0112	–0.0118	0.0006
0.7	0.9	–0.0258	–0.0248	0.0009	0.7	0.9	–0.0095	–0.0102	0.0008
0.9	0.9	–0.0159	–0.0153	0.0007	0.9	0.9	–0.0040	–0.0044	0.0004

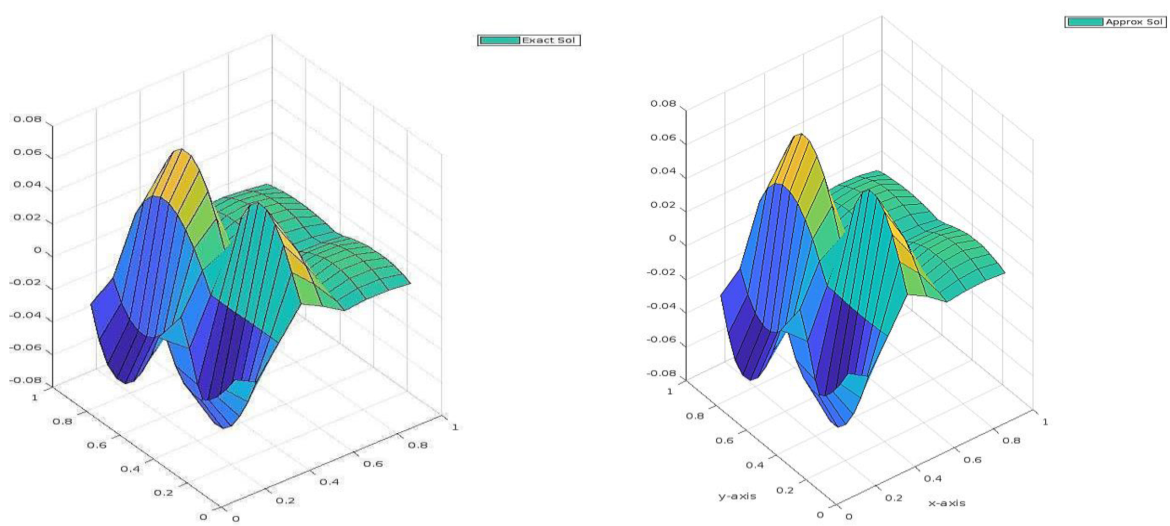

Fig. 3. Shows the exact (LHS) and the approximate vector solutions (RHS) at $\mathbf{t} = 0.5$.

$$-0.4\sqrt{e^{-0.11t}}\cos(x_1x_2)\left[x_2(x_2-1)-x_1(x_1-1)\right]$$
$$+0.2(x_1-1)(x_2-1)\sin(x_1x_2)\left[\sqrt{e^{-0.11t}}\left(1+x_1^2\right.\right.$$
$$\left.+x_2^2-0.2\cos\left((x_1-1)(x_2-1)\sin(x_1x_2)\sqrt{e^{-0.11t}}\right)\right)$$
$$\left.-0.055\sqrt{e^{-0.11t}}/\sqrt{e^{-0.11t}}\right]$$

The exact solutions are $u(\vec{x},t) = 0.3\tan(1-x_2)\sin(2\pi x_1)$ and $u_2(\vec{x},t) = x_1x_2(1-x_1)(1-x_2)$.

This problem is solved using the MGIFDM for $M = 9$, $NT = 20$ *and* $T = 1$, then the approximate vector solution $\vec{U}^n$ and its exact vector solution at x_1 and x_2 are given at the time $\hat{t} = 0.5$ in the Table 2 and are shown in Fig. 3, the absolute maximum error is (0.0024), and is shown in Fig. 4.

Remark 1: It is important to mention here; the above two examples were solved by the MGIFDM for different values for M, NT and T and the absolute maximum error were given in Table 3.

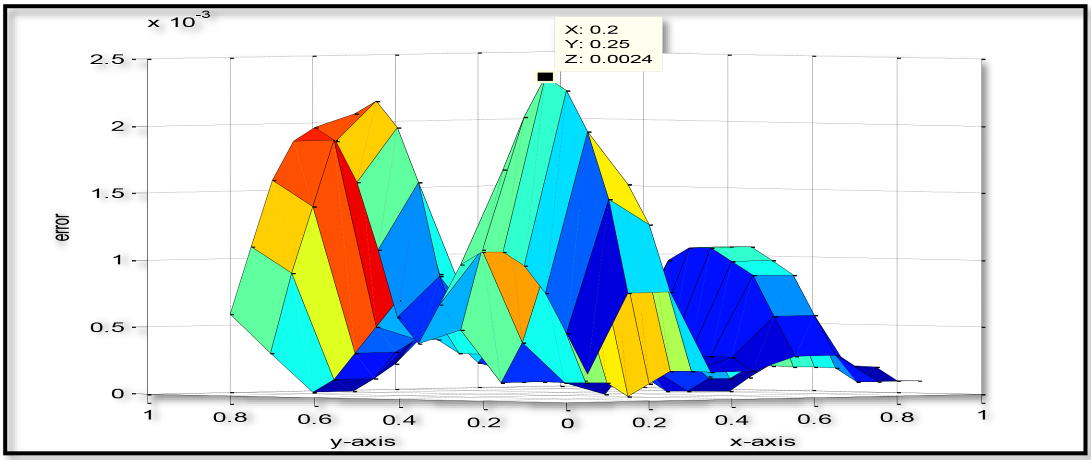

Fig. 4. Shows the absolute error between the exact and the approximate solutions at **t** = 0.5.

Table 2. Comparison between the exact and the approximate vector solutions

x_1	x_2	EXS	APPS	Absolute error	x_1	x_2	EXS	APPS	Absolute error
0.1	0.1	−0.0114	−0.0116	0.0002	0.1	0.1	0.0016	0.0014	0.0002
0.3	0.1	−0.0291	−0.0299	0.0008	0.3	0.1	0.0037	0.0032	0.0005
0.5	0.1	−0.0381	−0.0392	0.0011	0.5	0.1	0.0044	0.0037	0.0007
0.7	0.1	−0.0353	−0.0362	0.0010	0.7	0.1	0.0037	0.0031	0.0006
0.9	0.1	−0.0167	−0.0171	0.0004	0.9	0.1	0.0016	0.0013	0.0002
0.1	0.3	−0.0184	−0.0192	0.0008	0.1	0.3	0.0037	0.0033	0.0004
0.3	0.3	−0.0470	−0.0491	0.0020	0.3	0.3	0.0086	0.0076	0.0010
0.5	0.3	−0.0616	−0.0639	0.0024	0.5	0.3	0.0102	0.0089	0.0013
0.7	0.3	−0.0571	−0.0588	0.0017	0.7	0.3	0.0085	0.0074	0.0011
0.9	0.3	−0.0270	−0.0275	0.0005	0.9	0.3	0.0036	0.0032	0.0005
0.1	0.5	0	−0.0008	0.0008	0.1	0.5	0.0044	0.0042	0.0002
0.3	0.5	0	−0.0016	0.0016	0.3	0.5	0.0102	0.0096	0.0006
0.5	0.5	0	−0.0015	0.0015	0.5	0.5	0.0120	0.0114	0.0007
0.7	0.5	0	−0.0009	0.0009	0.7	0.5	0.0100	0.0094	0.0006
0.9	0.5	0	−0.0001	0.0001	0.9	0.5	0.0042	0.0040	0.0002
0.1	0.7	0.0184	0.0182	0.0002	0.1	0.7	0.0037	0.0037	0.0000
0.3	0.7	0.0470	0.0468	0.0002	0.3	0.7	0.0085	0.0086	0.0001
0.5	0.7	0.0616	0.0616	0.0000	0.5	0.7	0.0100	0.0101	0.0001
0.7	0.7	0.0571	0.0572	0.0001	0.7	0.7	0.0082	0.0084	0.0002
0.9	0.7	0.0270	0.0271	0.0001	0.9	0.7	0.0034	0.0035	0.0001
0.1	0.9	0.0114	0.0115	0.0002	0.1	0.9	0.0016	0.0016	0.0000
0.3	0.9	0.0291	0.0294	0.0003	0.3	0.9	0.0036	0.0038	0.0001
0.5	0.9	0.0381	0.0384	0.0003	0.5	0.9	0.0042	0.0044	0.0002
0.7	0.9	0.0353	0.0354	0.0002	0.7	0.9	0.0034	0.0036	0.0002
0.9	0.9	0.0167	0.0166	0.0001	0.9	0.9	0.0014	0.0015	0.0001

Table 3. Solving the two examples for different values of M, T, Nt resp.

Value of M	Value of T	Value of Nt	Max. Error For Example 1	Max. Error For Example 2
9	1	20	0.0032	0.0024
9	1	50	0.0030	0.0016
9	1	100	0.0029	0.0017
9	2	50	0.0059	0.0073
9	2	100	0.0058	0.0037
19	1	20	0.0029	0.0028
29	1	20	0.0014	0.0019
19	2	20	0.0063	0.022
29	2	20	0.0031	0.0011

Conclusion

The proposed method “MGIFDM” is used for solving CNPBVPCC. Two numerical examples have been used to examine the efficiency and the accuracy of the GIFDM. The results in Tables 1 and 2 show the maximum errors (Figs. 2 and 4) between the approximate and the exact vector solutions for the considered problems (Figs. 1 and 3), it is observed that the proposed method is efficient and accurate. The transformed system of equations (the GLAS) was solved by Cholesky decomposition; this method is faster than the other methods like Gauss elimination and the Haar wavelets methods because it saves a lot of number of calculations. The GFEM was applied easily and the elements in the GNAS are in analytic form (exact) and this way reduces the time of calculation for the elements compared with other methods that the elements will be in a full discrete form which needs more steps for calculations. It is important to mention here the approximate vector solution are given at the value of $\hat{t} = 0.5$ to brief the size of the paper, in fact same results with same accuracy were obtained at any value of $\hat{t}$ provided this value belong to I. In addition the two examples were solved for different values of the M, NT and T, the results given in Table 3 show the accuracy and the efficiency increase slowly with increasing the division “step length” of the interval of time, and the division of the space variable of the method, and this explains why detailed results are given for the values determined in Examples 1 and 2 and presented in Tables 1 and 2.

Author’s declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Mustansiriyah University.

Author’s contributions statement

J. A. A. Al-Hawasy and W. A. Ibrahim contributed to the design and implementation of the research, solving the problem, the proof of the theorem, and the writing of the manuscript.

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مزج طريقتي كاليركن – الفروقات الضمنية لحل زوج من نظام مكافيء غير خطي ذات معاملات ثابتة

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الخلاصة

في هذا البحث، تم استخدام طريقة الفروقات الضمنية – كاليركن الممزجة لحل زوج نظام غيرخطي من المعادلات التفاضلية الجزئية المكافئة مكافئ ذات معاملات ثابتة والتي تختصر بـ (CNPSCC) . في البداية تم ايجاد الصيغة الضعيفة لزوج النظام المكافيء الغير خطي ومن ثم تم صياغة الصيغة المتقطعة له باستخدام الطريقة المقترحة , الطريقة هذه تمزج طريقة كاليركن للعناصر المنتهية بالنسبة لمتغير الفضاء مع الطريقة الضمنية للفروقات المنتهية بالنسبة لمتغير الزمن ولهذا السبب سميت ب (MGIFDM). عند اي زمن متقطع tj حولت الطريقة المقترحة زوج النظام المكافئ الغير خطي ذات معاملات ثابتة الى زوج من نظام كاليركن الجبري الغير خطي، والذي تم حله باستخدام تقنية التنبأ والتصحيح حيث تقوم هذه التقنية بتحويل زوج النظام الجبري الغيرخطي الى زوج النظام كاليركن الجبري الخطي عند اي زمن tj والذي تم حله باستخدام طريقة جولسكي. تم برهان مبرهنة التقارب لبيان تقارب الحلول في هذه المسالة القترحة . تم اعطاء مثالين لبيان واختبار كفاءة ودقة الطريقة المقترحة، و النتائج اعطيت على شكل جداول وبينت بالرسومات.

الكلمات المفتاحية: زوج من نظام مكافئ غير الخطي، طريقة كاليركن، طريقة جولسكي، طريقة الفروقات الضمنية، تقنية التنبأ والتصحيح.