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RESEARCH ARTICLE

Solving Hierarchical Variational Inequality for Almost Mean Nonexpansive Mappings

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ABSTRACT

A clear common method in solving some types of nonlinear problems is to exchange the original problem with a collection of regularized problems and all these regularized problems have exactly one solution. A particular solution of the original problem will be obtained as a limit of these unique solutions of the regularized problems. This idea is used to provide a method for the hierarchical fixed point approach to solving variational inequality problems (VIPs). In this work, we intend to study two new iterative schemes by examining their strong convergence to a common fixed point for mappings defined for a nonempty closed and convex $\mathfrak{D}$ subset of a real Hilbert space Ξ. These iterative schemes are constructed for sequences of almost mean nonexpansive mappings and nonexpansive mappings under some control conditions. Firstly, strong convergence results are established for two iterative schemes for three mappings: the first $\Gamma\colon \mathfrak{D} \to \Xi$ is a contraction mapping, the second $\mathcal{P}_n\colon \mathfrak{D} \to \Xi$ is a sequence of nonexpansive mappings, and the third $\mathcal{K}_n\colon \mathfrak{D} \to \mathfrak{D}$ is a sequence of almost mean nonexpansive mappings. Secondly, when the constraints on parameters of two iterative schemes are relaxed, this yields other strong convergence results which also are solutions of hierarchical fixed point problem (HFPP). Finally, a solution of the quadratic minimization problem is found as a special case and this convergence is unique. Our results contain the former studies as particular statuses, and can be seen as a rededication and amelioration of many corresponding familiar results of hierarchical variational inequality problems (HVIP).

Keywords: Contraction mapping, Hierarchical variational inequality, Real hilbert space, Sequence of nonexpansive mappings, Strong convergence

Introduction

The well-known variational inequality problem (VIP) is produced from the manner of nonlinear programming and mathematical physics. It has huge applications in several domains, such as, engineering, mechanics, physics, control theory, economic decision, and others. It is a system of partial differential equations. It was first introduced by Stampacchia[1] in 1964 for modeling in the mechanics problem and it is defined in below in Eq. (1).

There are many sorts of VIPs[2–5] have been generalized, developed, and extended in many different techniques over time.[6–9]

Consider a real Hilbert space Ξ, with inner product and norm denoted by $\langle .,. \rangle$ and $\|\cdot\|$, respectively. Assume that $\mathfrak{D}$ closed and convex and $\emptyset \neq \mathfrak{D} \subseteq \Xi$. A mapping $\mathcal{K}:\ \mathfrak{D} \to \Xi$ is said to be

i. Strongly monotone[10,11] if there exists a constant $\tau > 0$ which in

$$\left\langle \mathcal{K}\mathfrak{f}^{\#} - \mathcal{K}t,\ \mathfrak{f}^{\#} - t \right\rangle \geq \tau\ \|\mathfrak{f}^{\#} - t\|^2,\ \forall\ \mathfrak{f}^{\#},\ t \in \mathfrak{D},$$

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ii. Monotone[10] if

$$\langle \mathcal{K}f^{\#} - \mathcal{K}t,\ f^{\#} - t \rangle \geq 0,\ \forall\, f^{\#},\ t \in \mathfrak{D},$$

iii. Contraction[12] if there exists a constant $\Lambda \in (0,\ 1)$ such that

$$\|\mathcal{K}f^{\#} - \mathcal{K}t\| \leq \Lambda\ \|f^{\#} - t\|,\ \forall\, f^{\#}, t \in \mathfrak{D},$$

iv. Nonexpansive[12] if

$$\|\mathcal{K}f^{\#} - \mathcal{K}t\| \leq\ \|f^{\#} - t\|,\ \forall\, f^{\#}, t \in \mathfrak{D},$$

v. Mean nonexpansive[13] if $\forall f,\ t \in \mathfrak{D}$

$$\|\mathcal{K}f^{\#} - \mathcal{K}t\| \leq \gamma \|f^{\#} - t\| + \delta \|f - \mathcal{K}t\|,\ \gamma, \delta \geq 0,\ \gamma + \delta\ \leq 1, \forall\, f^{\#}, t \in \mathfrak{D}.$$

The Authors studied, developed and proved many results about above mappings.[14–16]

Let $\mathfrak{B}(\mathfrak{D})$ denotes of the collection of all bounded subsets of $\mathfrak{D}$. Let $\mathcal{K}_1,\ \mathcal{K}_2 : \mathfrak{D} \rightarrow \Xi$ be two mappings. The deviation between $\mathcal{K}_1,\ \mathcal{K}_2$ on $B \in \mathfrak{B}(\mathfrak{D})$[17] denoted by $\|\mathcal{K}_1 - \mathcal{K}_2\|_{\infty,B}$, it is defined by

$$\|\mathcal{K}_1 - \mathcal{K}_2\|_{\infty,B} = \sup\left\{\|\mathcal{K}_1\left(f^{\#}\right) - \mathcal{K}_2\left(f^{\#}\right)\|\ :\ f^{\#} \in B\right\}$$

It is famous that the mapping $(I - \mathcal{K})$ is monotone, if $\mathcal{K}$ is a nonexpansive mapping.

If there exists a point $f^{\#} \in \mathfrak{D}$ such that $f^{\#} = \mathcal{K}\, f^{\#}$, then $f^{\#}$ is called be fixed point of $\mathcal{K}$. It is well famous that $\mathcal{F}(\mathcal{K})$ is closed and convex if $\mathcal{K}$ is nonexpansive. There are many Authors studied a fixed point as,[18,19] also many applications and related papers for this field such as.[20,21]

A VIP in a real Hilbert space Ξ is finding a point $f^{\hat{}} \in \mathfrak{D}$ such that

$$\langle \mathcal{K}f^{\hat{}},\ f^{\#} - f^{\hat{}} \rangle \geq 0, \forall\, f^{\#} \in \mathfrak{D}, \tag{1}$$

where $\mathcal{K}$ is a nonlinear mapping. The set of solution of Eq. (1) denote by VI($\mathfrak{D}$, $\mathcal{K}$), that is

$$\mathrm{VI}\,(\mathfrak{D},\ \mathcal{K}) = \{\langle \mathcal{K}f^{\hat{}},\ t - f^{\hat{}} \rangle \geq 0,\ \forall\, t \in \mathfrak{D}\})$$

It is well-known that the VIP in Eq. (1) is equivalent to the fixed point equation

$$f^{\hat{}} = P\mathfrak{D}\,((I - \mu\mathcal{K})\, f^{\hat{}}) \tag{2}$$

where $\mu > 0$ and $P\mathfrak{D}$ is the metric projection of Ξ onto $\mathfrak{D}$ which assigns, to each $f^{\#} \in \Xi$, the unique point in $\mathfrak{D}$, denoted $P\mathfrak{D}\ (f^{\#})$, such that

$$\|f^{\#} - P\mathfrak{D}\,(f)\,\| = \inf\left\{\|f^{\#} - t\|:\ \forall t \in \mathfrak{D}\right\}$$

So, fixed point algorithms can be utilized to disband VIPs. It is noticed that $P\mathfrak{D} : \mathfrak{D} \rightarrow \Xi$ is a nonexpansive and monotone mapping from Ξ onto $\mathfrak{D}$ (see[22] for another properties of projection operators).

The next problem is denominate a hierarchical fixed point problem: Find $f^{\#}\ \in \mathcal{F}(\mathcal{K})$ which in

$$\langle f^{\hat{}} -\ \mathcal{P}f^{\hat{}},\ f^{\#} -\ f^{\hat{}} \rangle \geq 0,\ \forall f^{\#} \in \mathcal{F}\,(\mathcal{K})\,. \tag{3}$$

where $\mathcal{P} : \mathfrak{D} \rightarrow \Xi$ be a mapping. The hierarchical fixed point approach was however introduced recently (see[23]). It's generalized and developed by several Authors as.[24,25]

In the literature there are several papers in which iterative methods for solving Eq. (1):

In[26] Moudafi proved a weak convergence's result for nonexpansive mappings.

On the other side, Yao et al.[27] show that converges strongly to unique solution of VIP for contraction mapping to dilate the range by using the metric projection, nonexpansive mapping, and a countable family of nonexpansive mappings.

Also, Sahu and Kang[28] proved strongly convergence's results for unique solution of VIP for contraction mapping, sequence of nonexpansive mappings, and a sequence of nearly nonexpansive mappings.
Dadashi and Amjadi[29] introduced strongly convergence's result for contraction mapping, and two sequence of nonexpansive mappings to find solution of VIP.
Throughout this paper, we studied the hierarchical VIP via the set of fixed points of a mapping and proved it is convergence and uniqueness of the following two schemes of sequences with general condition:

$$t_n = P\mathfrak{D}\left[(1-\Upsilon_n)\,f^{\#} + \Upsilon_n \mathcal{P}_n f^{\#}\right], \tag{4}$$

$$f^{\#}_{n+1} = P\mathfrak{D}\left[\varsigma_n \Gamma(t_n) + \sum_{i=1}^{n} (\varsigma_{i-1} - \varsigma_i)\mathcal{K}_i t_n\right],\ n \in \mathbb{N}$$

where $\varsigma_0 = 1,\ \{\Upsilon_n\},\ \{\varsigma_n\}$ are sequences in $(0,1)$.
Our results generalize and motivated the results of[26–29] and many other related works.
Now using the following symbols:
Ξ := real Hilbert space.
$\mathfrak{D}$:= non-empty convex closed subset of Ξ.
$(\rightarrow)$ for a strong convergence.
$(\rightharpoonup)$ for a weak convergence.
$\omega_w(f^{\#}_n) = \{f^{\#} :\ f^{\#}_{ni} \rightharpoonup f^{\#}\}$ denotes the weak ω-limit set of $\{f^{\#}_n\}$.
I the identity mapping of Ξ.
$f^{\wedge}$ is the unique solution of the VIP defined by Eq. (13).
$\mathcal{F}(\mathcal{K})$:= the set of all fixed points of $\mathcal{K}$.
In the following, recall the needed lemmas:

Lemma 1:[30] *If $f^{\#} \in \Xi$ and $t \in \mathfrak{D}$, then $t = P\mathfrak{D}(f^{\#})$ if and only if the next inequality satisfies:*

$$\langle f^{\#} - t,\ \beth - t\rangle \le 0,\ \forall \beth \in \mathfrak{D}.$$

Lemma 2:[31] *Let $\Gamma :\ \mathfrak{D} \rightarrow \Xi$ be a Λ-contraction mapping and $\mathcal{K} : \mathfrak{D} \rightarrow \mathfrak{D}$ be a nonexpansive mapping. Then,*

(i) The mapping $I - \Gamma$ is $(1-\Lambda)$–strongly monoton, i.e.

$$\langle (I-\Gamma)f^{\#} - (I-\Gamma)t, f^{\#} - t\rangle \ge (1-\Lambda)\,\| f^{\#} - t\|^2,\ \forall f^{\#},\ t \in \mathfrak{D}.$$

(ii) The mapping $I - \mathcal{K}$ is monotone, i.e.

$$\langle (I-\mathcal{K})f^{\#} - (I-\mathcal{K})t,\ f^{\#} - t\rangle \ge 0,\ \forall f^{\#}, t \in \mathfrak{D}.$$

Lemma 3:[32] *Let $\{\daleth_n\}$ and $\{\sigma_n\}$ be the sequences of nonnegative real numbers such that*

$$\daleth_{n+1} \le (1-\Delta_n)\daleth_n + \varepsilon_n + \sigma_n, \forall n \in \mathbb{N},$$

where $\{\Delta_n\}$ is a real number sequence in $(0,\ 1)$ and $\{\varepsilon_n\}$ is a real number sequence. Suppose that the next conditions satisfy:
(i) $\sum_{n=1}^{\infty} \sigma_n < \infty$;
(ii) $\sum_{n=1}^{\infty} \Delta_n = \infty$; and $\lim\limits_{n\to\infty} \sup \frac{\varepsilon_n}{\Delta_n} \le 0$.
Then, $\lim\limits_{n\to\infty} \daleth_n = 0$.

Lemma 4:[33] *If $f^n \rightharpoonup t \in \Xi$, then for any $t' \in \Xi, t \ne t'$ the following inequality holds*

$$\lim_n inf\|f^n - t'\| > \lim_n inf\|f^n - t\|.$$

Results and discussion

Assume that Ξ be real Hilbert space and $\emptyset \neq \mathfrak{D} \subseteq \Xi$ where $\mathfrak{D}$ is closed and convex. Also, let $\Gamma\colon \mathfrak{D} \to \Xi$ be a Λ-contraction, $\mathcal{J} = \{\mathcal{K}_n\}_{n=1}^{\infty}\colon \mathfrak{D} \to \mathfrak{D}$ be a sequence of almost mean nonexpansive mappings which in $\mathcal{F}(\mathcal{J}) \neq \emptyset$, and $\mathcal{K} : \mathfrak{D} \to \mathfrak{D}$ be a mapping defined by $\mathcal{K}f^{\#} = \lim_{n\to\infty} \mathcal{K}_n f^{\#},\ \forall f^{\#} \in \mathfrak{D}$. Assume that $\mathcal{F}(\mathcal{K}) = \cap_{n=1}^{\infty}\mathcal{F}(\mathcal{K}_n)$. Suppose that $\mathcal{P}_n : \mathfrak{D} \to \Xi$ be a sequence of nonexpansive mappings, and $\mathcal{P} : \mathfrak{D} \to \Xi$ be a nonexpansive mapping such that $\lim_{n\to\infty} \mathcal{P}_n f^{\#} = \mathcal{P}f^{\#}, \forall f \in \mathfrak{D}$. For arbitrary $f_1^{\#} \in \mathfrak{D}$, consider the sequence $\{f_n^{\#}\}$ defined by Eq. (4) and let recount some hypotheses which needs in:

(**E1**) $\lim_{n\to\infty} \varsigma_n = 0,\ \sum_{n=1}^{\infty} \varsigma_n = \infty;$

(**E2**) $\lim_{n\to\infty} \frac{\Upsilon_n}{\varsigma_n} = \varphi \in (0,\ \infty),\ \Upsilon_n \leq \sigma\varsigma_n$

(**E3**) $\sum_{n=1}^{\infty} (\varsigma_{n-1} - \varsigma_n) < \infty,\ \ \sum_{n=1}^{\infty} |\Upsilon_{n-1} - \Upsilon_n| < \infty;$

(**E4**) $\lim_{n\to\infty} \frac{(|\Upsilon_{n-1}-\Upsilon_n|+(\varsigma_{n-1}-\varsigma_n))}{\Upsilon_n\varsigma_n} = 0;$

(**E5**) there exists a constant $\Upsilon > 0$ such that $\frac{1}{\varsigma_n}\left|\frac{1}{\Upsilon_n} - \frac{1}{\Upsilon_{n-1}}\right| \leq \Upsilon;$

(**E6**) either $\sum_{n=1}^{\infty} \|\mathcal{P}_{n+1} - \mathcal{P}_n\|_{\infty,B} < \infty$ *or* $\lim_{n\to\infty} \frac{\|\mathcal{P}_{n+1}-\mathcal{P}_n\|_{\infty,B}}{\varsigma_n} = 0, \forall B \in \mathfrak{B}(\mathfrak{D}),$

Important new definitions and proposition are presented below:

Definition 1: A mapping $\mathcal{K} :\ \mathfrak{D} \to \mathfrak{D}$ is called almost mean nonexpansive if

$$\|\mathcal{K}^n f^{\#} - \mathcal{K}^n t\| \leq \gamma_n\|f^{\#} - t\| + \delta_n\|f^{\#} - \mathcal{K}^n t\|,\ \gamma_n, \delta_n \geq 0,\ \gamma_n + \delta_n \leq 1,\ n \in \mathbb{N}.$$

Example 1: *Let* $\mathcal{K}^n : \mathbb{R} \to \mathbb{R}$, *define* $\mathcal{K}^n(f^{\#})$ *by:* **(I)** $\mathcal{K}^n(f^{\#}) = \frac{f^{\#}}{7} + \frac{1}{2^n}$

$$\begin{aligned}\|\mathcal{K}^n f^{\#} - \mathcal{K}^n t\| &= \frac{1}{6}\left\|\frac{6}{7}\left(f^{\#} + \frac{1}{2^n}\right) - \frac{6}{7}\left(t + \frac{1}{2^n}\right)\right\| = \frac{1}{6}\left\|\frac{6}{7}f^{\#} - \frac{6}{7}t\right\| \\ &= \frac{1}{6}\left\|f^{\#} - \frac{f^{\#}}{7} - \frac{1}{2^n} - t + \frac{t}{7} + \frac{1}{2^n} - \frac{t}{7} - \frac{1}{2^n} + \frac{t}{7} + \frac{1}{2^n} - f^{\#} + f^{\#}\right\| \\ &\leq \frac{1}{6}\|f^{\#} - t\| + \frac{2}{6}\|f^{\#} - \mathcal{K}_n t\| + \frac{1}{6}\|\mathcal{K}_n f^{\#} - \mathcal{K}_n t\|\end{aligned}$$

$$\|\mathcal{K}^n f^{\#} - \mathcal{K}^n t\| \leq \frac{1}{5}\|f^{\#} - t\| + \frac{2}{5}\|f^{\#} - \mathcal{K}_n t\|$$

(II) $\mathcal{K}^n(f^{\#}) = 1$

$$\begin{aligned}\|\mathcal{K}^n f^{\#} - \mathcal{K}^n t\| &= \|1 - 1\| = 0 \\ &\leq \left\|\frac{f^{\#}}{8} + \frac{7}{8}f^{\#} - \frac{t}{8} - \frac{7}{8}\right\| \\ &\leq \frac{1}{8}\|f^{\#} - t\| + \frac{7}{8}\|f^{\#} - 1\| \\ \|\mathcal{K}^n f^{\#} - \mathcal{K}^n t\| &\leq \frac{1}{8}\|f^{\#} - t\| + \frac{7}{8}\|f^{\#} - \mathcal{K}^n t\|\end{aligned}$$

Definition 2: Let $\{\mathcal{K}_n\}$: $\mathfrak{D} \to \mathfrak{D}$ be a sequence of mappings. Then the sequence $\{\mathcal{K}_n\}$ is denominated a sequence of almost mean nonexpansive mappings if

$$\|\mathcal{K}_n f^{\#} - \mathcal{K}_n t\| \leq \gamma_n\|f^{\#} - t\| + \delta_n\|f^{\#} - \mathcal{K}_n t\|,\ \gamma_n, \delta_n \geq 0,\ \gamma_n + \delta_n \leq 1,\ n \in \mathbb{N}.$$

Definition 3: Let $\mathcal{J} = \{\mathcal{K}_n\}_{n=1}^{\infty}$: $\mathfrak{D} \to \mathfrak{D}$ be a sequence of almost mean nonexpansive mappings such that $\cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n) \neq \phi$. Let $\mathcal{K}$: $\mathfrak{D} \to \mathfrak{D}$ be a mapping such that $\mathcal{K}\mathfrak{f}^{\#} = \lim_{n\to\infty} \mathcal{K}_n\mathfrak{f}^{\#}, \forall \mathfrak{f}^{\#} \in \mathfrak{D}$ with $\mathcal{F}(\mathcal{K}) = \cap_{n=1}^{\infty}\mathcal{F}(\mathcal{K}_n)$. Then $\{\mathcal{K}_n\}$ is called hold condition (DS) if for each sequence $\{\mathfrak{f}_n^{\#}\}$ in $\mathfrak{D}$ with $\mathfrak{f}_n^{\#} \rightharpoonup w$ and $\mathfrak{f}_n^{\#} - \mathcal{K}_i\mathfrak{f}_n^{\#} \to 0, \forall i \in \mathbb{N}$, yield to $w \in \mathcal{F}(\mathcal{K})$.

It is plain to see that the sequence of nonexpansive mappings is a sequence of almost mean nonexpansive, but the converse is not hold.

Example 2: *Let* $\mathfrak{D} = [0,\ 1],\ \mathcal{K}_n : \mathfrak{D} \to \mathfrak{D}$, *defined by*

$$\mathcal{K}_n = \frac{\mathfrak{f}^{\#}}{5} + \frac{1}{\sqrt{n}},\ if\ \mathfrak{f}^{\#} \in \left[0, 1/2\right),\ 5 \leq n < \infty$$

and

$$\mathcal{K}_n = \frac{\mathfrak{f}^{\#}}{6} + \frac{1}{\sqrt{n}},\ if\ \mathfrak{f}^{\#} \in \left[1/2, 1\right],\ 1 < \mathrm{n} < 5.$$

Then $\mathcal{K}_n$ is discontinuous at $\mathfrak{f}^{\#} = 1/2$, so, $\mathcal{K}_n$ is not sequence of nonexpansive mapping. Now, let's prove that $\mathcal{K}_n$ is sequence of almost mean nonexpansive.

Case 1: $\mathfrak{f}^{\#}, t \in [0, 1/2)$

$$\begin{aligned}\|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\| &= \frac{1}{4}\left\|\frac{4}{5}\left(\mathfrak{f}^{\#} + \frac{1}{\sqrt{n}}\right) - \frac{4}{5}\left(t + \frac{1}{\sqrt{n}}\right)\right\| = \frac{1}{4}\left\|\frac{4}{5}\mathfrak{f}^{\#} - \frac{4}{5}t\right\| \\ &= \frac{1}{4}\left\|\mathfrak{f}^{\#} - \frac{\mathfrak{f}^{\#}}{5} - \frac{1}{\sqrt{n}} - t + \frac{t}{5} + \frac{1}{\sqrt{n}} - \frac{t}{5} - \frac{1}{\sqrt{n}} + \frac{t}{5} + \frac{1}{\sqrt{n}} - \mathfrak{f}^{\#} + \mathfrak{f}^{\#}\right\| \\ &\leq \frac{1}{4}\|\mathfrak{f}^{\#} - t\| + \frac{1}{2}\|\mathfrak{f}^{\#} - \mathcal{K}_n t\| + \frac{1}{4}\|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\|\end{aligned}$$

$$\|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\| \leq \frac{1}{3}\|\mathfrak{f}^{\#} - t\| + \frac{2}{3}\|\mathfrak{f}^{\#} - \mathcal{K}_n t\|$$

Case 2: If $\mathfrak{f} \in [0, 1/2)$ and $t \in [1/2, 1]$,

$$\begin{aligned}\|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\| &= \left\|\frac{\mathfrak{f}^{\#}}{5} + \frac{1}{\sqrt{n}} - \frac{t}{6} - \frac{1}{\sqrt{n}}\right\| = \left\|\frac{\mathfrak{f}^{\#}}{5} - \frac{\mathcal{K}_n\mathfrak{f}^{\#}}{5} + \frac{\mathcal{K}_n\mathfrak{f}^{\#}}{5} - \frac{\mathcal{K}_n t}{5} + \frac{\mathcal{K}_n t}{5} - \frac{t}{6}\right\| \\ &\leq \frac{2}{5}\|\mathfrak{f}^{\#} - \mathcal{K}_n t\| + \frac{2}{5}\|\mathcal{K}_n t - \mathcal{K}_n\mathfrak{f}^{\#}\| + \frac{1}{5}\|t - \mathfrak{f}^{\#}\|\end{aligned}$$

$$\|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\| \leq \frac{1}{3}\|\mathfrak{f}^{\#} - t\| + \frac{2}{3}\|\mathfrak{f}^{\#} - \mathcal{K}_n t\|$$

Case 3: If $t \in [0, 1/2)$ and $\mathfrak{f}^{\#} \in [1/2, 1]$, then the proof by the same way of case 2

$$\begin{aligned}\|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\| &\leq \frac{2}{5}\|\mathfrak{f}^{\#} - \mathcal{K}_n t\| + \frac{2}{5}\|\mathcal{K}_n t - \mathcal{K}_n\mathfrak{f}^{\#}\| + \frac{1}{5}\|t - \mathfrak{f}^{\#}\| \\ &\leq \frac{1}{3}\|\mathfrak{f}^{\#} - t\| + \frac{2}{3}\|\mathfrak{f}^{\#} - \mathcal{K}_n t\|\end{aligned}$$

Case 4: if $\mathfrak{f}^{\#}, t \in [^1\!/_2, 1]$, then the proof by the same way of case 1

$$\|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\| \le \frac{1}{4}\|\mathfrak{f}^{\#} - t\| + \frac{1}{2}\|\mathfrak{f}^{\#} - \mathcal{K}_n t\| + \frac{1}{4}\|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\|$$
$$\le \frac{1}{3}\|\mathfrak{f}^{\#} - t\| + \frac{2}{3}\|\mathfrak{f}^{\#} - \mathcal{K}_n t\|$$

So, $\gamma_n = \frac{1}{3}, \delta_n = \frac{2}{3}\gamma_n$

Proposition 1: Assume that Ξ be real Hilbert space and $\emptyset \neq \mathfrak{D} \subseteq \Xi$ where $\mathfrak{D}$ is closed and convex. Suppose that $\{\mathcal{K}_n\}_{n=1}^{\infty} : \mathfrak{D} \to \mathfrak{D}$ be a sequence of almost mean nonexpansive mappings such that $\cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n) \neq \phi$. Then

$$\langle (I - \mathcal{K}_n)\mathfrak{f}^{\#} - (I - \mathcal{K}_n)t,\ \mathfrak{f}^{\#} - t\rangle \ge 0,\ \forall\ \mathfrak{f}^{\#},\ t \in \mathfrak{D} \text{ and } n \in \mathbb{N}.$$

Proof: Let $\mathfrak{f}^{\#}, t^{\#} \in \mathfrak{D}$ and $r^{\#} \in \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$

$$\left\langle (I - \mathcal{K}_n)\,\mathfrak{f}^{\#} - (I - \mathcal{K}_n)\,t, \mathfrak{f}^{\#} - t\right\rangle = \|\mathfrak{f}^{\#} - t\|^2 - \left\langle \mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t, \mathfrak{f}^{\#} - t\right\rangle$$
$$\ge \|\mathfrak{f}^{\#} - t\|^2 - \|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\|\|\mathfrak{f}^{\#} - t\| \tag{5}$$

$$\|\mathcal{K}_n\mathfrak{f}^{\#} - \mathcal{K}_n t\| \le \gamma_n\|\mathfrak{f}^{\#} - t\| + \delta_n\|\mathfrak{f}^{\#} - \mathcal{K}_n t\|$$
$$\le \gamma_n\|\mathfrak{f}^{\#} - t\| + \delta_n\left[\|\mathfrak{f}^{\#} - r^{\#}\| - \left(\gamma_n\|t - r^{\#}\| + \delta_n\|t - r^{\#}\|\right)\right]$$
$$\le \gamma_n\|\mathfrak{f}^{\#} - t\| + \delta_n\left[\|\mathfrak{f}^{\#} - r^{\#}\| - \|t - r^{\#}\|\right]$$
$$\le \|\mathfrak{f}^{\#} - t\| \tag{6}$$

Putting Eq. (6) into Eq. (5),

$$\left\langle (I - \mathcal{K}_n)\,\mathfrak{f}^{\#} - (I - \mathcal{K}_n)\,t,\ \mathfrak{f}^{\#} - t\right\rangle$$
$$\ge \|\mathfrak{f}^{\#} - t\|^2 - \|\mathfrak{f}^{\#} - t\|\|\mathfrak{f}^{\#} - t\| \ge 0.$$

Proposition 2: Let $\mathcal{J} = \{\mathcal{K}_n\}_{n=1}^{\infty} : \Xi \to \Xi$ be a sequence almost mean nonexpansive mappings such that $\mathcal{F}(\mathcal{J}) \neq \emptyset$, *and* $t \in \Xi$ be a cluster point of a sequence $\{\mathfrak{f}_n^{\#}\}_{n=0}^{\infty}$. If $\|\mathcal{K}_n\mathfrak{f}_n^{\#} - \mathfrak{f}_n^{\#}\| \to 0$, $\forall n \in \mathbb{N}$, then $t \in \mathcal{F}(\mathcal{J})$.

Proof: Let $\{\mathfrak{f}_{n\varkappa}^{\#}\}_{n=0}^{\infty}$ be a subsequence of $\{\mathfrak{f}_n^{\#}\}_{n=0}^{\infty}$ s.t $\mathfrak{f}_{n\varkappa}^{\#} \rightharpoonup t$ and $\|\mathcal{K}_n\mathfrak{f}_n^{\#} - \mathfrak{f}_n^{\#}\| \to 0$.
Assume that $\mathcal{K}_n t \neq t$. By definition of a sequences of almost mean nonexpansive

$$\gamma_n\|\mathfrak{f}_{n\varkappa}^{\#} - t\| + \delta_n\|\mathfrak{f}_{n\varkappa}^{\#} - \mathcal{K}_n t\| \ge \|\mathcal{K}_n\mathfrak{f}_{n\varkappa}^{\#} - \mathcal{K}_n t\|$$
$$\gamma_n\|\mathfrak{f}_{n\varkappa}^{\#} - t\| + \delta_n\|\mathfrak{f}_{n\varkappa}^{\#} - \mathcal{K}_n t\|$$
$$\ge \gamma_n\|\mathfrak{f}_{n\varkappa}^{\#} - t\| + \delta_n\left[\|\mathfrak{f}_{n\varkappa}^{\#} - \mathcal{K}_n r^{\#}\| - \|\mathcal{K}_n t - \mathcal{K}_n r^{\#}\|\right]$$
$$\ge \gamma_n\|\mathfrak{f}_{n\varkappa}^{\#} - t\| + \delta_n[\|\mathfrak{f}_{n\varkappa}^{\#} - r^{\#}\| - \left(\gamma_n\|t - r^{\#}\| + \delta_n\|t - \mathcal{K}_n r^{\#}\|\right)]$$
$$\ge \gamma_n\|\mathfrak{f}_{n\varkappa}^{\#} - t\| + \delta_n\left[\|\mathfrak{f}_{n\varkappa}^{\#} - r^{\#}\| - \|t - r^{\#}\|\right]$$
$$\ge \gamma_n\|\mathfrak{f}_{n\varkappa}^{\#} - t\| + \delta_n\left[\left|\ \|\mathfrak{f}_{n\varkappa}^{\#} - r^{\#}\| - \|t - r^{\#}\|\ \right|\right]$$
$$\ge \gamma_n\|\mathfrak{f}_{n\mathfrak{f}_{n\varkappa}^{\#}\varkappa} - t\| + \delta_n\|\mathfrak{f}_{n\varkappa}^{\#} - t\|$$
$$\ge \|\mathfrak{f}_{n\varkappa}^{\#} - t\| \ge$$

Thus

$$\|\mathfrak{f}^{\#}_{n\varkappa}-t\| \geq \|\mathcal{K}_n\mathfrak{f}^{\#}_{n\varkappa}-\mathcal{K}_n t\|$$

But by Lemma 4 and triangle inequality, implies that

$$\begin{aligned}\lim_n inf\|\mathfrak{f}^{\#}_{n\varkappa}-t\| &\geq \lim_n inf\|\mathcal{K}_n\mathfrak{f}^{\#}_{n\varkappa}-\mathcal{K}_n t\|.\\ &= \lim_n inf\|\mathcal{K}_n\mathfrak{f}^{\#}_{n\varkappa}-\mathfrak{f}^{\#}_{n\varkappa}+\mathfrak{f}^{\#}_{n\varkappa}-\mathcal{K}_n t\|\\ &\geq \lim_n inf\left(\|\mathfrak{f}^{\#}_{n\varkappa}-\mathcal{K}_n t\|-\|\mathcal{K}_n\mathfrak{f}^{\#}_{n\varkappa}-\mathfrak{f}^{\#}_{n\varkappa}\|\right)\\ &= \lim_n inf\|\mathfrak{f}^{\#}_{n\varkappa}-\mathcal{K}_n t\|\\ &> \lim_n inf\|\mathfrak{f}^{\#}_{n\varkappa}-t\|\end{aligned}$$

A contradiction proves, so $t \in \mathcal{F}(\mathcal{J})$.

Proposition 3: Assume that $(\boldsymbol{E1})$– $(\boldsymbol{E2})$ holding, then $\{\mathfrak{f}^{\#}_n\}$ defined by Eq. (4) is bounded.

Proof: From Eq. (4)

$$\begin{aligned}\|t_n-r^{\#}\| &= \|P\mathfrak{D}\left[(1-\Upsilon_n)\mathfrak{f}^{\#}_n+\Upsilon_n\mathcal{P}_n\mathfrak{f}^{\#}_n\right]-P\mathfrak{D}\left(r^{\#}\right)\|\\ &\leq \|\left[(1-\Upsilon_n)\mathfrak{f}^{\#}_n+\Upsilon_n\mathcal{P}_n\mathfrak{f}^{\#}_n\right]-r^{\#}\|\\ &\leq (1-\Upsilon_n)\|\mathfrak{f}^{\#}_n-r^{\#}\|+\Upsilon_n\left(\|\mathcal{P}_n\mathfrak{f}^{\#}_n-\mathcal{P}_n r^{\#}\|+\|\mathcal{P}_n r-r^{\#}\|\right)\\ &\leq \|\mathfrak{f}^{\#}_n-r^{\#}\|+\Upsilon_n\|\mathcal{P}_n r-r^{\#}\|\end{aligned}$$

It follows that

$$\begin{aligned}\|\mathfrak{f}^{\#}_{n+1}-r^{\#}\| &= \left\|P\mathfrak{D}\left[\varsigma_n\Gamma(t_n)+\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)\mathcal{K}_i t_n\right]-P\mathfrak{D}\left(r^{\#}\right)\right\|\\ &\leq \left\|\left[\varsigma_n\Gamma(t_n)+\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)\mathcal{K}_i t_n\right]-r^{\#}\right\|\\ &\leq \varsigma_n\left(\|\Gamma(t_n)-\Gamma\left(r^{\#}\right)\|+\|\Gamma\left(r^{\#}\right)-r^{\#}\|\right)+\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)\|\mathcal{K}_i t_n-\mathcal{K}_i r^{\#}\|\\ &\leq \varsigma_n\left(\Lambda\|t_n-r^{\#}\|+\|\Gamma\left(r^{\#}\right)-r^{\#}\|\right)+(1-\varsigma_n)\|t_n-r^{\#}\|\\ &\leq \varsigma_n\left(\Lambda\|\mathfrak{f}^{\#}_n-r^{\#}\|+\Upsilon_n\Lambda\|\mathcal{P}_n r^{\#}-r^{\#}\|+\|\Gamma\left(r^{\#}\right)-r^{\#}\|\right)\\ &\quad+(1-\varsigma_n)\left(\|\mathfrak{f}^{\#}_n-r^{\#}\|+\Upsilon_n\|\mathcal{P}_n r^{\#}-r^{\#}\|\right)\\ &\leq [1-\varsigma_n(1-\Lambda)]\|\mathfrak{f}^{\#}_n-r^{\#}\|+\varsigma_n\|\Gamma\left(r^{\#}\right)-r^{\#}\|+\Upsilon_n(1+\varsigma_n\Lambda)\|\mathcal{P}_n r^{\#}-r^{\#}\|\end{aligned}$$

Since $\lim_{n\to\infty}\frac{\Upsilon_n}{\varsigma_n}=\varphi\in(0,\ \infty)$, and there is a constant $\mathcal{W}>0$, s.t.

$$\frac{\varsigma_n\|\Gamma\left(r^{\#}\right)-r^{\#}\|+\Upsilon_n(1+\varsigma_n\Lambda)\|\mathcal{P}_n r^{\#}-r^{\#}\|}{\varsigma_n}\leq\mathcal{W},\forall n\in\mathbb{N}.$$

Thus,

$$\begin{aligned}\|\mathfrak{f}^{\#}_{n+1}-r^{\#}\| &\leq [1-\varsigma_n(1-\Lambda)]\|\mathfrak{f}^{\#}_n-r^{\#}\|+\varsigma_n\mathcal{W}\\ &\leq \max\left\{\|\mathfrak{f}^{\#}_n-r^{\#}\|,\frac{\mathcal{W}}{1-\Lambda}\right\}\forall n\in\mathbb{N}\end{aligned}$$

Hence $\{\mathfrak{f}_n^{\#}\}$ is bounded. So $\{\Gamma(\mathfrak{f}_n^{\#})\}$, $\{t_n\}$, $\{\mathcal{K}_i\mathfrak{f}_n^{\#}\}$, and $\{\mathcal{K}_i t_n\}$ are bounded.

Proposition 4: Suppose that the hypothesis $(\boldsymbol{E1})$ and $(\boldsymbol{E6})$ holding. Then $\|\mathfrak{f}_{n+1}^{\#} - \mathfrak{f}_n^{\#}\| \to 0$, as $n \to \infty$.

Proof: Set $\wedge_n := \varsigma_n\Gamma(t_n) + \sum_{i=1}^{n}(\varsigma_{i-1} - \varsigma_i)\mathcal{K}_i t_n$, $\forall n \in \mathbb{N}$.
and $\mathcal{R} := sup_{n>1}\{\|\Gamma(t_{n-1})\| + \|\mathcal{K}_n t_{n-1}\| + \|\mathcal{P}_{n-1}\mathfrak{f}_{n-1}^{\#}\| + \|\mathfrak{f}_{n-1}^{\#}\|\}$
From Eq. (4), yields

$$\begin{aligned}
\|\mathfrak{f}_{n+1}^{\#} - \mathfrak{f}_n^{\#}\| &= \|P\mathfrak{D}\left(\wedge_n\right) - P\mathfrak{D}\left(\wedge_{n-1}\right)\| \le \|\wedge_n - \wedge_{n-1}\| \\
&= \Bigg\| \varsigma_n\Gamma\left(t_n\right) + \sum_{i=1}^{n}\left(\varsigma_{i-1} - \varsigma_i\right)\mathcal{K}_i t_n - \left[\varsigma_{n-1}\Gamma\left(t_{n-1}\right) + \sum_{i=1}^{n}\left(\varsigma_{i-1} - \varsigma_i\right)\mathcal{K}_i t_{n-1}\right] \\
&\quad + \varsigma_n\Gamma\left(t_{n-1}\right) - \varsigma_n\Gamma\left(t_{n-1}\right)\Bigg\| \\
&\le \varsigma_n\|\Gamma\left(t_n\right) - \Gamma\left(t_{n-1}\right)\| + \left(\varsigma_{n-1} - \varsigma_n\right)\left(\|\Gamma\left(t_{n-1}\right)\| + \|\mathcal{K}_n t_{n-1}\|\right) + \sum_{i=1}^{n}\left(\varsigma_{i-1} - \varsigma_i\right)\|\mathcal{K}_i t_n - \mathcal{K}_i t_{n-1}\| \\
&\le \varsigma_n\|\Gamma\left(t_n\right) - \Gamma\left(t_{n-1}\right)\| + \left(\varsigma_{n-1} - \varsigma_n\right)\left(\|\Gamma\left(t_{n-1}\right)\| + \|\mathcal{K}_n t_{n-1}\|\right) \\
&\quad + \sum_{i=1}^{n}\left(\varsigma_{i-1} - \varsigma_i\right)\left(\gamma_n\|t_n - t_{n-1}\| + \delta_n\|t_n - \mathcal{K}_i t_{n-1}\|\right) \\
&\le \varsigma_n\Lambda\|t_n - t_{n-1}\| + \left(\varsigma_{n-1} - \varsigma_n\right)\mathcal{R} + \left(1 - \varsigma_n\right)\|t_n - t_{n-1}\| \\
&= \left[1 - \varsigma_n\left(1 - \Lambda\right)\right]\|t_n - t_{n-1}\| + \left(\varsigma_{n-1} - \varsigma_n\right)\mathcal{R} \qquad (7)
\end{aligned}$$

Set $B := \{\mathfrak{f}_n\}$
From Eq. (4)

$$\begin{aligned}
\|t_n - t_{n-1}\| &= \left\|(1 - \Upsilon_n)\mathfrak{f}_n^{\#} + \Upsilon_n\mathcal{P}_n\mathfrak{f}_n^{\#} - \left[(1 - \Upsilon_{n-1})\mathfrak{f}_{n-1}^{\#} + \Upsilon_{n-1}\mathcal{P}_{n-1}\mathfrak{f}_{n-1}^{\#}\right]\right\| \\
&\le (1 - \Upsilon_n)\|\mathfrak{f}_n^{\#} - \mathfrak{f}_{n-1}^{\#}\| + |\Upsilon_{n-1} - \Upsilon_n|\,\|\mathfrak{f}_{n-1}^{\#} + \mathcal{P}_{n-1}\mathfrak{f}_{n-1}^{\#}\| + \Upsilon_n\|\mathfrak{f}_n^{\#} - \mathfrak{f}_{n-1}^{\#}\| + \Upsilon_n\|\mathcal{P}_n - \mathcal{P}_{n-1}\|_{\infty,B} \\
&= \|\mathfrak{f}_n^{\#} - \mathfrak{f}_{n-1}^{\#}\| + |\Upsilon_{n-1} - \Upsilon_n|\,\mathcal{R} + \Upsilon_n\|\mathcal{P}_n - \mathcal{P}_{n-1}\|_{\infty,B} \qquad (8)
\end{aligned}$$

Putting Eq. (8) into Eq. (7), and then use $(\boldsymbol{E1})$, $(\boldsymbol{E3})$, $(\boldsymbol{E6})$, and by Lemma 3 to get the following:

$$\begin{aligned}
\|\mathfrak{f}_{n+1}^{\#} - \mathfrak{f}_n^{\#}\| &\le \|\wedge_n - \wedge_{n-1}\| \\
&\le \left[1 - \varsigma_n\left(1 - \Lambda\right)\right]\left[\|\mathfrak{f}_n^{\#} - \mathfrak{f}_{n-1}^{\#}\| + |\Upsilon_{n-1} - \Upsilon_n|\,\mathcal{R} + \Upsilon_n\|\mathcal{P}_n - \mathcal{P}_{n-1}\|_{\infty,B}\right] + \left(\varsigma_{n-1} - \varsigma_n\right)\mathcal{R} \\
&\le \left[1 - \varsigma_n\left(1 - \Lambda\right)\right]\|\mathfrak{f}_n^{\#} - \mathfrak{f}_{n-1}^{\#}\| + \left[|\Upsilon_{n-1} - \Upsilon_n| + \left(\varsigma_{n-1} - \varsigma_n\right)\right]\mathcal{R} + \Upsilon_n\|\mathcal{P}_n - \mathcal{P}_{n-1}\|_{\infty,B} \qquad (9)
\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\|\mathfrak{f}_{n+1}^{\#} - \mathfrak{f}_n^{\#}\| = 0, \qquad (10)$$

Proposition 5: Assume that $(\boldsymbol{E1})$, $(\boldsymbol{E2})$, $(\boldsymbol{E4})$ and $(\boldsymbol{E6})$ holding. Then:

i. $\lim_{n\to\infty}\|\mathfrak{f}_n^{\#} - \mathcal{K}_i\mathfrak{f}_n^{\#}\| = 0$,
ii. $\|t_n - \mathcal{K}_i t_n\| \to 0$, as $n \to \infty$, $\forall i \in \mathbb{N}$,
iii. $\lim_{n\to\infty}\frac{\|\mathfrak{f}_{n+1}^{\#} - \mathfrak{f}_n^{\#}\|}{\Upsilon_n} = 0$ and $\lim_{n\to\infty}\frac{\|\wedge_n - \wedge_{n-1}\|}{\Upsilon_n} = \frac{\|\wedge_n - \wedge_{n-1}\|}{\varsigma_n} = 0$,
iv. $\|t_n - \Gamma(t_n)\| \le \|\mathfrak{f}_n^{\#} - \Gamma(\mathfrak{f}_n^{\#})\|$. when $n \to \infty$.

Proof for (i):

$$
\begin{aligned}
\|ƀ_n^{\#} - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\| &\le \|ƀ_n^{\#} - ƀ_{n+1}^{\#}\| + \|ƀ_{n+1}^{\#} - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\| \\
&\le \|ƀ_n^{\#} - ƀ_{n+1}^{\#}\| + \varsigma_n \|\Gamma(t_n) - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\| + \sum_{\mathrm{i}=1}^{n} (\varsigma_{\mathrm{i}-1} - \varsigma_{\mathrm{i}}) \|\mathcal{K}_{\mathrm{i}} t_n - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\| \\
&\le \|ƀ_n^{\#} - ƀ_{n+1}^{\#}\| + \varsigma_n \|\Gamma(t_n) - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\| + (1-\varsigma_n)\left[\gamma_n \|t_n - ƀ_n^{\#}\| + \delta_n \|t_n - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\|\right]
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|ƀ_n^{\#} - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\| \le{}& \frac{1}{(1-(1-\varsigma_n)\delta_n(1-\Upsilon_n))}\Big[\|ƀ_n^{\#} - ƀ_{n+1}^{\#}\| + \varsigma_n \|\Gamma(t_n) - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\| \\
&+ (1-\varsigma_n)\left[\gamma_n \Upsilon_n \|\mathcal{P}_n ƀ_n^{\#} - ƀ_n^{\#}\| + \delta_n \Upsilon_n \|\mathcal{P}_n ƀ_n^{\#} - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\|\right]\Big]
\end{aligned}
$$

By (***E*1**), (***E*2**), and Eq. (10) it follows that $\Upsilon_n \to 0$, as $n \to \infty$, $\lim_{n\to\infty} \|ƀ_n^{\#} - \mathcal{K}_{\mathrm{i}} ƀ_n^{\#}\| = 0, \ \forall \mathrm{i} \in \mathbb{N}$.

Proof for (ii): $\Upsilon_n \to 0$ as $n \to \infty$, by $\lim_{n\to\infty} \frac{\Upsilon_n}{\varsigma_n} = \varphi \in (0, \infty)$, and condition (***E*1**).

$$
\|t_n - ƀ_n^{\#}\| = \Upsilon_n \|\mathcal{P}_n ƀ_n^{\#} - ƀ_n^{\#}\| \to 0 \text{ as } n \to \infty. \tag{11}
$$

From Eqs. (10) and (11)

$$
\begin{aligned}
\|t_n - \mathcal{K}_{\mathrm{i}} t_n\| &\le \|t_n - ƀ_n^{\#}\| + \|ƀ_n^{\#} - ƀ_{n+1}^{\#}\| + \|ƀ_{n+1}^{\#} - \mathcal{K}_{\mathrm{i}} t_n\| \\
&\le \|t_n - ƀ_n^{\#}\| + \|ƀ_n^{\#} - ƀ_{n+1}^{\#}\| + \varsigma_n \|\Gamma(t_n) - \mathcal{K}_{\mathrm{i}} t_n\| + (1-\varsigma_n)\|\mathcal{K}_{\mathrm{i}} t_n - \mathcal{K}_{\mathrm{i}} t_n\| \to 0,
\end{aligned}
$$

as $n \to \infty, \ \forall \mathrm{i} \in \mathbb{N}$.

Proof for (iii): From Eq. (9)

$$
\begin{aligned}
\frac{\|ƀ_{n+1}^{\#} - ƀ_n^{\#}\|}{\Upsilon_n} &= \frac{\|P\mathfrak{D}(\wedge_n) - P\mathfrak{D}(\wedge_{n-1})\|}{\Upsilon_n} \le \frac{\|\wedge_n - \wedge_{n-1}\|}{\Upsilon_n} \\
&\le [1 - \varsigma_n(1-\Lambda)] \frac{\|ƀ_n^{\#} - ƀ_{n-1}^{\#}\|}{\Upsilon_n} + \left[\frac{|\Upsilon_{n-1} - \Upsilon_n|}{\Upsilon_n} + \frac{(\varsigma_{n-1} - \varsigma_n)}{\Upsilon_n} + \right] \mathcal{R} + \|\mathcal{P}_n - \mathcal{P}_{n-1}\|_{\infty,B} \\
&= [1 - \varsigma_n(1-\Lambda)] \frac{\|ƀ_n^{\#} - ƀ_{n-1}^{\#}\|}{\Upsilon_{n-1}} + [1 - \varsigma_n(1-\Lambda)] \left(\frac{\|ƀ_n^{\#} - ƀ_{n-1}^{\#}\|}{\Upsilon_n} - \frac{\|ƀ_n^{\#} - ƀ_{n-1}^{\#}\|}{\Upsilon_{n-1}}\right) \\
&\quad + \left[\frac{|\Upsilon_{n-1} - \Upsilon_n|}{\Upsilon_n} + \frac{(\varsigma_{n-1} - \varsigma_n)}{\Upsilon_n}\right] \mathcal{R} + \|\mathcal{P}_n - \mathcal{P}_{n-1}\|_{\infty,B}
\end{aligned} \tag{12}
$$

Hence

$$
(1 - \varsigma_n(1-\Lambda))\left(\frac{1}{\Upsilon_n} - \frac{1}{\Upsilon_{n-1}}\right) \le \varsigma_n \frac{1}{\varsigma_n}\left|\frac{1}{\Upsilon_n} - \frac{1}{\Upsilon_{n-1}}\right| \le \varsigma_n \Upsilon
$$

Let $u_n = \varsigma_n(1-\Lambda)$

$$
v_n = \varsigma_n \Upsilon \|ƀ_n^{\#} - ƀ_{n-1}^{\#}\| + \left[\frac{|\Upsilon_{n-1} - \Upsilon_n|}{\Upsilon_n} + \frac{(\varsigma_{n-1} - \varsigma_n)}{\Upsilon_n}\right] \mathcal{R},
$$

$$
e_n = \|\mathcal{P}_n - \mathcal{P}_{n-1}\|_{\infty,B}
$$

By Eq. (12)

$$\frac{\|\mathfrak{f}^{\#}_{n+1}-\mathfrak{f}^{\#}_{n}\|}{\Upsilon_n} \le \frac{\|\Lambda_n-\Lambda_{n-1}\|}{\Upsilon_n}$$

$$\le \ (1-\varsigma_n(1-\Lambda))\frac{\|\mathfrak{f}^{\#}_{n}-\mathfrak{f}^{\#}_{n-1}\|}{\Upsilon_{n-1}}+\varsigma_n\Upsilon\|\mathfrak{f}^{\#}_{n}-\mathfrak{f}^{\#}_{n-1}\|+\left[\frac{|\Upsilon_{n-1}-\Upsilon_n|}{\Upsilon_n}+\frac{(\varsigma_{n-1}-\varsigma_n)}{\Upsilon_n}\right]\mathcal{R}+\|\mathcal{P}_n-\mathcal{P}_{n-1}\|_{\infty,B}$$

$$\le (1-\varsigma_n(1-\Lambda))\frac{\|\Lambda_{n-1}-\Lambda_{n-2}\|}{\Upsilon_{n-1}}+\varsigma_n\Upsilon\|\mathfrak{f}^{\#}_{n}-\mathfrak{f}^{\#}_{n-1}\|+\left[\frac{|\Upsilon_{n-1}-\Upsilon_n|}{\Upsilon_n}+\frac{(\varsigma_{n-1}-\varsigma_n)}{\Upsilon_n}\right]\mathcal{R}+\|\mathcal{P}_n-\mathcal{P}_{n-1}\|_{\infty,B}$$

$$\le (1-u_n)\frac{\|\Lambda_{n-1}-\Lambda_{n-2}\|}{\Upsilon_{n-1}}+v_n+e_n$$

By conditions $(\boldsymbol{E1}),(\boldsymbol{E4}),(\boldsymbol{E6})$ and by applying Lemma 3 $\lim_{n\to\infty}\frac{\|\mathfrak{f}^{\#}_{n+1}-\mathfrak{f}^{\#}_{n}\|}{\Upsilon_n}=0$, and $\lim_{n\to\infty}\frac{\|\Lambda_n-\Lambda_{n-1}\|}{\Upsilon_n}=\frac{\|\Lambda_n-\Lambda_{n-1}\|}{\varsigma_n}=0$.

Proof for (iv): $\Upsilon_n\to 0$ as $n\to\infty$, by $\lim_{n\to\infty}\frac{\Upsilon_n}{\varsigma_n}=\varphi\in(0,\ \infty)$, and condition $(\boldsymbol{E1})$.

$$\|\mathfrak{f}^{\#}_{n}-t_n\|=\Upsilon_n\|\mathfrak{f}^{\#}_{n}-\mathcal{P}_n\mathfrak{f}^{\#}_{n}\|\to 0 \text{ as } n\to\infty.$$

$$\|t_n-\Gamma(t_n)\|\le\|t_n-\Gamma(\mathfrak{f}^{\#}_{n})\|+\|\Gamma(\mathfrak{f}^{\#}_{n})-\Gamma(t_n)\|$$

$$=\|\mathfrak{f}^{\#}_{n}-\Gamma(\mathfrak{f}^{\#}_{n})\|-\Upsilon_n\|\mathfrak{f}^{\#}_{n}-\Gamma(\mathfrak{f}^{\#}_{n})\|+\Upsilon_n\|\mathcal{P}_n\mathfrak{f}^{\#}_{n}-\Gamma(\mathfrak{f}^{\#}_{n})\|+\Lambda\Upsilon_n\|\mathfrak{f}^{\#}_{n}-\mathcal{P}_n\mathfrak{f}^{\#}_{n}\|$$

$$=\|\mathfrak{f}^{\#}_{n}-\Gamma(\mathfrak{f}^{\#}_{n})\|,\ \text{when } n\to\infty.$$

Theorem 1: *Let* $(\boldsymbol{E1})-(\boldsymbol{E6})$ *are holding. Then* $\mathfrak{f}^{\#}_{n}\to\mathfrak{f}^{\wedge}$, *as* $n\to\infty$ *where* $\{\mathfrak{f}^{\#}_{n}\}$ *is defined by* Eq. (4) *and* $\mathfrak{f}^{\wedge}\in\cap_{n=1}^{\infty}\mathcal{F}(\mathcal{K}_n)$ *is the unique solution of the VIP defined by*

$$\left\langle\frac{1}{\varphi}(I-\Gamma)\ \mathfrak{f}^{\wedge}+(I-\mathcal{P})\,\mathfrak{f}^{\wedge},\mathfrak{f}-\mathfrak{f}^{\wedge}\right\rangle\ge 0,\ \forall\,\mathfrak{f}\ \in\ \cap_{n=1}^{\infty}\mathcal{F}(\mathcal{K}_n). \tag{13}$$

Proof: It's evidenced in[27] the VIP has unique solution.

Now, to prove the convergence

For any $r^{\#}\epsilon\cap_{n=1}^{\infty}\ \mathcal{F}(\mathcal{K}_n)$, computing $\langle\sigma_n,\ \mathfrak{f}^{\#}_{n}-r^{\#}\rangle$

where $\sigma_n:=\frac{\|\mathfrak{f}^{\#}_{n}-\mathfrak{f}^{\#}_{n+1}\|}{(1-\varsigma_n)\Upsilon_n}$ and $d_n=\varsigma_n\Gamma(\mathfrak{f}^{\#}_{n})+\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)\mathcal{K}_i t_n,\ \forall n\in\mathbb{N}$.

Then, By Eq. (4)

$$\mathfrak{f}_{n+1}=P\mathfrak{D}(\Lambda_n)-\Lambda_n+\Lambda_n+(1-\varsigma_n)t_n-\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)t_n$$

$$=P\mathfrak{D}(\Lambda_n)-\Lambda_n+\varsigma_n\Gamma(t_n)+\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)(\mathcal{K}_i t_n-t_n)+(1-\varsigma_n)t_n$$

This yields that

$$\mathfrak{f}^{\#}_{n}-\mathfrak{f}^{\#}_{n+1}=\mathfrak{f}^{\#}_{n}-\varsigma_n\mathfrak{f}^{\#}_{n}+\varsigma_n\mathfrak{f}^{\#}_{n}-\left(P\mathfrak{D}(\Lambda_n)-\Lambda_n+\varsigma_n\Gamma(t_n)+\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)(\mathcal{K}_i t_n-t_n)+(1-\varsigma_n)t_n\right)$$

$$=(1-\varsigma_n)(\mathfrak{f}^{\#}_{n}-t_n)+(\Lambda_n-P\mathfrak{D}(\Lambda_n))+\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)(t_n-\mathcal{K}_i t_n)+\varsigma_n(\mathfrak{f}^{\#}_{n}-\Gamma(t_n))$$

$$\frac{\mathfrak{f}_n^{\#} - \mathfrak{f}_{n+1}^{\#}}{(1-\varsigma_n)\Upsilon_n} = \left(\mathfrak{f}_n^{\#} - \mathcal{P}_n \mathfrak{f}_n^{\#}\right) + \frac{1}{(1-\varsigma_n)\Upsilon_n}\left(\wedge_n - P\mathfrak{D}\left(\wedge_n\right)\right) + \frac{1}{(1-\varsigma_n)\Upsilon_n}\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)(t_n - \mathcal{K}_i t_n)$$
$$+ \frac{\varsigma_n}{(1-\varsigma_n)\Upsilon_n}\left(\mathfrak{f}_n^{\#} - \Gamma\left(\mathfrak{f}_n^{\#}\right)\right) - \frac{\varsigma_n}{(1-\varsigma_n)\Upsilon_n}\left[(I-\Gamma)\mathfrak{f}_n^{\#}) - (I-\Gamma)t_n\right] + \frac{\varsigma_n}{(1-\varsigma_n)\Upsilon_n}\left(\mathfrak{f}_n^{\#} - t_n\right)$$

Hence $\mathfrak{f}_{n+1}^{\#} = P\mathfrak{D}(\wedge_n)$. For any $r^{\#} \in \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$,

$$\left\langle \sigma_n,\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle = \left\langle \mathfrak{f}_n^{\#} - \mathcal{P}_n\mathfrak{f}_n^{\#}, \mathfrak{f}_n^{\#} - r^{\#}\right\rangle + \frac{1}{(1-\varsigma_n)\Upsilon_n}\left\langle \wedge_n - P\mathfrak{D}\left(\wedge_n\right), P\mathfrak{D}\left(\wedge_{n-1}\right) - r^{\#}\right\rangle$$
$$+ \frac{1}{(1-\varsigma_n)\Upsilon_n}\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)\left\langle t_n - \mathcal{K}_i t_n,\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle + \frac{\varsigma_n}{(1-\varsigma_n)\Upsilon_n}\left\langle (I-\Gamma)\mathfrak{f}_n^{\#},\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle$$
$$- \frac{\varsigma_n}{(1-\varsigma_n)\Upsilon_n}\left\langle (I-\Gamma)\mathfrak{f}_n^{\#} - (I-\Gamma)t_n,\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle$$
$$+ \frac{\varsigma_n}{(1-\varsigma_n)\Upsilon_n}\left\langle \mathfrak{f}_n^{\#} - t_n,\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle \quad (14)$$

Using Lemma 2, and Proposition 1

$$\left\langle \mathfrak{f}_n^{\#} - \mathcal{P}_n\mathfrak{f}_n^{\#}, \mathfrak{f}_n^{\#} - r^{\#}\right\rangle = \left\langle (I-\mathcal{P}_n)\mathfrak{f}_n^{\#} - (I-\mathcal{P}_n)r^{\#}, \mathfrak{f}_n^{\#} - r^{\#}\right\rangle + \left\langle (I-\mathcal{P}_n)r^{\#}, \mathfrak{f}_n^{\#} - r^{\#}\right\rangle$$
$$\geq \left\langle (I-\mathcal{P}_n)r^{\#}, \mathfrak{f}_n^{\#} - r^{\#}\right\rangle \quad (15)$$

$$\left\langle t_n - \mathcal{K}_i t_n,\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle = \left\langle (I-\mathcal{K}_i)t_n - (I-\mathcal{K}_i)r^{\#},\ \mathfrak{f}_n^{\#} - t_n\right\rangle + \left\langle (I-\mathcal{K}_i)t_n - (I-\mathcal{K}_i)r^{\#},\ t_n - r^{\#}\right\rangle$$
$$\geq \left\langle (I-\mathcal{K}_i)t_n - (I-\mathcal{K}_i)r^{\#},\ \mathfrak{f}_n^{\#} - t_n\right\rangle$$
$$= \left\langle (I-\mathcal{K}_i)t_n,\ \mathfrak{f}_n^{\#} - t_n\right\rangle$$
$$= \Upsilon_n\left\langle (I-\mathcal{K}_i)t_n,\ \mathfrak{f}_n^{\#} - \mathcal{P}_n\mathfrak{f}_n^{\#})\right\rangle \quad (16)$$

$$\left\langle (I-\Gamma)\mathfrak{f}_n^{\#},\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle = \left\langle (I-\Gamma)\mathfrak{f}_n^{\#} - (I-\Gamma)r^{\#},\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle + \left\langle (I-\Gamma)r^{\#},\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle$$
$$\geq (1-\Lambda)\|\mathfrak{f}_n^{\#} - r^{\#}\|^2 + \left\langle (I-\Gamma)r^{\#},\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle \quad (17)$$
$$- \left\langle (I-\Gamma)\mathfrak{f}_n^{\#} - (I-\Gamma)t_n,\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle$$
$$\geq \|(I-\Gamma)\mathfrak{f}_n^{\#} - (I-\Gamma)t_n\|\|\mathfrak{f}_n^{\#} - r^{\#}\| \quad (18)$$

$$\left\langle \mathfrak{f}_n^{\#} - t_n,\ \mathfrak{f}_n^{\#} - r^{\#}\right\rangle \geq \Upsilon_n\|\mathfrak{f}_n^{\#} - \mathcal{P}_n\mathfrak{f}_n\|\|\mathfrak{f}_n^{\#} - r^{\#}\| \quad (19)$$

Applying Lemma 1, to get

$$\left\langle \wedge_n - P\mathfrak{D}\left(\wedge_n\right), P\mathfrak{D}\left(\wedge_{n-1}\right) - r^{\#}\right\rangle$$
$$= \left\langle \wedge_n - P\mathfrak{D}\left(\wedge_n\right), P\mathfrak{D}\left(\wedge_{n-1}\right) - P\mathfrak{D}\left(\wedge_n\right)\right\rangle + \left\langle \wedge_n - P\mathfrak{D}\left(\wedge_n\right), P\mathfrak{D}\left(\wedge_n\right) - r^{\#}\right\rangle$$
$$\geq \left\langle \wedge_n - P\mathfrak{D}\left(\wedge_n\right), P\mathfrak{D}\left(\wedge_{n-1}\right) - \mathfrak{D}\left(\wedge_n\right)\right\rangle$$
$$\geq \left\langle \wedge_n - P\mathfrak{D}\left(\wedge_n\right), P\mathfrak{D}\left(\wedge_{n-1}\right) - P\mathfrak{D}\left(\wedge_n\right)\right\rangle \quad (20)$$

Substituting Eqs. (15) to (20) into Eq. (14)

$$\begin{aligned}\langle \sigma_n,\ \not{f}_n^{\#} - r^{\#}\rangle \geq\ & \langle (I-\mathcal{P}_n) r^{\#}, \not{f}_n^{\#} - r^{\#}\rangle \\ &+ \frac{1}{(1-\varsigma_n)\ell_n}\langle \wedge_n - P\mathfrak{D}(\wedge_n), P\mathfrak{D}(\wedge_{n-1}) - P\mathfrak{D}(\wedge_n)\rangle \\ &+ \frac{1}{(1-\varsigma_n)}\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)\langle (I-\mathcal{K}_i)t_n,\ \not{f}_n^{\#} - \mathcal{P}_n\not{f}_n^{\#})\rangle \\ &+ \frac{\varsigma_n}{(1-\varsigma_n)\Upsilon_n}\|(I-\Gamma)\not{f}_n^{\#} - (I-\Gamma)t_n\|\|\not{f}_n^{\#} - r\| \\ &+ \frac{\varsigma_n(1-\Lambda)}{(1-\varsigma_n)\Upsilon_n}\|\not{f}_n^{\#} - r^{\#}\|^2 + \frac{\varsigma_n}{(1-\varsigma_n)\Upsilon_n}\langle (I-\Gamma)r,\ \not{f}_n^{\#} - r^{\#}\rangle \\ &+ \frac{\varsigma_n\Upsilon_n}{(1-\varsigma_n)\Upsilon_n}\|\not{f}_n^{\#} - \mathcal{P}_n\not{f}_n^{\#}\|\|\not{f}_n^{\#} - r^{\#}\|\end{aligned} \tag{21}$$

From Eq. (21)

$$\begin{aligned}\|\not{f}_n^{\#} - r^{\#}\|^2 \leq\ & \frac{(1-\varsigma_n)\Upsilon_n}{\varsigma_n(1-\Lambda)}\left[\langle \sigma_n,\ \not{f}_n^{\#} - r^{\#}\rangle - \langle (I-\mathcal{P}_n)r, \not{f}_n^{\#} - r^{\#}\rangle\right] \\ &- \frac{1}{(1-\Lambda)}\|(I-\Gamma)\not{f}_n^{\#} - (I-\Gamma)t_n\|\|\not{f}_n^{\#} - r^{\#}\| \\ &- \frac{\Upsilon_n}{\varsigma_n(1-\Lambda)}\sum_{i=1}^{n}(\varsigma_{i-1}-\varsigma_i)\langle (I-\mathcal{K}_i)t_n,\ \not{f}_n^{\#} - \mathcal{P}_n\not{f}_n^{\#})\rangle \\ &- \frac{1}{\varsigma_n(1-\Lambda)}\|\wedge_n - P\mathfrak{D}(\wedge_n)\|\|P\mathfrak{D}(\wedge_{n-1}) - P\mathfrak{D}(\wedge_n)\| \\ &- \frac{\Upsilon_n}{(1-\Lambda)}\|\not{f}_n^{\#} - \mathcal{P}_n\not{f}_n^{\#}\|\|\not{f}_n^{\#} - r^{\#}\| - \frac{1}{(1-\Lambda)}\langle (I-\Gamma)r,\ \not{f}_n^{\#} - r^{\#}\rangle\end{aligned}$$

Hence, from proposition Proposition 5 (iii)
$\sigma_n = \frac{\|\not{f}_n^{\#} - \not{f}_{n+1}^{\#}\|}{(1-\varsigma_n)\Upsilon_n} \to 0$, $\frac{\|\wedge_n - \wedge_{n-1}\|}{\varsigma_n} \to 0$ and $\forall i \in \mathbb{N}$, $t_n - \mathcal{K}_i t_n \to 0$, as $n \to \infty$.
$\widetilde{\not{f}}$ is the weak cluster point of the sequence $\{\not{f}_n^{\#}\}$
i.e. $\not{f}_n^{\#} \rightharpoonup \widetilde{\not{f}}$, to show that $\widetilde{\not{f}}$ is also strong cluster point, i.e. $\not{f}_n^{\#} \to \widetilde{\not{f}}$
From, Proposition 3 since the sequence $\{\not{f}_n^{\#}\}$ is bounded, thus there is a subsequence $\{\not{f}_{n\varkappa}^{\#}\}$ of $\{\not{f}_n^{\#}\}$ such that converges to $\hat{\not{f}} \in \mathfrak{D}$.
Also, have $\forall i \in \mathbb{N}$, $\not{f}_n^{\#} - \mathcal{K}_i\, \not{f}_n^{\#} \to 0$, as $n \to \infty$.
Thus $\hat{\not{f}} \in \mathcal{F}(\mathcal{K})$, by condition (DS)
Therefore, $\forall r^{\#} \in \cap_{n=1}^{\infty}\mathcal{F}(\mathcal{K}_n)$

$$\begin{aligned}&\langle (I-\Gamma)\not{f}_{n\varkappa}^{\#},\ \not{f}_{n\varkappa}^{\#} - r^{\#}\rangle \\ &\leq \frac{(1-\varsigma_{n\varkappa})\Upsilon_{n\varkappa}}{\varsigma_{n\varkappa}}\langle \sigma_{n\varkappa},\ \not{f}_{n\varkappa}^{\#} - r^{\#}\rangle - \frac{(1-\varsigma_{n\varkappa})\Upsilon_{n\varkappa}}{\varsigma_{n\varkappa}}\langle (I-\mathcal{P}_{n\varkappa})r^{\#}, \not{f}_{n\varkappa}^{\#} - r^{\#}\rangle \\ &- \frac{\Upsilon_{n\varkappa}}{\varsigma_{n\varkappa}}\sum_{i=1}^{n\varkappa}(\varsigma_{i-1}-\varsigma_i)\langle (I-\mathcal{K}_i)t_{n\varkappa},\ \not{f}_{n\varkappa}^{\#} - \mathcal{P}_{n\varkappa}\not{f}_{n\varkappa}^{\#})\rangle - \frac{1}{\varsigma_{n\varkappa}}\|\wedge_{n\varkappa} - P\mathfrak{D}(\wedge_{n\varkappa})\|\|\wedge_{n\varkappa-1} - \wedge_{n\varkappa}\| \\ &- \|(I-\Gamma)\not{f}_{n\varkappa}^{\#} - (I-\Gamma)t_{n\varkappa}\|\|\not{f}_{n\varkappa}^{\#} - r^{\#}\| \\ &- \Upsilon_n\|\not{f}_{n\varkappa}^{\#} - \mathcal{P}_{n\varkappa}\not{f}_{n\varkappa}^{\#}\|\|\not{f}_{n\varkappa}^{\#} - r^{\#}\| \\ &\leq \frac{(1-\varsigma_{n\varkappa})\Upsilon_{n\varkappa}}{\varsigma_{n\varkappa}}\langle \sigma_{n\varkappa},\ \not{f}_{n\varkappa}^{\#} - r^{\#}\rangle - \frac{(1-\varsigma_{n\varkappa})\Upsilon_{n\varkappa}}{\varsigma_{n\varkappa}}\langle (I-\mathcal{P}_{n\varkappa})r^{\#}, \not{f}_{n\varkappa}^{\#} - r^{\#}\rangle\end{aligned}$$

$$- \frac{\Upsilon_{n\varkappa}}{\varsigma_{n\varkappa}} \sum_{i=1}^{n\varkappa} (\varsigma_{i-1} - \varsigma_i) \langle (I - \mathcal{K}_i) t_{n\varkappa}, \ f^{\#}_{n\varkappa} - \mathcal{P}_{n\varkappa} f^{\#}_{n\varkappa}) \rangle - \frac{1}{\varsigma_{n\varkappa}} \| \Lambda_{n\varkappa} - P\mathfrak{D}(\Lambda_{n\varkappa}) \| \| \Lambda_{n\varkappa - 1} - \Lambda_{n\varkappa} \|$$
$$- \| (I - \Gamma) f^{\#}_{n\varkappa} - (I - \Gamma) f^{\#}_{n\varkappa} \| \| f^{\#}_{n\varkappa} - r^{\#} \|$$
$$- \Upsilon_n \| f^{\#}_{n\varkappa} - \mathcal{P}_{n\varkappa} f^{\#}_{n\varkappa} \| \| f^{\#}_{n\varkappa} - r^{\#} \|$$

when $\varkappa \to \infty$,

$$\langle (I - \Gamma) \hat{f}, \hat{f} - r^{\#} \rangle \leq - \varphi \langle (I - \mathcal{P}) r^{\#}, \hat{f} - r^{\#} \rangle, \ \forall r^{\#} \in \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n),$$

But the VIP in Eq. (13) has the unique solution, then $\omega_w(f^{\#}_n) = \{\tilde{f}\}$.
Hence every weak cluster point of the sequence $\{f^{\#}_n\}$ is also strong cluster point.
Then, $\lim_{n\to\infty} f^{\#}_n = \tilde{f}$, i.e. $f^{\#}_n \to \tilde{f}$.
In Theorem 1, if $\mathcal{P}_n$ is a self mapping of $\mathfrak{D}$ into itself. Then the next result is obtained.

Corollary 1: *Assume that Ξ be real Hilbert space and $\emptyset \neq \mathfrak{D} \subseteq \Xi$* where $\mathfrak{D}$ *is closed and convex. Also, let $\Gamma : \mathfrak{D} \to \Xi$ be be a Λ-contraction, $\mathcal{J} = \{\mathcal{K}_n\}_{n=1}^{\infty} : \mathfrak{D} \to \mathfrak{D}$ be a sequence of almost mean nonexpansive mappings such that $\mathcal{F}(\mathcal{J}) \neq \emptyset$, and $\mathcal{K} : \mathfrak{D} \to \mathfrak{D}$ be a mapping defined by $\mathcal{K} f^{\#} = \lim_{n\to\infty} \mathcal{K}_n f^{\#}, \ \forall f^{\#} \in \mathfrak{D}$. Assume that $\mathcal{F}(\mathcal{K}) = \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$. Assume that $\mathcal{P}_n : \mathfrak{D} \to \mathfrak{D}$ be a sequence of nonexpansive mappings, and $\mathcal{P} : \mathfrak{D} \to \mathfrak{D}$ be a nonexpansive mapping such that $\lim_{n\to\infty} \mathcal{P}_n f^{\#} = \mathcal{P} f^{\#}, \ \forall f^{\#} \in \mathfrak{D}$. For arbitrary $f^{\#}_1 \in \mathfrak{D}$, define a sequence $\{f^{\#}_n\}$ by*

$$t_n = (1 - \Upsilon_n) f^{\#}_n + \Upsilon_n \mathcal{P}_n f^{\#}_n,$$
$$f^{\#}_{n+1} = P\mathfrak{D} \left[\varsigma_n \Gamma(t_n) + \sum_{i=1}^{n} (\varsigma_{i-1} - \varsigma_i) \mathcal{K}_i t_n \right], \ n \in \mathbb{N} \tag{22}$$

where $\varsigma_0 = 1$, $\{\Upsilon_n\}, \{\varsigma_n\}$ are sequences in $(0, 1)$ holding the same conditions of the sequence $\{f^{\#}_n\}$ defined by Eq. (4). Then $\{f^{\#}_n\}$ converges strongly to $\hat{f} \in \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$.
The next corollary is obtained from the result of ([29], Theorem 3.4).

Corollary 2: *Let $\mathfrak{D}$ be a nonempty closed convex subset of a real Hilbert space Ξ. Assume that $\Gamma : \mathfrak{D} \to \Xi$ be be a Λ-contraction, and $\mathcal{J} = \{\mathcal{K}_n\}_{n=1}^{N} : \mathfrak{D} \to \mathfrak{D}$ be a nonexpansive mappings such that $\mathcal{F}(\mathcal{J}) \neq \emptyset$, and $\mathcal{K} : \mathfrak{D} \to \mathfrak{D}$ be a mapping defined by $\mathcal{K} f^{\#} = \lim_{n\to\infty} \mathcal{K}_n f^{\#}, \ \forall f^{\#} \in \mathfrak{D}$. Assume that $\mathcal{F}(\mathcal{K}) = \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$, and $\mathcal{P} : \mathfrak{D} \to \mathfrak{D}$ be a nonexpansive mapping. For arbitrary $f^{\#}_1 \in \mathfrak{D}$, define a sequence $\{f^{\#}_n\}$ by*

$$t_n = (1 - \Upsilon_n) f^{\#}_n + \Upsilon_n \mathcal{P} f^{\#}_n,$$
$$f^{\#}_{n+1} = P\mathfrak{D} [\varsigma_n \Gamma(t_n) + (1 - \varsigma_n) \mathcal{K}_n t_n], \ n \in \mathbb{N} \tag{23}$$

where $\{\Upsilon_n\}, \{\varsigma_n\}$ are asequence in $(0, 1)$ holding the conditions of the Theorem 1. Then $\{f^{\#}_n\}$ converges strongly to $\hat{f} \in \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$.
Once more, the next corollary is obtained from the result of ([28], Theorem 3.1).

Corollary 3: *Assume that Ξ be real Hilbert space and $\emptyset \neq \mathfrak{D} \subseteq \Xi$* where $\mathfrak{D}$ *is closed and convex. Also, let $\Gamma : \mathfrak{D} \to \Xi$ be be a Λ-contraction. Suppose that $\mathcal{J} = \{\mathcal{K}_n\}_{n=1}^{\infty} : \mathfrak{D} \to \mathfrak{D}$ be a sequence of uniformly continuous nearly nonexpansive mappings which in $\mathcal{F}(\mathcal{J}) \neq \emptyset$. Let $\mathcal{K} : \mathfrak{D} \to \mathfrak{D}$ be a mapping defined by $\mathcal{K} f^{\#} = \lim_{n\to\infty} \mathcal{K}_n f^{\#}, \forall f \in \mathfrak{D}$. Assume that $\mathcal{F}(\mathcal{K}) = \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$. Let $\mathcal{P}_n : \mathfrak{D} \to \mathfrak{D}$ be a sequence of nonexpansive mappings, $\mathcal{P} : \mathfrak{D} \to \mathfrak{D}$ be a nonexpansive mapping such that $\lim_{n\to\infty} \mathcal{P}_n f^{\#} = \mathcal{P} f^{\#}, \ \forall f^{\#} \in \mathfrak{D}$.*

For arbitrary $f_1^{\#} \in \mathfrak{D}$, define a sequence $\{f_n^{\#}\}$ by

$$t_n = (1-\Upsilon_n) f_n^{\#} + \Upsilon_n \mathcal{P}_n f_n^{\#},$$
$$f_{n+1}^{\#} = P\mathfrak{D}\left[\varsigma_n \Gamma\left(f_n^{\#}\right) + \sum_{i=1}^{n} (\varsigma_{i-1} - \varsigma_i)\mathcal{K}_i t_n\right],\ n \in \mathbb{N} \tag{24}$$

where $\{\Upsilon_n\}$ is a sequence in $(0,1)$, $\varsigma_0 = 1$, and $\{\varsigma_n\}$ is a strictly decreasing sequence in $(0,1)$ hold the conditions of the Theorem 1. Then $\{f_n^{\#}\}$ converges strongly to $f\hat{} \in \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$.

We relax the result of Theorem 1 as following:

Theorem 2: *Let $\mathfrak{D}$ be a nonempty closed convex subset of a real Hilbert space Ξ. Let $\Gamma : \mathfrak{D} \to \Xi$ be be a Λ -contraction, and $\mathcal{J} = \{\mathcal{K}_n\}_{n=1}^{\infty} : \mathfrak{D} \to \mathfrak{D}$ be a sequence of almost mean nonexpansive mappings such that $\mathcal{F}(\mathcal{J}) \neq \emptyset$. Let $\mathcal{K} : \mathfrak{D} \to \mathfrak{D}$ be a mapping defined by $\mathcal{K}f^{\#} = \lim_{n\to\infty} \mathcal{K}_n f^{\#}$, $\forall f \in \mathfrak{D}$. Assume that $\mathcal{F}(\mathcal{K}) = \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$. Let $\mathcal{P}_n : \mathfrak{D} \to \Xi$ be a sequence of nonexpansive mappings. Suppose that $\mathcal{P} : \mathfrak{D} \to \Xi$ be a nonexpansive mapping such that $\lim_{n\to\infty} \mathcal{P}_n f^{\#} = \mathcal{P}f^{\#}$, $\forall f^{\#} \in \mathfrak{D}$, for arbitrary $f_1^{\#} \in \mathfrak{D}$, define a sequence $\{f_n^{\#}\}$ by*

$$t_n = P\mathfrak{D}\left[(1-\Upsilon_n) f_n^{\#} + \Upsilon_n \mathcal{P}_n f_n^{\#}\right],$$
$$f_{n+1}^{\#} = P\mathfrak{D}\left[\varsigma_n \Gamma(t_n) + \sum_{i=1}^{n} (\varsigma_{i-1} - \varsigma_i)\mathcal{K}_i t_n\right],\ n \in \mathbb{N} \tag{25}$$

where $\varsigma_0 = 1$, $\{\Upsilon_n\}$, $\{\varsigma_n\}$ are sequences in $(0,1)$ holding the next conditions:

(E1) $\lim_{n\to\infty} \varsigma_n = 0$, $\sum_{n=1}^{\infty} \varsigma_n = \infty$;
(E2) $\lim_{n\to\infty} \frac{\Upsilon_n}{\varsigma_n} = 0$;
(E3) $\sum_{n=1}^{\infty} (\varsigma_{n-1} - \varsigma_n) < \infty$, $\sum_{n=1}^{\infty} |\Upsilon_{n-1} - \Upsilon_n| < \infty$.

Then $\{f_n^{\#}\}$ converges strongly to $f\hat{} \in \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$ is the unique solution of the VIP

$$\left\langle (I-\Gamma) f\hat{},\ f\hat{} - r^{\#}\right\rangle \geq 0,\ \forall\, r^{\#} \in \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n). \tag{26}$$

Proof: It's evidenced in[27] the VIP has unique solution.

By a similar controversy of Theorem 1, we can inference that $\{f_n^{\#}\}$ bounded, $\|f_{n+1}^{\#} - f_n^{\#}\| \to 0$, and $\lim_{n\to\infty} \|f_n^{\#} - \mathcal{K}_i f_n^{\#}\| \to 0$, as $n \to \infty$.

Now, to prove that $\lim_{n\to\infty} sup\langle \Gamma(f\hat{}) - f\hat{}, f_n^{\#} - f\hat{}\rangle \leq 0$,

Hence $\{f_n^{\#}\}$ bounded, we can choose a subsequence $\{f_{n_\varkappa}^{\#}\}$ of $\{f_n^{\#}\}$ s.t $f_{n_\varkappa}^{\#} \rightharpoonup \tilde{f}$ and

$$\lim_{n\to\infty} sup\langle \Gamma(f\hat{}) - f\hat{}, f_{n_\varkappa}^{\#} - f\hat{}\rangle = \lim_{\varkappa\to\infty} \langle \Gamma(f\hat{}) - f\hat{}, f_{n_\varkappa}^{\#} - f\hat{}\rangle$$

$\tilde{f} \in \mathcal{F}(\mathcal{K}_n)$, $\forall\, n \geq 1$, i.e. that $\tilde{f} \in \cap_{n=1}^{\infty} \mathcal{F}(\mathcal{K}_n)$, from Proposition 3 and $\lim_{n\to\infty} \|f_n - \mathcal{K}_i f_n\| \to 0$,

Then, $\lim_{\varkappa\to\infty} \langle \Gamma(f\hat{}) - f\hat{}, f_{n_\varkappa}^{\#} - f\hat{}\rangle = \langle \Gamma(f\hat{}) - f\hat{}, \tilde{f} - f\hat{}\rangle \leq 0$.

By Lemma 1

$$\left\langle P\mathfrak{D}\left(\wedge_n\right) - d_n, P\mathfrak{D}\left(\wedge_n\right) - r^{\#}\right\rangle \leq 0$$

Also,

$$\sum_{i=1}^{n} (\varsigma_{i-1} - \varsigma_i)\left\langle \mathcal{K}_i t_n - r^{\#}, f_{n+1}^{\#} - r^{\#}\right\rangle \leq \sum_{i=1}^{n} (\varsigma_{i-1} - \varsigma_i)\, \|\mathcal{K}_i t_n - r^{\#}\| \|f_{n+1}^{\#} - r^{\#}\|$$

$$\leq \sum_{i=1}^{n} (\varsigma_{i-1} - \varsigma_i)\left[\gamma_n \|t_n - r^{\#}\| + \delta_n \|t_n - \mathcal{K}_i r^{\#}\|\right] \|f_{n+1}^{\#} - r^{\#}\|$$

$$\leq \sum_{i=1}^{n} (\varsigma_{i-1} - \varsigma_i) \|t_n - r\| \, \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|$$
$$= (1 - \varsigma_n) \|t_n - r^{\#}\| \, \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|$$
$$\leq (1 - \varsigma_n) \|\mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\| + (1 - \varsigma_n) \Upsilon_n \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\| \, \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\| \tag{27}$$

Therefore,

$$\|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|^2 = \left\langle P\mathfrak{D}\left(\wedge_n\right) - d_n, P\mathfrak{D}\left(\wedge_n\right) - r^{\#}\right\rangle + \left\langle \wedge_n - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle$$
$$\leq \left\langle \wedge_n - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle$$
$$= \varsigma_n \left\langle \Gamma\ (t_n) - \Gamma\ (r^{\#}), \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle + \varsigma_n \left\langle \Gamma\ (r^{\#}) - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle$$
$$+ \sum_{i=1}^{n} (\varsigma_{i-1} - \varsigma_i) \left\langle \mathcal{K}_i t_n - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle$$
$$\leq \varsigma_n \Lambda \|t_n - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\| + \varsigma_n \left\langle \Gamma\ (r^{\#}) - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle + (1 - \varsigma_n) \|\mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|$$
$$+ (1 - \varsigma_n) \Upsilon_n \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|$$
$$= [1 - \varsigma_n (1 - \Lambda)] \|\mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\| + \left[(1 - \varsigma_n) \wedge_n + \varsigma_n \Lambda \Upsilon_n\right] \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|$$
$$+ \varsigma_n \left\langle \Gamma\ (r^{\#}) - r, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle$$
$$\leq \frac{1 - \varsigma_n (1 - \Lambda)}{2} \left[\|\mathfrak{f}_n^{\#} - r^{\#}\|^2 + \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|^2\right] + [(1 - \varsigma_n) \Upsilon_n + \Lambda \varsigma_n \Upsilon_n] \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\|$$
$$\times \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\| + \varsigma_n \left\langle \Gamma\ (r^{\#}) - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle \tag{28}$$
$$\left[1 - \frac{1 - \varsigma_n (1 - \Lambda)}{2}\right] \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|^2$$
$$\leq \frac{1 - \varsigma_n (1 - \Lambda)}{2} \left[\|\mathfrak{f}_n^{\#} - r^{\#}\|^2\right] + [(1 - \varsigma_n) \Upsilon_n + \Lambda \varsigma_n \Upsilon_n] \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|$$
$$+ \varsigma_n \left\langle \Gamma\ (r^{\#}) - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle$$
$$= \left[1 - \frac{2 (1 - \Lambda) \varsigma_n}{1 + (1 - \Lambda) \varsigma_n}\right] \|\mathfrak{f}_n^{\#} - r^{\#}\|^2 + \frac{2 [(1 - \varsigma_n) \Upsilon_n + \Lambda \varsigma_n \Upsilon_n]}{1 + (1 - \Lambda) \varsigma_n} \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|$$
$$+ \frac{2\varsigma_n}{1 + (1 - \Lambda) \varsigma_n} \varsigma_n \left\langle \Gamma\ (r) - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle$$
$$= \left[1 - \frac{2 (1 - \Lambda) \varsigma_n}{1 + (1 - \Lambda) \varsigma_n}\right] \|\mathfrak{f}_n^{\#} - r\|^2 + \frac{2 (1 - \Lambda) \varsigma_n}{1 + (1 - \Lambda) \varsigma_n} \left\{\frac{(1 - \varsigma_n) \Upsilon_n}{(1 - \Lambda) \varsigma_n} \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|\right.$$
$$\left. + \frac{\Lambda \varsigma_n \Upsilon_n}{(1 - \Lambda) \varsigma_n} \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\| + \frac{1}{(1 - \Lambda)} \left\langle \Gamma\ (r^{\#}) - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle\right\}$$

Suppose that $\Delta_n = \frac{2(1-\Lambda)\varsigma_n}{1+(1-\Lambda)\varsigma_n}$,

$$\varepsilon_n = \frac{2 (1 - \Lambda) \varsigma_n}{1 + (1 - \Lambda) \varsigma_n} \left\{\frac{(1 - \varsigma_n) \Upsilon_n}{(1 - \Lambda) \varsigma_n} \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\| + \frac{\Lambda \varsigma_n \Upsilon_n}{(1 - \Lambda) \varsigma_n} \|\mathcal{P}_n \mathfrak{f}_n^{\#} - r^{\#}\| \|\mathfrak{f}_{n+1}^{\#} - r^{\#}\|\right.$$
$$\left. + \frac{1}{(1 - \Lambda)} \left\langle \Gamma\ (r^{\#}) - r^{\#}, \mathfrak{f}_{n+1}^{\#} - r^{\#}\right\rangle\right\}, \ \ \sigma_n = 0$$

$$\lim_{n\to\infty} sup \left\{ \frac{(1-\varsigma_n)\Upsilon_n}{(1-\Lambda)\varsigma_n} \|\mathcal{P}_n f_n^{\#} - r^{\#}\| \|f_{n+1}^{\#} - r^{\#}\| + \frac{\Upsilon_n \varsigma_n \Lambda}{(1-\Lambda)\varsigma_n} \|\mathcal{P}_n f_n^{\#} - r^{\#}\| \|f_{n+1}^{\#} - r^{\#}\| + \frac{1}{(1-\Lambda)} \left\langle \Gamma\ (r^{\#}) - r^{\#}, f_{n+1}^{\#} - r^{\#} \right\rangle \right\} \le 0,$$

Since, $\sum_{n=1}^{\infty} \varsigma_n = \infty$; and $\frac{2(1-\Lambda)\varsigma_n}{1+(1-\Lambda)\varsigma_n} \ge (1-\Lambda)\varsigma_n$,
$\sum_{n=1}^{\infty} \sigma_n = 0$; $\lim_{n\to\infty} sup \frac{\varepsilon_n}{\Delta_n} = 0$, and
Thus, by Lemma 3, $\lim_{n\to\infty} \|f_n^{\#} - r^{\#}\| = 0$.

Remark 1: In Eq. (4), if $\Gamma = \mathbf{0}$, then $f_n^{\#} \to r = P_{\mathcal{F}}\mathbf{0}$. Thus, it follows from Eq. (26)

$$\left\langle f\hat{},\, f\hat{} - r^{\#} \right\rangle \le 0, \forall\, f\hat{} \in \mathcal{F}(\mathcal{K}_n)$$

i.e. $\|f\hat{}\|^2 \le \left\langle f\hat{},\ r^{\#} \right\rangle \le \|f\hat{}\|\|r\|, \forall\, r \in \mathcal{F}(\mathcal{K}_n)$

There'refore, $f\hat{}$ is the unique solution of the quadratic minimization problem

$$f\hat{} = \arg\min_{f \in F} \|r\|^2.$$

Conclusion

The effect of this work betokens that the two iterative schemes satisfying some appropriate conditions have a hierarchically common fixed point. Also the sequence $\{f_n^{\#}\}$ is asymptotic regularity and demiclosedness principle which gives us strong convergence results for two iterative schemes of sequences of almost mean nonexpansive mappings, and it has been indicated that the sequence of nonexpansive mappings is a sequence of almost mean nonexpansive. Through this work when the parameter conditions are relaxed in Theorem 1, other strong convergence results are found which also solutions of (HFPP). The solution of an iterative scheme by the hierarchically a common fixed point by showing the conditions referred to above warranting the convergence of the manner.

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Authors' declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

Authors' contribution statement

S.S.A visualized of the presented idea and supervised the findings of this study. N.S.T checked the analytical methods. All authors discussed the scores and participated to the final manuscript.

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حل مسألة متراجحة التغاير الهرمية للتطبيقات الوسطية اللاتوسيعية تقريبا

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الخلاصة

احدى الطرق الشائعة الواضحة في حل بعض انواع المسائل غير الخطية هي استبدال المسألة الاصلية بمجموعة من من المسائل المنتظمة وكل هذه المسائل المنتظمة لها حل واحد مضبوط. سيتم الحصول على حل معين للمسألة الاصلية كغاية لهذه الحلول الوحيدة للمسائل المنتظمة. تُستخدم هذه الفكرة لتوفير طريقة لنهج النقطة الصامدة الهرمية لحل مسائل متراجحة التغاير (VIPs).في هذا العمل، نعتزم دراسة مخططيين تكراريين جديدين من خلال دراسة تقاربهما القوي مع نقطة صامدة مشتركة لتطبيقات معرفة على مجموعة جزئية $\mathfrak{D}$ غير خالية مغلقة ومحدبة من فضاء هلبرت الحقيقي Ξ. يتم انشاء هذه المخططات التكرارية لمتتابعات من التطبيقات الوسطية اللاتوسيعية تقريبا وتطبيقات لاتوسيعية في ظل بعض الشروط المُحكمة.أولاً، تم إنشاء نتائج تقارب قوية لمخططين تكراريين لثلاثة تطبيقات: الأول Γ : $\mathfrak{D} \longrightarrow \Xi$: هو تطبيق انكماشي، الثاني $\mathcal{P}_n$: $\mathfrak{D} \longrightarrow \Xi$: هو عبارة عن متتابعات من التطبيقات اللاتوسيعية، والثالث $\mathfrak{D}\mathcal{K}_n$: $\mathfrak{D} \longrightarrow \mathfrak{D}$: هو تطبيقات من المتتابعات الوسطية اللاتوسيعية تقريبا.ثانيًا، عندما يتم تخفيف القيود المفروضة على معلمات مخططين تكراريين، فإننا نحصل على نتائج تقارب قوية أخرى والتي تعد أيضًا حلولاً لمسألة النقطة الصامدة الهرمية (HFPP). وأخيراً تم إيجاد حل لمسألة التصغير التربيعية كحالة خاصة وهذا التقارب وحيد.تحتوي نتائجنا على الدراسات السابقة كحالة خاصة ويمكن اعتبارها تنقيح واعادة تحسين للعديد من النتائج المألوفة المقابلة لمسأل متراجحة التغاير الهرمية (HVIP) .

الكلمات المفتاحية : تطبيق انكماشي، مسألة متراجحة التغاير الهرمية، فضاء هلبرت الحقيقي، متتابعة من التطبيقات اللاتوسيعية، تقارب قوي.