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Strong Convergence of a New Iterative Scheme in Generalized (α, β) – Mean Nonexpansive Mappings

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Abstract: In this paper, introduced new accelerated iterative algorithm in generalized (α, β) -mean nonexpansive mappings in Banach spaces and present some results for convergence to fixed point in this mapping, we using new iterative scheme in new style mappings is generalized (α, β) -mean nonexpansive mapping, and transfer the idea of this class of mapping from generalized (α, β) -mean nonexpansive mappings in modular function spaces to Banach spaces.

Keywords: Strong convergence, mean nonexpansive mappings, Banach spaces, iterative scheme, fixed point.

التقارب القوي لمخطط تكراري جديد في تعميم (α, β) متوسط الدوال غير التوسعية

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المستخلص: في هذا البحث، نقدم خوارزميه تكراريه جديده في تعميم $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ متوسط الدوال غير التوسعية في فضائات بناخ ونقدم بعض النتائج للتقارب الى النقطه الصامده في هذه الدوال، نستخدم المخطط التكراري الجديد مع نوع دوال جديد هو تعميم $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ متوسط الدوال غير التوسعية حيث ننقل فكره هذا النمط من الدوال من الفضائات المعياريه الى فضائات بناخ.

الكلمات المفتاحية: تقارب قوي، متوسط الدوال الغير توسعيه، فضائات بناخ، مخطط تكراري، نقطه صامده.

1-Introduction:

If f the self mapping of E and $s \in E$ then s side to be fixed point if $s = f(s)$, Banach in 1922 [1] proved that every contraction mapping defined on a complete metric spaces has unique fixed point, and Banach discussed Picard iterative algorithm this one of the earliest iterative scheme used to approximate the solution of a fixed point problems, researchers have found many new types of iterative schemes that appropriate the type of mapping and spaces. The most prominent of these sequences are Mann [2], Ishikawa [3], Noor [4], Agarwal et al [5], Kadioglu et al [6], Abbas and Nazir [7], Thakur et al [8], Karakaya et al [9], Ullah and Arshad [10] A, and Hassan et al [11]. In general, to solve fixed point problem must be followed approximate methods because from an analytical point of view solving them is almost impossible, Therefore, fixed point has become a thriving field of scientific research, while it is considered the fixed point theory provides very useful tools to solve most of the problems, that have many applications in different fields [12].

Now, Present the iterative algorithm that will deal with in this paper
let $T: E \longrightarrow E$, and E nonempty convex subset of Banach spaces, here, we introduced the sequence $\{x_n\}$ by the algorithm following.

$x_1 \in E$
$h_n = (1-\beta_n)x_n + \beta_n T x_n$
$y_n = T h_n$
$J_n = (1-\alpha_n)y_n + \alpha_n T y_n$
$f_{n+1} = T J_n, n \epsilon N$
where $\{\alpha_n\}$and $\{\beta_n\}$ in (0,1) (1)

2-Prelimineareies

In this section review some important definitions and lemmas that we can use in the results

Definition 2-1 [14]:Let $T: E \longrightarrow E$ a mapping and E is nonempty subset of Banach space said to be nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|$$

Definition 2-2 [15]:Let $T: E \longrightarrow E$ a mapping and E is nonempty subset of Banach space said to be qausi nonexpansive mapping if there exist s fixed point and

$$\|Tx - s\| \leq \|x - s\|$$

Note that: Every nonexpansive mapping with fixed point is qausi nonexpansive mapping but the convers is not true for example.

Example 2-3: Let $T: E \longrightarrow E$ and E is nonempty subset of Banach space the function

$$Tx = \begin{cases} 0 & if\ x \neq 2 \\ 1 & if\ x = 2 \end{cases}$$

T is ρ-quasi nonexpansive mapping but not ρ-nonexpansive since if $x = 1.9, y = 2$ then $\|Tx - Ty\| = \|0 - 1\| = 1 \nleq \|1.9 - 2\| = 0.1$.

Definition 2-4 [14]: Let $T: E \longrightarrow E$ is mapping, a sequence $\{x_n\}$ in E is said to be Fajer monotone if $\|x_{n+1} - s\| \leq \|x_n - s\|$ for all s fixed point.

Lemma 2-5 [16]**:** Let X satisfy uniformly convex Banach spaces and let $\{t_n\}$ in (0,1) be bounded away from 0 and 1, if there exists $m > 0$ such that

$$\lim sup_{n\to\infty} \rho(x_n) \leq m, \lim sup_{n\to\infty} \rho(y_n) \leq m$$

And $lim_{n\to\infty} \rho(t_n x_n + (1-t_n)y_n) = m$, then $\lim_{n\to\infty} \rho(x_n - y_n) = 0$

Lemma 2-6 [17]

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Let $\{\rho_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$and $\{\zeta_n\}_{n=1}^{\infty}$ nonnegative sequence such that

$$\rho_{n+1} \leq (1-\theta_n)\rho_n + \zeta_n$$

Where $\{\theta_n\}$ sequence in $(0,1)$ and $\{\zeta_n\}$ sequence in real number such that

$\sum_{n=1}^{\infty} \theta_n < \infty$ and $\sum_{n=1}^{\infty} \zeta_n < \infty$, then $\lim_{n\longrightarrow\infty} \rho_n$ is exists.

Definition 2-7 [18] **:** A Banach space X is said to be uniformly convex if $\psi_\eta(\epsilon) = \inf\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_\eta, \|x-y\| \geq \epsilon\} > 0$ for all $0 < \varepsilon \leq 2$ say that uniformly convex Banach spaces has power P and $P \geq 1$ there exists constant c such that $\psi_\eta(\epsilon) \geq c\epsilon^P$ for all $0 < \varepsilon \leq 2$.

Definition 2-8 [14]: Let X satisfy uniformly convex Banach spaces, let E be nonempty convex subset of X, let $T: E \longrightarrow E$ said to be satisfy condition (I) if there exsist a nondecreasing function $\phi: [0,\infty) \longrightarrow [0,\infty)$ such that $\phi(0) = 0$ and $\phi(t) > 0$ for all t in $[0,\infty)$ and $\|x - Tx\| \geq \phi(dist(x, F_p(T))$ for all $x \in E$, where $dist(x, F_p(T))$ denotes the distance from x to $F_p(T)$.

3- Main Rustles

Salman.B.B, and Abed.S.S,in (2023) [13] introduced generalized (λ, δ, ρ) -mean nonexpansive mappings in modular function spaces, will study this concept in Banach space

Definition 3-1: Let X satisfy uniformly convex Banach spaces, let E be nonempty convex subset of X, $T: E \longrightarrow E$ is mapping said to be generalized (α, β)-mean nonexpansive mapping if

$$\|Tx - Ty\| \leq \max\{R(x,y), Q(x,y), Z(x,y)\}$$

Where

$$R(x,y) = \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha+\beta))\|x - y\|$$

$$Q(x,y) = \alpha\|x - y\| + \beta\|x - Ty\|$$

$$Z(x,y) = \alpha\|x - y\| + \beta\|Tx - y\|$$

And x, y in E, α, β in $[0,1]$, $\alpha + \beta < 1$**.**

Lemma 3-2: Every generalized (α, β)-mean nonexpansive mapping is qausi nonexpansive mapping.

Proof: let s is fixed point for T to prove T is qausi

$$\|Tx - s\| \leq \max\{R(x,s), Q(x,s), Z(x,s)\}$$

If $R(x,s)$ is maximum

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$$\|Tx - s\| \leq \alpha\|s - Tx\| + \beta\|x - s\| + (1 - (\alpha + \beta))\|x - s\|$$

$$(1 - \alpha)\|Tx - s\| \leq (1 - \alpha)\|x - s\|$$

Then $\|Tx - s\| \leq \|x - s\|$

If $Q(x, s)$ is maximum

$$\|Tx - s\| \leq \alpha\|x - s\| + \beta\|x - Ts\|$$
$$\|Tx - s\| \leq (\alpha + \beta)\|x - s\|$$

Then $\|Tx - s\| \leq \|x - s\|$ since $\alpha + \beta < 1$

Finally, If $Z(x, s)$ is maximum

$$\|Tx - s\| \leq \alpha\|x - s\| + \beta\|Tx - s\|$$

$$(1 - \beta)\|Tx - s\| \leq \alpha\|x - s\|$$

Then $\|Tx - s\| \leq \|x - s\|$ since $\alpha < 1 - \beta$

Theorem 3-3: Let X satisfy uniformly convex Banach spaces, let E be nonempty convex subset of X, let $T: E \longrightarrow E$ be generalized (α, β)-mean nonexpansive mapping and x_n in E define by (1) then $\lim_{n\longrightarrow\infty} \|x_n - s\|$ exists for all s fixed point of T in E.

Proof: T is generalized (α, β)-mean nonexpansive mapping

If $R(f, g)$ is maximum

By definition (3-1), and convexity

$\|x_{n+1} - s\| = \|TJ_n - s\|$

$$\leq \alpha\|s - TJ_n\| + \beta\|J_n - Ts\| + (1 - (\alpha + \beta))\|J_n - s\|$$

$(1 - \alpha)\|x_{n+1} - s\| \leq (1 - \alpha)\|J_n - s\|$

$\|x_{n+1} - s\| \leq \|J_n - s\|$ (2)

Also, $\|J_n - s\| = \|(1 - \alpha_n)y_n + \alpha_n Ty_n - s\|$

$$\leq (1 - \alpha_n)\|y_n - s\| + \alpha_n\|Ty_n - s\|$$

We find $\|Ty_n - s\|$

$$\|Ty_n - s\| \leq \alpha\rho(s - Ty_n) + \beta\|y_n - Ts\| + (1 - (\alpha + \beta))\|y_n - s\|$$
$$\leq \|y_n - s\|$$

Substituting in equation we get

$\|J_n - s\| \leq \|y_n - s\|$ (3)

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By the same way,

$$\|y_n - s\| = \|Th_n - s\|$$

$$\leq \alpha\|s - Th_n\| + \beta\|h_n - Ts\| + (1 - (\alpha + \beta))\|h_n - s\|$$

$$(1 - \alpha)\|y_n - s\| \leq (1 - \alpha)\|h_n - s\|$$

$$\|y_n - s\| \leq \|h_n - s\| \qquad (4)$$

Similarity,

$$\|h_n - s\| = \|(1 - \beta_n)x_n + \beta_n Tx_n - s\|$$

$$\leq (1 - \beta_n)\|x_n - s\| + \beta_n\|Tx_n - s\|$$

We find $\|Tx_n - s\|$

$$\|Tx_n - s\| \leq \alpha\|s - Tx_n\| + \beta\|x_n - Ts\| + (1 - (\alpha + \beta))\|x_n - s\|$$

$$\leq \|x_n - s\|$$

Substituting in equation we get

$$\|h_n - s\| \leq \|x_n - s\| \qquad (5)$$

By (2),(3),(4) and (5) $\|x_{n+1} - s\| \leq \|x_n - s\|$

And by Lemma 2-6 then $\lim_{n\to\infty} \|x_n - s\|$exists

If $Q(f,g)$ is maximum, By Definition (3-1), Lemma 3-2, convexity and $\alpha + \beta < 1$ then

$$\|x_{n+1} - s\| = \|TJ_n - s\|$$

$$\leq \alpha\|J_n - s\| + \beta\|J_n - Ts\|$$

$$\leq (\alpha + \beta)\|J_n - s\|$$

$$\leq \|J_n - s\| \qquad (6)$$

Also, $\|J_n - s\| = \|(1 - \alpha_n)y_n + \alpha_n Ty_n - s\|$

$$\leq (1 - \alpha_n)\|y_n - s\| + \alpha_n\|Ty_n - s\|$$

We find $\|Ty_n - s\|$

$$\|Ty_n - s\| \leq \alpha\|y_n - s\| + \beta\|y_n - Ts\|$$

$$\leq \|y_n - s\|$$

Substituting in equation we get

$$\|J_n - s\| \leq \|y_n - s\| \qquad (7)$$

By the same way,

$$\|y_n - s\| = \|Th_n - s\|$$
$$\leq \alpha\|h_n - s\| + \beta\|h_n - Ts\|$$
$$\leq (\alpha + \beta)\|h_n - s\|$$
$$\leq \|h_n - s\|$$
$$\|y_n - s\| \leq \|h_n - s\| \tag{8}$$

Similarity,

$$\|h_n - s\| = \|(1 - \beta_n)y_n + \beta_n Tx_n - s\|$$
$$\leq (1 - \beta_n)\|x_n - s\| + \beta_n\|Tx_n - s\|$$

We find $\|Tx_n - s\|$

$$\|Tx_n - s\| \leq \alpha\|x_n - s\| + \beta\|x_n - Ts\|$$
$$\leq \|x_n - s\|$$

Substituting in equation we get

$$\|h_n - s\| \leq \|x_n - s\| \tag{9}$$

By (6),(7),(8) and (9) $\|x_{n+1} - s\| \leq \|x_n - s\|$

And by Lemma 2-6 then $\lim_{n\to\infty}\|x_n - s\|$ exists

If $Z(f,g)$ is maximum, By Definition (3-1), Lemma 3-2, convexity and $\alpha < 1 - \beta$ then

$$\|x_{n+1} - s\| = \|TJ_n - s\|$$
$$\leq \alpha\|J_n - s\| + \beta\|TJ_n - s\|$$
$$(1 - \beta)\|x_{n+1} - s\| \leq \alpha\|J_n - s\|$$
$$\|x_{n+1} - s\| \leq \|J_n - s\| \tag{10}$$

Also, $\|J_n - s\| = \|(1 - \alpha_n)y_n + \alpha_n Ty_n - s\|$

$$\leq (1 - \alpha_n)\|y_n - s\| + \alpha_n\|Ty_n - s\|$$

We find $\|Ty_n - s\|$

$$\|Ty_n - s\| \leq \alpha\|y_n - s\| + \beta\|Ty_n - s\|$$
$$(1 - \beta)\|Ty_n - s\| \leq \beta\|y_n - s\|$$
$$\|Ty_n - s\| \leq \|y_n - s\|$$

Substituting in equation we get

$$\|J_n - s\| \leq \|y_n - s\| \tag{11}$$

By the same way,

$$\|y_n - s\| = \|Th_n - s\|$$
$$\leq \alpha\|h_n - s\| + \beta\|Th_n - s\|$$
$$(1-\beta)\|y_n - s\| \leq \beta\|h_n - s\|$$
$$\|y_n - s\| \leq \|h_n - s\| \qquad (12)$$

Similarity,

$$\|h_n - s\| = \|(1-\beta_n)x_n + \beta_n Tx_n - s\|$$
$$\leq (1-\beta_n)\|x_n - s\| + \beta_n\|Tx_n - s\|$$

We find $\|Tx_n - s\|$

$$\|Tx_n - s\| \leq \alpha\|x_n - s\| + \beta\|Tx_n - s\|$$
$$\leq \|x_n - s\|$$

Substituting in equation we get

$$\|h_n - s\| \leq \|x_n - s\| \qquad (13)$$

By (10),(11),(12) and (13) $\|x_{n+1} - s\| \leq \|x_n - s\|$

Also, Lemma 2-6 implies that $\lim_{n\to\infty}\|x_n - s\|$ exists

Theorem 3-4: Let X satisfy uniformly convex Banach spaces, let E be nonempty convex subset of X, let $T: E \longrightarrow E$ be generalized (α,β)-mean nonexpansive mapping and x_n in E define by (1) is Fajer monotone.

Proof: By Theorem 3-3 $\|x_{n+1} - s\| \leq \|x_n - s\|$

And by Definition 2-4, x_n in E define by (1) is Fajer monotone.

Theorem 3-5: Let X satisfy uniformly convex Banach spaces, let E be nonempty convex subset of X, let $T: E \longrightarrow E$ be generalized (α,β)-mean nonexpansive mapping and x_n in E define by (1) then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof: By Theorem 3-3 $\lim_{n\to\infty}\|x_n - s\|$ exists

Let $\lim_{n\to\infty}\|x_n - s\| = k$ such that $k \geq 0$ (14)

By (5),(6) and (13)

$$\|h_n - s\| \leq \|x_n - s\| = k \qquad (15)$$
$$\|x_{n+1} - s\| = k = \lim_{n\to\infty}\|x_n - s\|$$

By (2),(6) and (10)

$\|x_{n+1} - s\| = \|TJ_n - s\| \leq \|J_n - s\|$

By (4),(8) and (12)

$\|J_n - s\| \leq \|y_n - s\|$

By (4),(8) and (12)

$\|y_n - s\| \leq \|h_n - s\|$

Then $\|x_{n+1} - s\| \leq \|h_n - s\| \Rightarrow k \leq \|h_n - s\|$ (16)

By (15) and (16) $\lim\limits_{n\to\infty} \|h_n - s\| = k$ (17)

T is generalized (α, β)-mean nonexpansive mapping

If $R(f, g)$ is maximum

$$\|Tx_n - s\| \leq \alpha\|s - Tx_n\| + \beta\|x_n - Ts\| + (1 - (\alpha + \beta))\|x_n - s\|$$
$$\leq \|x_n - s\| \tag{18}$$

If $Q(f, g)$ is maximum

$$\|Tx_n - s\| \leq \alpha\|x_n - s\| + \beta\|x_n - Ts\|$$
$$\leq \|x_n - s\| \tag{19}$$

If $Z(f, g)$ is maximum

$$\|Tx_n - s\| \leq \alpha\|x_n - s\| + \beta\|Tx_n - s\|$$
$$\leq \|x_n - s\| \tag{20}$$

By (18),(19) and (20)

$\|Tx_n - s\| \leq \|x_n - s\|$

$\lim\limits_{n\to\infty} \|Tx_n - s\| \leq \lim\limits_{n\to\infty} \|x_n - s\|$

So, $\lim\limits_{n\to\infty} \|Tx_n - s\| \leq k$ (21)

Since $\lim\limits_{n\to\infty} \|h_n - s\| = k$ then

$\lim\limits_{n\to\infty} \|(1 - \alpha_n)x_n + \alpha_n Tx_n - s\| = k$

$\lim\limits_{n\to\infty} \|(1 - \alpha_n)x_n + \alpha_n Tx_n - s\| = k$

$\lim\limits_{n\to\infty} \|(1 - \alpha_n)(x_n - s) + \alpha_n(Tx_n - s)\| = k$ (22)

By (14),(21) , (22) and by using Lemma 2-5 then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$..

Theorem 3-6: Let X satisfy uniformly convex Banach spaces, let E be nonempty convex subset of X, let $T: E \longrightarrow E$ be generalized (α, β)-mean nonexpansive mapping and x_n in E define by (1), x_0 is unique fixed point in T then x_n ρ-strongly convergence to fixed point of T in E.

Proof: T is generalized (α, β)-mean nonexpansive mapping

If $R(f, g)$ is maximum

By definition (3-1), Lemma 3-2, convexity

$$\|x_{n+1} - x_0\| = \|TJ_n - x_0\|$$

$$\leq \alpha\|x_0 - TJ_n\| + \beta\|J_n - Tx_0\| + (1 - (\alpha + \beta))\|J_n - x_0\|$$

$$(1 - \alpha)\|x_{n+1} - x_0\| \leq (1 - \alpha)\|J_n - x_0\|$$

$$\|x_{n+1} - x_0\| \leq \|J_n - x_0\| \tag{23}$$

Also, $\|J_n - x_0\| = \|(1 - \alpha_n)y_n + \alpha_n Ty_n - x_0\|$

$$\leq (1 - \alpha_n)\|y_n - x_0\| + \alpha_n\|Ty_n - x_0\|$$

We find $\|Ty_n - x_0\|$

$$\|Ty_n - x_0\| \leq \alpha\|x_0 - Ty_n\| + \beta\|y_n - Tx_0\| + (1 - (\alpha + \beta))\|y_n - x_0\|$$

$$\leq \|y_n - x_0\|$$

Substituting in equation we get

$$\|J_n - x_0\| \leq \|y_n - x_0\| \tag{24}$$

By the same way,

$$\|y_n - x_0\| = \|Th_n - x_0\|$$

$$\leq \alpha\|x_0 - Th_n\| + \beta\|h_n - Tx_0\| + (1 - (\alpha + \beta))\|h_n - x_0\|$$

$$(1 - \alpha)\|y_n - x_0\| \leq (1 - \alpha)\|h_n - x_0\|$$

$$\|y_n - x_0\| \leq \|h_n - x_0\| \tag{25}$$

Similarity,

$$\|h_n - x_0\| = \|(1 - \beta_n)y_n + \beta_n Tx_n - x_0\|$$

$$\leq (1 - \beta_n)\|x_n - x_0\| + \beta_n\|Tx_n - x_0\|$$

We find $\|Tx_n - x_0\|$

$$\big\|\|Tx_n - x_0\|\big\| \leq \alpha\|x_0 - Tx_n\| + \beta\|x_n - Tx_0\| + (1 - (\alpha + \beta))\|x_n - x_0\|$$

$$\leq \|x_n - x_0\|$$

Substituting in equation we get

$$\|h_n - x_0\| \leq \|x_n - x_0\| \tag{26}$$

By (23),(24),(25) and (26)

$$\|x_{n+1} - s\| \leq \|x_n - s\|$$

By the same way if $Q(f,g)$ or $Z(f,g)$ is maximum

Furthermore it

$$\|x_n - x_0\| \leq \|x_{n-1} - x_0\|$$

Since $\|x_1 - x_0\| \leq \|x_0 - x_0\|$, so ,$\|x_n - x_0\| \leq \|x_0 - x_0\|$

$\|x_n - x_0\| \leq \|0\| = 0$, then $x_n \longrightarrow x_0$

Theorem 3-7: Let X satisfy uniformly convex Banach spaces, let E be nonempty convex subset of X, let $T: E \longrightarrow E$ be generalized (α, β)-mean nonexpansive mapping and satisfy (I) condition, x_n in E define by (1), then x_n strongly convergence to fixed point of T in E.

Proof: : By Theorem 3-3 $lim_{n\to\infty}\|x_n - s\|$ exists for all s is fixed point, if $lim_{n\to\infty}\|x_n - s\| = 0$, nothing to prove,

if $lim_{n\to\infty}\|x_n - s\| = k, k \geq 0$

Since $\|x_{n+1} - s\| \leq \|x_n - s\|$, then $dist\big(x_{n+1}, F_p(T)\big) \leq dist_\rho\big(x_n, F_p(T)\big)$

So $lim_{n\to\infty} dist_\rho\big(x_n, F_p(T)\big)$ exists, by applying condition (I) and Theorem 3-3

$$lim_{n\to\infty}\emptyset(dist\left(x_n, F_p(T)\right) \leq lim_{n\to\infty}\ dist\|x_n - Tx_n\| = 0$$

Since $\emptyset(0) = 0$, hence $lim_{n\to\infty} dist\big(x_n, F_p(T)\big) = 0$

By Theorem 3-3 $lim_{n\to\infty}\|x_n - s\|$ exists, then $lim_{n\to\infty}\big\|x_n - F_p(T)\big\|$ exists and $s \in F_p(T)$

Suppose that x_{n_k} subsequence of x_n, and u_k sequence in $F_p(T)$

$$\left\|x_{n_k} - u_k\right\| \leq \frac{1}{2^k} \quad \text{since } \lim inf_{n\to\infty}\ dist\big(x_n, F_p(T)\big) = 0$$

$$\left\|x_{n_{+1}} - u_k\right\| \leq \|x_n - u_k\| \leq \frac{1}{2^k}$$

$$\|u_{k+1} - u_k\| \leq \|u_{k+1} - x_{n+1}\| + \|x_{n+1} - u_k\|$$

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$$\leq \frac{1}{2^{k+1}} + \frac{1}{2^k}$$

$$\leq \frac{1}{2^{k-1}} \tag{5-26}$$

$\|u_{k+1} - u_k\| \longrightarrow 0$ as $k \longrightarrow \infty$

u_k is ρ-Cauchy, $F_p(T)$, So, u_k is ρ-converge to $F_p(T)$, then $\|u_k - s\| \longrightarrow 0$

Now,

$\|x_{n_k} - s\| \leq \|x_{n_k} - u_k\| + \|u_k - s\|$, hence, x_n converge to fixed point s in $F_p(T)$.

4- Conclusion

In research paper, we proved that the iterative scheme that presented in equation (1) convergence to the fixed point with the generalized (α, β)-mean nonexpansive mapping in Banach spaces, and proved this through the theorems above. Suggest that to reader work on the iteration that presented in equation (1) with other class of mapping or it is possible to work with other spaces, Finally, it is possible to use style mapping in definition 3-1 with onther iteration scheme and prove it is convergence,

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