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RESEARCH ARTICLE

A Study of the Atangana Baleanu Fractional Differential Equation with an Application in Gas Dynamic Problem

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ABSTRACT

This article presents an analytical approximation for the nonlinearity of the fractional order gas dynamics equation in both homogeneous and nonhomogeneous cases, with a particular emphasis on shock fronts. The approach utilizes the fractional differentiation operator proposed by Atangana and Baleanu. By employing this operator, an approximate solution for the fractional order gas dynamic equations in scenarios involving shock fronts is derived. The solution process primarily involves an iterative procedure that leverages the Laplace transform for numerical computations. The use of the Laplace transform minimizes round-off errors, resulting in a solution that is both precise and straightforward to implement. The method's accuracy and reliability were validated using specific criteria of existence and uniqueness. Additionally, a table is included to demonstrate the method's effectiveness and capability, showing the absolute errors for particular values. Graphical illustrations are also provided to depict the solution's behavior and variations across different values. These visualizations aid in understanding the dynamics and complexities of the solutions obtained using the Atangana-Baleanu operator. Overall, the method proves to be a robust and efficient tool for solving fractional order gas dynamic equations, offering significant precision and ease of application.

Keywords: Atangana-Baleanu operator, Fractional differential equations, Fractional gas dynamic equations, Iterative Laplace transform method, Numerical solutions

Introduction

Gas dynamic equations encompass mathematical formulations that define the fundamental physical principles of conservation of energy, preservation of mass, and preservation of momentum. These nonlinear fractional-order equations find application in scenarios involving shock fronts, intricate fractions, and connectivity disruptions. Within the realm of fluid dynamics, gas dynamics delves into the examination of the movement of gas and its effects, based on the fundamentals of fluid dynamics of fluids. In 2023, Alexander Zlotnik[1] introduced a finite-difference method for addressing linearized two-dimensional and three-dimensional gas dynamic equations, incorporating kinetic-type regularization. The study of gas flow

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phenomena covers a range of scenarios, including choked flows within nozzles and pipelines, the movement of gas propellants in rocket engines, the thermal impacts on vehicles re-entering the atmosphere, and the formation of shock waves around aircraft. According to Kuzenov et al., the use of adaptive composite block-structured grids is crucial for accurately calculating the gas-dynamic characteristics of an aircraft moving through a gas environment, underscoring the significance of precise simulations in this area of research.[2]

Fractional calculus holds significant significance, primarily owing to its crucial contributions across diverse domains. It has gained prominence over integral calculus in shaping numerous everyday models and facilitating their study. Over the years, various methods have been established by scholars and scientists to attain convergent numerical solutions for fractional partial differential equations with nonlinearity. Recent methodologies encompass analytical and numerical solutions, including the Crank–Nicholson finite difference scheme method,[3] Homotopy analysis method, finite difference method, Sumudu Transform, and various iterative approaches.[4–6] Lately, researchers have utilized the studies conducted by Biazar and Eslami,[7] to derive solutions for equations related to time-fractional gas dynamics in both homogenous and non-homogenous cases. Additionally, Rasulov et al.[8] employed the finite difference technique to ascertain solutions for nonlinear equations in gas dynamic scenarios characterized by discontinuous functions. In their study,[9] Prakash and Kumar immersed themselves in exploring a computational approach to tackle time-fractional gas dynamic equations. The exploration of efficient methods to acquire solutions for gas dynamics equations with fractional orders is undertaken by Iqbal et al. in,[10] while Mohamed[11] introduced a novel iterative method for dealing with coupled Burger's and fractional gas dynamics equations. The realm of fractional differential equations encompasses various operators, including the Caputo operator, and Caputo-Fabrizio operator,[12–14] each characterized by power law kernels, exponential kernels, and singularities. However, these operators exhibit limitations when it comes to modeling certain physical problems. To surmount this challenge, Atangana and Baleanu recently proposed a different operator utilizing the Mittag-Leffler function characterized by a nonlocal and non-singular kernel, which is recognized as the Atangana-Baleanu operator.[15] The aim of this study is focused on using the iterative approach of the Laplace transform methodology.

Future research could explore extending these models to more complex nonlinear systems, where fractional dynamics play a crucial role. Investigating numerical stability and accuracy for higher-order fractional differential equations is another promising direction. Moreover, applying the fractional-order models to real-world scenarios, such as gas dynamics under extreme conditions (e.g., in propulsion systems or re-entry vehicles), can provide valuable insights. The exploration of newer operators, such as the Atangana-Baleanu operator, for modelling intricate physical phenomena also presents an avenue for further study.

The structure of the document is as follows: Section 2 covers an outline is presented that summarizes key findings in the field of fractional calculus. Section 3 outlines the operational guidelines for the iterative Laplace transform technique. Within Section 4, criteria regarding the presence and singular nature of solutions for equations of gas dynamics with fractional temporal derivatives, both in homogeneous and non-homogeneous forms are expounded upon. Section 5 showcases numerical simulations alongside tables and visual representations of the derived solutions. Ultimately, our conclusions are presented in Section 6.

Preliminary

Definition 1: The definitions of the Fractional derivatives using the Atangana-Baleanu operator in the Caputo sense of order μ, are as follows:

$$ {}^{ABC}_{\ a}D^{\mu}_{\xi} f(\xi) = \frac{B(\mu)}{1-\mu} \int_a^{\xi} \frac{df(t)}{dt} E_{\mu}\left[-\frac{\mu}{1-\mu} (\xi - t)^{\mu} \right] dt. $$

where $B(\mu) = (1-\mu) + \frac{\mu}{\Gamma(\mu)}$ is a normalization function, and $E_{\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k+1)}$ is the Mittag-Leffler function.

Definition 2: Atangana-Baleanu fractional integral with order μ is defined as follows:

$$ {}^{AB}I^{\mu}_{\xi} f(\xi) = \frac{1-\mu}{B(\mu)} f(\xi) + \frac{\mu}{B(\mu)\Gamma(\mu)} \int_a^{\xi} f(t)(\xi - t)^{\mu-1} dt. $$

When $f(\xi)$ takes on a constant value, the integral will result in an outcome of zero.

Definition 3: Laplace Transform of function with Atangana-Baleanu fractional derivative has the following definition:[16]

$$L\left\{{}^{AB}D_t^{\mu} f(\xi,t)\right\} = \frac{s^{\mu} L\left\{f(\xi,t)\right\} - s^{\mu-1} f(\xi,0)}{s^{\mu}(1-\mu)+\mu}.$$

Core concept of the method of Laplace transform iteration

To predict solutions for gas dynamic equations with fractional time derivatives, both in homogeneous and non-homogeneous forms that involve the Atangana-Baleanu operator. The equations are presented below:

$$\frac{{}^{ABC}\partial^{\mu}\psi}{\partial t^{\mu}} + \psi\frac{\partial\psi}{\partial\xi} - \psi(1-\psi)\log\lambda = 0 \qquad \lambda > 0, \quad \mu \in (0,1), \tag{1}$$

including the initial condition $\psi(\xi,\ 0) = \lambda^{-\xi}$ and

$$\frac{{}^{ABC}\partial^{\mu}\psi}{\partial t^{\mu}} + \psi\frac{\partial\psi}{\partial\xi} + (1+t)^2\psi^2 = \xi^2 \quad \mu \in (0,1), \tag{2}$$

including the initial condition $\psi(\xi,0) = \xi$.
where ψ is the function of independent variables ξ and t.

In this context, the foundational steps of the method presented by Daftardar-Gejji and Jafari[17] are demonstrated. This is achieved by investigating a general differential equation of fractional order that incorporates the Atangana-Baleanu operator.

$${}^{ABC}_{\ a}D_{\xi}^{\mu}\psi(\xi,t) + \mathcal{P}\psi(\xi,t) + \mathcal{Q}\psi(\xi,t) = f(\xi,t), \tag{3}$$

including the initial state

$$\psi(\xi,0) = u(\xi,t). \tag{4}$$

Here, $\mathcal{P}$ and $\mathcal{Q}$ represent linear and non-linear operators, respectively, defined on an appropriate function space, such as $L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ represents the spatial domain. The term $f(\xi,t)$ denotes the source function.

By employing the Laplace transform to Eq. (3), the result is achieved...

$$L\{\psi(\xi,t)\} - \frac{1}{s}\psi(\xi,\ 0) + \frac{1}{B(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)\left(L\{\mathcal{P}\psi(\xi,t)\} + L\{\mathcal{Q}\psi(\xi,t)\} - L\left\{f(\xi,t)\right\}\right) = 0. \tag{5}$$

After regrouping terms, the result

$$L\{\psi(\xi,t)\} = \frac{1}{s}u(\xi,t) - \frac{1}{B(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)\left(L\{\mathcal{P}\psi(\xi,t)\} + L\{\mathcal{Q}\psi(\xi,t)\} - L\left\{f(\xi,t)\right\}\right). \tag{6}$$

Moreover, through the application of the Laplace transform's inverse to Eq. (6), the point is reached where

$$\psi(\xi,t) = u(\xi,t) + L^{-1}\left\{\frac{-1}{B(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)\left(L\{\mathcal{P}\psi(\xi,t)\} + L\{\mathcal{Q}\psi(\xi,t)\} - L\left\{f(\xi,t)\right\}\right)\right\}. \tag{7}$$

Subsequently, the iterative approach is employed described in[15] to derive the series solution as presented below:

$$\psi(\xi,t) = \sum_{j=0}^{\infty}\psi_j(\xi,t). \tag{8}$$

As $\mathcal{P}$ is linear,

$$\mathcal{P}\left(\sum_{j=0}^{\infty}\psi_j(\xi,t)\right)=\sum_{j=0}^{\infty}\mathcal{P}\left(\psi_j(\xi,t)\right). \quad (9)$$

The nonlinear operator $\mathcal{Q}$ is decomposed as...

$$\mathcal{Q}\left(\psi_j(\xi,t)\right)=\mathcal{Q}\left(\psi_0(\xi,t)\right)+\sum_{j=0}^{\infty}\left\{\mathcal{Q}\left(\sum_{i=0}^{j}\psi_i(\xi,t)\right)-\mathcal{Q}\left(\sum_{i=0}^{j-1}\psi_i(\xi,t)\right)\right\} \quad (10)$$

$$=\sum_{i=0}^{\infty}M_i. \quad (11)$$

Where $M_0=\mathcal{Q}(\psi_0(\xi,t))$ and $M_i=\{\mathcal{Q}(\sum_{j=0}^{i}\psi_j(\xi,t))-\mathcal{Q}(\sum_{j=0}^{i-1}\psi_j(\xi,t))\},\quad 1\le i.$
By using Eqs. (8) to (10), the Eq. (3) transforms to

$$\psi(\xi,t)=u(\xi,t)+L^{-1}\left\{\frac{-1}{B(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)\left[\begin{array}{l}L\left\{\sum_{j=0}^{\infty}\mathcal{P}\left(\psi_j(\xi,t)\right)\right\}\\+L\left\{\mathcal{Q}\left(\psi_0(\xi,t)\right)+\sum_{j=1}^{\infty}\left\{\mathcal{Q}\left(\sum_{i=0}^{j}\psi_j(\xi,t)\right)-\mathcal{Q}\left(\sum_{i=0}^{j-1}\psi_i(\xi,t)\right)\right\}\right\}\\-L\left\{f(\xi,t)\right\}\end{array}\right]\right\}. \quad (12)$$

Furthermore, let's investigate the subsequent recursive relationship.

$$\psi_0(\xi,t)=u(\xi,t), \quad (13)$$

$$\psi_1(\xi,t)=L^{-1}\left\{\frac{-1}{B(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)\left(L\{\mathcal{P}(\psi_0(\xi,t))\}+L\{\mathcal{Q}(\psi_0(\xi,t))\}\right)\right\}, \quad (14)$$

$$\vdots$$

$$\psi_{n+1}(\xi,t)=L^{-1}\left\{\frac{-1}{B(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)\left(L\{\mathcal{P}(\psi_n(\xi,t))\}+L\left\{\mathcal{Q}\left(\sum_{i=0}^{n}\psi_i(\xi,t)\right)-\mathcal{Q}\left(\sum_{i=0}^{n-1}\psi_i(\xi,t)\right)\right\}\right)\right\}. \quad (15)$$

The solution that approximates the n-term is provided by

$$\psi=\psi_0+\psi_1+\psi_2\ldots\ \ldots\ \ldots+\psi_{n-1}. \quad (16)$$

Existence and Uniqueness condition for solutions to fractional gas dynamics equation

The presence of solutions to the fractional gas dynamics equation with homogenous characteristics:

Within this segment, conditions have been established for both scenarios the existence and uniqueness of solutions to the gas dynamics equation by employing the fixed-point theorem. [18] Let's examine the gas dynamics equation with a fractional time derivative Eq. (1) employing the operator for fractional derivatives following the Atangana-Baleanu approach, as depicted below:

$$^{ABC}D^{\mu}\psi(\xi,t)=\mathcal{W}(\xi,t,\psi(\xi,t))=-\psi\frac{\partial\psi}{\partial\xi}+\psi(1-\psi)\,log\lambda. \quad (17)$$

Where $^{ABC}D^{\mu}$ signifies the Atangana-Baleanu-Caputo (ABC) type of fractional operator having fractional order μ; where $0<\mu<1$, subjected to the given initial constraints $\psi_0(\xi,t)=\psi(\xi,0)$.

By utilizing the ABC fractional integral, one can convert Eq. (17) into a Volterra-type integral equation. This can be expressed by referencing Definition 2, as illustrated below:

$$\psi(\xi,t)-\psi(\xi,0)=\frac{1-\mu}{M(\mu)}\mathcal{W}(\xi,t,\psi)+\frac{\mu}{M(\mu)\Gamma(\mu)}\int_a^t \mathcal{W}(\xi,\tau,\psi)(t-\tau)^{\mu-1}d\tau. \tag{18}$$

Theorem 1: *The kernel denoted as $\mathcal{W}(\xi,t,\psi)$ in Eq. (17) satisfy the conditions for Lipschitz continuity and contraction, given that the specified condition is fulfilled:*

$$0\leq\left(\frac{\eta}{2}(\varphi_1+\varphi_2)+log\lambda(1+\varphi_1+\varphi_2)\right)<1.$$

Proof: Consider

$$\mathcal{W}(\xi,t,\psi(\xi,t))=-\psi\frac{\partial\psi}{\partial\xi}+\psi(1-\psi)log\lambda.$$

Suppose there are two functions, ψ_1 and ψ_2. As a result, the following is obtained

$$\begin{aligned}
&\|\mathcal{W}(\xi,t,\psi_1(\xi,t))-\mathcal{W}(\xi,t,\psi_2(\xi,t))\|\\
&=\left\|-\psi_1(\xi,t)\frac{\partial\psi_1(\xi,t)}{\partial\xi}+\psi_2(\xi,t)\frac{\partial\psi_2(\xi,t)}{\partial\xi}+log\lambda(\psi_1(\xi,t)-\psi_2(\xi,t))+log\lambda\left(-\psi_1^2(\xi,t)+\psi_2^2(\xi,t)\right)\right\|\\
&\leq\frac{1}{2}\left\|\frac{\partial}{\partial\xi}\left[\psi_1^2(\xi,t)-\psi_2^2(\xi,t)\right]\right\|+log\lambda\|\psi_1(\xi,t)-\psi_2(\xi,t)\|\\
&\quad+log\lambda\|(\psi_1(\xi,t)+\psi_2(\xi,t))(\psi_1(\xi,t)-\psi_2(\xi,t))\|\\
&\leq\frac{\eta}{2}\|\psi_1^2(\xi,t)-\psi_2^2(\xi,t)\|+log\lambda\|\psi_1(\xi,t)-\psi_2(\xi,t)\|+log\lambda(\varphi_1+\varphi_2)\|\psi_1(\xi,t)-\psi_2(\xi,t)\|\\
&\leq\left(\frac{\eta}{2}(\varphi_1+\varphi_2)+log\lambda(1+\varphi_1+\varphi_2)\right)\|\psi_1(\xi,t)-\psi_2(\xi,t)\|\\
&\leq\Delta\|\psi_1(\xi,t)-\psi_2(\xi,t)\|.
\end{aligned}$$

Where $\eta=\frac{\partial}{\partial\xi}$ is the differential operator. ψ_1 and ψ_2 are bounded functions, as a result $\|\psi_1\|\leq\varphi_1$, $\|\psi_2\|\leq\varphi_2$ is obtained.

Moreover, $\Delta=(\frac{\eta}{2}(\varphi_1+\varphi_2)+log\lambda(1+\varphi_1+\varphi_2))$. Hence, it is derived that,

$$\|\mathcal{W}(\xi,t,\psi_1(\xi,t))-\mathcal{W}(\xi,t,\psi_2(\xi,t))\|\leq\Delta\|\psi_1(\xi,t)-\psi_2(\xi,t)\|. \tag{19}$$

Consequently, the Lipschitz condition is derived for $\mathcal{W}$. Additionally, it becomes evident that If $0\leq(\frac{\eta}{2}(\varphi_1+\varphi_2)+log\lambda(1+\varphi_1+\varphi_2))<1$ then this suggests the occurrence of contraction.
The Eq. (18) is described in the following manner,

$$\psi_n(\xi,t)=\frac{1-\mu}{M(\mu)}\mathcal{W}(\xi,t,\psi_{n-1})+\frac{\mu}{M(\mu)\Gamma(\mu)}\int_a^t\mathcal{W}(\xi,\tau,\psi_{n-1})(t-\tau)^{\mu-1}d\tau. \tag{20}$$

Next, the following expression gives the difference between the successive iterative terms,

$$\begin{aligned}
\pi_n(\xi,t)=\psi_n(\xi,t)-\psi_{n-1}(\xi,t)=&\frac{1-\mu}{M(\mu)}(\mathcal{W}(\xi,t,\psi_{n-1})-\mathcal{W}(\xi,t,\psi_{n-2}))\\
&+\frac{\mu}{M(\mu)\Gamma(\mu)}\int_a^t(\mathcal{W}(\xi,\tau,\psi_{n-1})-\mathcal{W}(\xi,t,\psi_{n-2}))(t-\tau)^{\mu-1}d\tau.
\end{aligned} \tag{21}$$

Note

$$\psi_n(\xi,t)=\sum_{i=1}^{n}\pi_i(\xi,t). \tag{22}$$

Next, applying Eqs. (20) and (21) yields:

$$\|\pi_n(\xi,t)\|\le\frac{1-\mu}{M(\mu)}\Delta\|\pi_{n-1}(\xi,t)\|+\frac{\mu}{M(\mu)\Gamma(\mu)}\Delta\int_0^t\|\pi_{n-1}(\xi,\tau)\|d\tau. \tag{23}$$

This is the required result of the theorem.

Theorem 2: *The existence and uniqueness of a solution for Eq. (17) are guaranteed, provided a value t_0 can be identified that fulfills the condition:*

$$\frac{1-\mu}{M(\mu)}\Delta+\frac{\mu t_0}{M(\mu)\Gamma(\mu)}\Delta<1.$$

Proof: If the function $\psi(\xi,t)$, which is constrained in its behavior, satisfies the Lipschitz criterion. By employing Eq. (21) and Eq. (23), the resulting equation

$$\|\pi_n(\xi,t)\|\le\|\psi_n(\xi,t)\|\left[\frac{1-\mu}{M(\mu)}\Delta+\frac{\mu t_0}{M(\mu)\Gamma(\mu)}\Delta\right]^n. \tag{24}$$

This demonstrates the effectiveness of the solution, and furthermore, establishes the validity of the acquired result. Additionally, the solution to Eq. (24) is verified to correspond with the solution provided in Eq. (17). To accomplish this, take into consideration,

$$\psi(\xi,t)-\psi(\xi,0)=\psi_n(\xi,t)-w_n(\xi,t). \tag{25}$$

Where the terms of the series solution are denoted as $w_n(\xi,t)$. Our objective is to demonstrate that these terms diminish toward zero as approach infinity, that is, $\|w_\infty(\xi,t)\|\to 0$.

$$\begin{aligned}\|w_n(\xi,t)\|&=\left\|\frac{1-\mu}{M(\mu)}(\mathcal{W}(\xi,t,\psi)-\mathcal{W}(\xi,t,\psi_{n-1}))\right.\\&\left.\quad+\frac{\mu}{M(\mu)\Gamma(\mu)}\int_0^t(\mathcal{W}(\xi,\tau,\psi)-\mathcal{W}(\xi,\tau,\psi_{n-1}))(t-\tau)^{\mu-1}d\tau\right\|\\&\le\frac{1-\mu}{M(\mu)}\|(\mathcal{W}(\xi,t,\psi)-\mathcal{W}(\xi,t,\psi_{n-1}))\|\\&\quad+\frac{\mu}{M(\mu)\Gamma(\mu)}\int_0^t\|(\mathcal{W}(\xi,\tau,\psi)-\mathcal{W}(\xi,\tau,\psi_{n-1}))\|(t-\tau)^{\mu-1}d\tau\\&\le\frac{1-\mu}{M(\mu)}\Delta\|\psi-\psi_{n-1}\|+\frac{\mu}{M(\mu)\Gamma(\mu)}\Delta\|\psi-\psi_{n-1}\|t.\end{aligned} \tag{26}$$

Thus, by following this recursive process, the result can be obtained at time t_0.

$$\|w_n(\xi,t)\|=\left(\frac{1-\mu}{M(\mu)}+\frac{\mu t_0}{M(\mu)\Gamma(\mu)}\right)^{n+1}\Delta^{n+1}P. \tag{27}$$

Where $P=\|\psi-\psi_{n-1}\|$. Upon evaluating the limit from both directions as $n\to\infty$, $\|w_n(\xi,t)\|\to 0$ obtained. Subsequently, it is crucial to verify the distinctiveness of the solution to the provided problem.

Assume that $\tilde{\psi}(\xi,t)$ represents an alternative solution. In this case, the result is

$$\psi(\xi,t)-\tilde{\psi}(\xi,t)=\frac{1-\mu}{M(\mu)}\left(\mathcal{W}(\xi,t,\psi)-\mathcal{W}\left(\xi,t,\psi^{*}\right)\right)$$
$$+\frac{\mu}{M(\mu)\Gamma(\mu)}\int_{0}^{t}\|\left(\mathcal{W}(\xi,\tau,\psi)-\mathcal{W}\left(\xi,\tau,\psi^{*}\right)\right)\|(t-\tau)^{\mu-1}d\tau. \quad (28)$$

Applying norms to both sides of the previously mentioned equation yields

$$\|\psi(\xi,t)-\tilde{\psi}(\xi,t)\|=\left\|\frac{1-\mu}{M(\mu)}\left(\mathcal{W}(\xi,t,\psi)-\mathcal{W}\left(\xi,t,\psi^{*}\right)\right)\right.$$
$$\left.+\frac{\mu}{M(\mu)\Gamma(\mu)}\right\|\left(\mathcal{W}(\xi,\tau,\psi)-\mathcal{W}\left(\xi,\tau,\psi^{*}\right)\right)\|(t-\tau)^{\mu-1}d\tau\|$$
$$\leq\frac{1-\mu}{M(\mu)}\|\psi(\xi,t)-\tilde{\psi}(\xi,t)\|+\frac{\mu}{M(\mu)\Gamma(\mu)}\Delta t\|\psi(\xi,t)-\tilde{\psi}(\xi,t)\|. \quad (29)$$

Following the process of simplification, the result is obtained as follows

$$\|\psi(\xi,t)-\tilde{\psi}(\xi,t)\|\left(1-\frac{1-\mu}{M(\mu)}\Delta-\frac{\mu t_{0}}{M(\mu)\Gamma(\mu)}\Delta\right)^{n}\leq 0. \quad (30)$$

From the inequality presented above, it follows that if

$$\left(1-\frac{1-\mu}{M(\mu)}\Delta-\frac{\mu t_{0}}{M(\mu)\Gamma(\mu)}\Delta\right)^{n}\geq 0, \quad (31)$$

then $\|\psi(\xi,t)-\tilde{\psi}(\xi,t)\|=0$.

Hence, Eq. (31) serves as a necessary requirement for ensuring uniqueness.

The presence of solutions to the fractional gas dynamics equation with non-homogenous characteristics

In this segment, the focus will be on exploring the time fractional non-homogeneous Eq. (2) within the context of fractional gas dynamics. The fractional derivative operator introduced by Atangana-Baleanu is utilized as illustrated below:

$$^{ABC}D^{\mu}\psi(\xi,t)=\mathcal{J}(\xi,t,\psi(\xi,t))=-\psi\frac{\partial\psi}{\partial\xi}-(1+t)^{2}\psi^{2}+\xi^{2}. \quad (32)$$

Where $^{ABC}D^{\mu}$ symbolizes the Atangana-Baleanu-Caputo (ABC) type fractional operator with a fractional order denoted as μ; where $0<\mu<1$, under the constraint of the initial condition $\psi_{0}(\xi,t)=\psi(\xi,0)$. The transformation of Eq. (32) into an Integral equation of the Volterra-type is achieved by employing the ABC fractional integral in a subsequent manner:

$$\psi(\xi,t)-\psi(\xi,0)=\frac{1-\mu}{M(\mu)}\mathcal{J}(\xi,t,\psi)+\frac{\mu}{M(\mu)\Gamma(\mu)}\int_{a}^{t}\mathcal{J}(\xi,\tau,\psi)(t-\tau)^{\mu-1}d\tau. \quad (33)$$

Theorem 3: *Kernels $\mathcal{J}(\xi,t,\psi)$ outlined in Eq. (17) meets the requirements of Lipschitz continuity condition and contraction, given that the following requirement is fulfilled:*

$$0\leq\left(\frac{\upsilon}{2}(\varphi_{3}+\varphi_{4})+(1+t)^{2}(\varphi_{3}+\varphi_{4})\right)<1.$$

Proof: Let $\mathcal{J}(\xi,t,\psi(\xi,t))=-\psi\frac{\partial\psi}{\partial y}-(1+t)^{2}\psi^{2}+\xi^{2}$, Consider the functions ψ_1 and ψ_2, the ensuing outcome is as follows:

$$\begin{aligned}
&\|\mathcal{J}(\xi, t, \psi_1(\xi, t)) - \mathcal{J}(\xi, t, \psi_2(\xi, t))\| \\
&\quad = \left\| -\psi_1(\xi, t)\frac{\partial \psi_1(\xi, t)}{\partial \xi} + \psi_2(\xi, t)\frac{\partial \psi_2(\xi, t)}{\partial \xi} - (1+t)^2\psi_1^2(\xi, t) + (1+t)^2\psi_2^2(\xi, t) \right\| \\
&\quad \le \frac{1}{2}\left\| \frac{\partial}{\partial y}\left[\psi_1^2(\xi, t) - \psi_2^2(\xi, t)\right] \right\| + (1+t)^2 \|(\psi_1(\xi, t) + \psi_2(\xi, t))(\psi_1(\xi, t) - \psi_2(\xi, t))\| \\
&\quad \le \frac{\upsilon}{2}\|\psi_1^2(\xi, t) - \psi_2^2(\xi, t)\| + (1+t)^2\|\psi_1^2(\xi, t) - \psi_2^2(\xi, t)\| \\
&\quad \le \left(\frac{\upsilon}{2}(\varphi_3 + \varphi_4) + (1+t)^2(\varphi_3 + \varphi_4)\right)\|\psi_1(\xi, t) - \psi_2(\xi, t)\| \\
&\quad \le \Theta\|\psi_1(\xi, t) - \psi_2(\xi, t)\|.
\end{aligned}$$

where $\upsilon = \frac{\partial}{\partial \xi}$ is the differential operator. ψ_1 and ψ_2 are bounded functions, it gives $\|\psi_1\| \le \varphi_3$, $\|\psi_2\| \le \varphi_4$.
Also, $\Theta = (\frac{\upsilon}{2}(\varphi_3 + \varphi_4) + (1+t)^2(\varphi_3 + \varphi_4))$. Hence

$$\|\mathcal{J}(\xi, t, \psi_1(\xi, t)) - \mathcal{J}(\xi, t, \psi_2(\xi, t))\| \le \Theta\|\psi_1(\xi, t) - \psi_2(\xi, t)\|. \tag{34}$$

This indicates that the Lipschitz condition is satisfied for $\mathcal{J}$. Furthermore, it is evident that if $0 \le (\frac{\upsilon}{2}(\varphi_3 + \varphi_4) + (1+t)^2(\varphi_3 + \varphi_4)) < 1$ in such a scenario, this leads to the concept of contraction. The recursive nature of Eq. (33) can be defined in a subsequent manner:

$$\psi_n(\xi, t) = \frac{1-\mu}{M(\mu)}\mathcal{J}(\xi, t, \psi_{n-1}) + \frac{\mu}{M(\mu)\Gamma(\mu)}\int_a^t \mathcal{J}(\xi, \tau, \psi_{n-1})(t-\tau)^{\mu-1}d\tau. \tag{35}$$

Subsequently, express the difference between consecutive iterative elements in the following manner:

$$\begin{aligned}
\sigma_n(\xi, t) = \psi_n(\xi, t) - \psi_{n-1}(\xi, t) &= \frac{1-\mu}{M(\mu)}(\mathcal{J}(\xi, t, \psi_{n-1}) - \mathcal{J}(\xi, t, \psi_{n-2})) \\
&\quad + \frac{\mu}{M(\mu)\Gamma(\mu)}\int_a^t (\mathcal{J}(\xi, \tau, \psi_{n-1}) - \mathcal{J}(\xi, t, \psi_{n-2}))(t-\tau)^{\mu-1}d\tau.
\end{aligned} \tag{36}$$

Be aware that

$$\psi_n(\xi, t) = \sum_{i=1}^{n}\sigma_i(\xi, t). \tag{37}$$

By utilizing Eq. (35) and then applying the norm to Eq. (36), the result is reached,

$$\|\sigma_n(\xi, t)\| = \frac{1-\mu}{M(\mu)}\Theta\|\sigma_{n-1}(\xi, t)\| + \frac{\mu}{M(\mu)\Gamma(\mu)}\Theta\int_0^t \|\sigma_{n-1}(\xi, \tau)\|d\tau. \tag{38}$$

with this, the theorem's proof concluded.

Theorem 4: *The uniqueness and existence of a solution for Eq. (32) are guaranteed, provided a value t_0 can be identified that fulfills the condition*

$$\frac{1-\mu}{M(\mu)}\Theta + \frac{\mu t_0}{M(\mu)\Gamma(\mu)}\Theta < 1.$$

Proof: If the function $\psi(\xi, t)$, which is constrained in its behavior, satisfies the Lipschitz criterion. By employing Eqs. (36) and (38), arrived at the subsequent equation,

$$\|\sigma_n(\xi, t)\| \le \|\psi_n(\xi, t)\|\left[\frac{1-\mu}{M(\mu)}\Theta + \frac{\mu t_0}{M(\mu)\Gamma(\mu)}\Theta\right]^n. \tag{39}$$

Hence, the solution displays smooth characteristics, with the existence of the derived solution being confirmed. Afterward, continue to illustrate that formula Eq. (39) functions as the answer to Eq. (32). To accomplish this, scrutinize...

$$\psi(\xi,t)-\psi(\xi,0)=\psi_n(\xi,t)-u_n(\xi,t). \tag{40}$$

within the framework of solution in the form of a series, considering $u_n(\xi,t)$ as for the remainder terms, our objective is to prove that these terms tend towards zero as $n \to 0$ that is, $\|u_\infty(\xi,t)\| \to 0$.

$$\begin{aligned}\|u_n(\xi,t)\| &= \left\| \frac{1-\mu}{M(\mu)}\left(\mathcal{J}(\xi,t,\psi)-\mathcal{J}(\xi,t,\psi_{n-1})\right) \right.\\ &\quad \left. +\frac{\mu}{M(\mu)\Gamma(\mu)}\int_0^t \left(\mathcal{J}(\xi,\tau,\psi)-\mathcal{J}(\xi,\tau,\psi_{n-1})\right)(t-\tau)^{\mu-1}d\tau \right\| \\ &\le \frac{1-\mu}{M(\mu)}\|\left(\mathcal{J}(\xi,t,\psi)-\mathcal{J}(\xi,t,\psi_{n-1})\right)\| \\ &\quad + \frac{\mu}{M(\mu)\Gamma(\mu)}\int_0^t \|\left(\mathcal{J}(\xi,\tau,\psi)-\mathcal{J}(\xi,\tau,\psi_{n-1})\right)\|(t-\tau)^{\mu-1}d\tau \\ &\le \frac{1-\mu}{M(\mu)}\Theta\|\psi-\psi_{n-1}\| + \frac{\mu}{M(\mu)\Gamma(\mu)}\Theta\|\psi-\psi_{n-1}\|t. \end{aligned} \tag{41}$$

Continuing recursively from t_0, the result is obtained as follows:

$$\|u_n(\xi,t)\| = \left(\frac{1-\mu}{M(\mu)}+\frac{\mu t_0}{M(\mu)\Gamma(\mu)}\right)^{n+1}\Theta^{n+1}P, \tag{42}$$

Where $Q=\|\psi-\psi_{n-1}\|$.

Upon taking the limit as n approaches infinity for both sides, $\|u_n(\xi,t)\| \to 0$ obtained.

Subsequently, it becomes imperative to illustrate the distinctiveness of the solution. To the provided issue.

Assume that $\tilde{\psi}(\xi,t)$ represents an alternative solution. In this case, obtaining,

$$\begin{aligned}\psi(\xi,t)-\tilde{\psi}(\xi,t) &= \frac{1-\mu}{M(\mu)}\left(\mathcal{J}(\xi,t,\psi)-\mathcal{J}\left(\xi,t,\psi^*\right)\right) \\ &\quad + \frac{\mu}{M(\mu)\Gamma(\mu)}\int_0^t \|\left(\mathcal{J}(\xi,\tau,\psi)-\mathcal{J}\left(\xi,\tau,\psi^*\right)\right)\|(t-\tau)^{\mu-1}d\tau. \end{aligned} \tag{43}$$

by applying norms to both sides of the equation mentioned earlier, the result is obtained

$$\begin{aligned}\|\psi(\xi,t)-\tilde{\psi}(\xi,t)\| &= \left\| \frac{1-\mu}{M(\mu)}\left(\mathcal{J}(\xi,t,\psi)-\mathcal{J}\left(\xi,t,\psi^*\right)\right) \right.\\ &\quad + \frac{\mu}{M(\mu)\Gamma(\mu)}\int_0^t \left\|\left(\mathcal{J}(\xi,\tau,\psi)-\mathcal{J}\left(\xi,\tau,\psi^*\right)\right)\right\|(t-\tau)^{\mu-1}d\tau \| \\ &\le \frac{1-\mu}{M(\mu)}\Theta\|\psi(\xi,t)-\tilde{\psi}(\xi,t)\| + \frac{\mu}{M(\mu)\Gamma(\mu)}\Theta t\|\psi(\xi,t)-\tilde{\psi}(\xi,t)\|. \end{aligned} \tag{44}$$

After simplification, the result obtained is

$$\|\psi(\xi,t)-\tilde{\psi}(\xi,t)\|\left(1-\frac{1-\mu}{M(\mu)}\Theta-\frac{\mu t_0}{M(\mu)\Gamma(\mu)}\Theta\right)^n \le 0. \tag{45}$$

Based on the inequality provided earlier, it follows that if

$$\left(1-\frac{1-\mu}{M(\mu)}\Theta-\frac{\mu t_0}{M(\mu)\Gamma(\mu)}\Theta\right)^n \ge 0, \tag{46}$$

Then $\|\psi(y,t)-\tilde{\psi}(y,t)\|=0$.
Hence, Eq. (46) serves as a necessary requirement for ensuring uniqueness.

Numerical simulation

Within this segment, the effectiveness of the method iterative Laplace transform is showcased by implementing it to solve the equation of gas dynamics involving time-fractional homogeneous & non-homogeneous cases. The calculations are executed utilizing the Mathematica software platform.

Approximate solution to the equation governing homogeneous gas dynamics with a time-fractional component

Examine gas dynamics Eq. (1) involving fractional time, along with its initial condition $\psi(\xi,0)=\lambda^{-\xi}$. The solution for Eq. (1) is explicitly provided in[19] as follows:

$$\psi\,(\xi,t)=\lambda^{t-\xi}. \tag{47}$$

by employing the Laplace transform from both sides of Eq. (1), the conclusion is:

$$L\{\psi\,(\xi,t)\}-\frac{1}{s}\psi\,(\xi,0)+\frac{1}{M\,(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)L\left\{\psi\frac{\partial\psi}{\partial\xi}-\psi\,(1-\psi)\,log\lambda\right\}=0. \tag{48}$$

by rearranging terms, the result is

$$L\{\psi\,(\xi,t)\}=\frac{1}{s}\left(\lambda^{-\xi}\right)-\frac{1}{M\,(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)L\left\{\psi\frac{\partial\psi}{\partial\xi}-\psi\,(1-\psi)\,log\lambda\right\}. \tag{49}$$

Furthermore, when the Laplace transform's inverse is employed on Eq. (49), the outcome is as bellow:

$$\psi\,(\xi,t)=\lambda^{-\xi}-L^{-1}\left\{\frac{1}{M\,(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)L\left\{\psi\frac{\partial\psi}{\partial\xi}-\psi\,(1-\psi)\,log\lambda\right\}\right\}, \tag{50}$$

the method yields a series solution as expressed by,

$$\psi\,(\xi,t)=\sum_{n=0}^{\infty}\psi_n\,(\xi,t)\,. \tag{51}$$

The term that is not linear in nature $\psi\frac{\partial\psi}{\partial\xi}$ can be expressed as $\psi_n\frac{\partial\psi_n}{\partial\xi}=\sum_{n=0}^{\infty}A_n$; Meanwhile, A_n is additionally broken down in the subsequent fashion

$$A_n=\sum_{i=0}^{n}\psi_i\frac{\partial}{\partial\xi}\left(\sum_{i=0}^{n}\psi_i\right)-\sum_{i=0}^{n-1}\psi_i\frac{\partial}{\partial\xi}\left(\sum_{i=0}^{n-1}\psi_i\right),$$

by using $\psi_0(\xi,t)=\lambda^{-\xi}$, the iterative equation arises in the following manner

$$\psi_n\,(\xi,t)=\psi_0\,(\xi,t)-L^{-1}\left\{\frac{1}{M\,(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)L\left\{\psi_n\frac{\partial\psi_n}{\partial\xi}-\psi_n\,(1-\psi_n)\,log\lambda\right\}\right\}. \tag{52}$$

The solution using the n-term approximation is expressed as

$$\psi\,(\xi,t)=\psi_0\,(\xi,t)+\psi_1\,(\xi,t)+\psi_2\,(\xi,t)+\cdots\;\cdots\;\cdots+\psi_{n-1}\,(\xi,t)\,. \tag{53}$$

Hence, by employing Eq. (52) below, the computation of the initial three components of the estimated solution to Eq. (1) is presented.

$$\psi_0 = \lambda^{-\xi},$$

$$\psi_1 = -\frac{t\Gamma(\mu)\, log(\lambda)\, \lambda^{-\xi}\left(-\mu + \frac{\mu t^{\mu}}{\Gamma(\mu+2)} + 1\right)}{\mu + \mu(-\Gamma(\mu)) + \Gamma(\mu)},$$

$$\psi_2 = \frac{\mu t^2\Gamma(\mu)\, log^2(\lambda)\,\lambda^{-\xi}}{\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)((\mu-1)\Gamma(\mu) - \mu)} - \frac{t^2\Gamma(\mu)\, log^2(\lambda)\,\lambda^{-\xi}}{2\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)((\mu-1)\Gamma(\mu) - \mu)}$$
$$- \frac{2\mu\Gamma(\mu)\, log^2(\lambda)\, t^{\mu+2}\lambda^{-\xi}}{\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)((\mu-1)\Gamma(\mu) - \mu)\Gamma(\mu+3)} - \frac{\mu^2 t^2\Gamma(\mu)\, log^2(\lambda)\,\lambda^{-\xi}}{2\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)((\mu-1)\Gamma(\mu) - \mu)}$$
$$- \frac{\mu^2\Gamma(\mu)\, log^2(\lambda)\, t^{2\mu+2}\lambda^{-\xi}}{\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)((\mu-1)\Gamma(\mu) - \mu)\Gamma(2\mu+3)} + \ldots$$

$$\psi_3 = \frac{\mu t^3\Gamma(\mu)^2 log^3(\lambda)\,\lambda^{-\xi}}{2\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)((\mu) + \mu(\Gamma(-\mu)) + \Gamma(\mu))^2} - \frac{t^3\Gamma(\mu)^2 log^3(\lambda)\,\lambda^{-\xi}}{6\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)(\mu + \mu(-\Gamma(\mu)) + \Gamma(\mu))^2}$$
$$+ \frac{\mu^3 t^3\Gamma(\mu)^2 log^3(\lambda)\,\lambda^{-\xi}}{6\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)(\mu + \mu(-\Gamma(\mu)) + \Gamma(\mu))^2} - \frac{\mu^2 t^3\Gamma(\mu)^2 log^3(\lambda)\,\lambda^{-\xi}}{2\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)(\mu + \mu(-\Gamma(\mu)) + \Gamma(\mu))^2}$$
$$- \frac{\mu^3\Gamma(\mu)^2 log^3(\lambda)\, t^{3\mu+3}\lambda^{-\xi}}{\left(-\mu + \frac{\mu}{\Gamma(\mu)} + 1\right)(\mu + \mu(-\Gamma(\mu)) + \Gamma(\mu))^2\Gamma(3\mu+4)} + \ldots$$

Following the same approach, the remaining terms of the iteration formula in Eq. (13) can be calculated.

Remark 1: According to Table 1 and Fig. 1, the computed outcomes for obtaining approximate solutions of Eq. (1) are presented. This method provides precise numerical solutions even with lower-order approximations. The precision of the technique in addressing nonlinear gas dynamics equations involving fractional time derivatives in a homogeneous setting is demonstrated by the absolute discrepancies between Eq. (1) and its exact solution. Fig. 1 displays surfaces representing the approximate solution of Eq. (1) and the precise solution for homogeneous fractional gas dynamics equations. Inspection reveals that the approximate and exact solutions closely align.

Table 1. Assessing the disparity in the discrepancy between the precise solution and the true solution and approximations using three terms acquired through the ILTM of Eq. (1) for $\mu = 1$ and numerical results for $\mu = 0.6,\ 0.8$.

t	ξ	$\mu = 0.6$	$\mu = 0.8$	$\mu = 1$	Absolute error $\lvert\psi_{exact} - \psi_{apprx}\rvert$ for $\mu = 1$
0.03	0	0.997213	0.998720	0.666626	0.005052
	1	0.845096	0.846373	0.847395	0.004282
	2	0.766183	0.717265	0.718131	0.003628
0.06	0	0.994109	0.997202	0.999702	0.010278
	1	0.842465	0.845086	0.847205	0.008710
	2	0.713954	0.716175	0.717970	0.007382
0.09	0	0.990782	0.995483	0.999330	0.015678
	1	0.839645	0.843630	0.846890	0.013286
	2	0.711564	0.714941	0.717703	0.012597

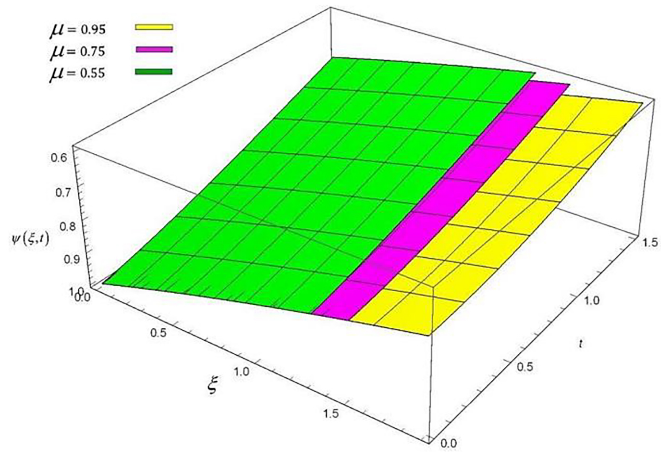

Fig. 1. An estimated solution to Eq. (1), for $\mu = 0.95$, $\mu = 0.75$, $\mu = 0.65$.

Approximate solution to the equation governing non-homogeneous gas dynamics with a time-fractional component

In this segment, time fractional non-homogeneous gas dynamics Eq. (2) is analyzed including the initial condition $\psi(\xi, 0) = \xi$.

The precise solution to the classical non-homogeneous gas dynamics equation is presented as [16]

$$\psi\,(\xi, t) = \frac{\xi}{1+t}. \tag{54}$$

by employing Laplace transform from both sides of Eq. (2), the conclusion is:

$$L\{\psi\,(\xi, t)\} - \frac{1}{s}\psi\,(\xi, 0) + \frac{1}{M\,(\mu)}\left(1 - \mu + \frac{\mu}{s^{\mu}}\right) L\left\{\psi\frac{\partial\psi}{\partial\xi} + (1+t)^2\psi^2 - \xi^2\right\} = 0, \tag{55}$$

by rearranging terms, the result is,

$$L\{\psi\,(\xi, t)\} = \frac{\xi}{s} - \frac{1}{M\,(\mu)}\left(1 - \mu + \frac{\mu}{s^{\mu}}\right) L\left\{\psi\frac{\partial\psi}{\partial\xi} + (1+t)^2\psi^2 - \xi^2\right\}. \tag{56}$$

Moreover, applying the Laplace transform's inverse to Eq. (56) leads to

$$\psi\,(\xi, t) = \xi - L^{-1}\left\{\frac{1}{M\,(\mu)}\left(1 - \mu + \frac{\mu}{s^{\mu}}\right) L\left\{\psi\frac{\partial\psi}{\partial\xi} + (1+t)^2\psi^2 - \xi^2\right\}\right\}, \tag{57}$$

the method yields a series solution as expressed by,

$$\psi\,(\xi, t) = \sum_{n=0}^{\infty}\psi_n\,(\xi, t)\,. \tag{58}$$

The term that is not linear in nature $\psi\frac{\partial\psi}{\partial\xi}$ can be expressed as $\psi_n\frac{\partial\psi_n}{\partial\xi} = \sum_{n=0}^{\infty} B_n$; Meanwhile B_n is additionally broken down in the following manner

$$B_n = \sum_{i=0}^{n}\psi_i\frac{\partial}{\partial\xi}\left(\sum_{i=0}^{n}\psi_i\right) - \sum_{i=0}^{n-1}\psi_i\frac{\partial}{\partial\xi}\left(\sum_{i=0}^{n-1}\psi_i\right).$$

by using $\psi_0(\xi,t)=\xi$. The recursive formula becomes evident in the following manner.

$$\psi_n(\xi,t)=\psi_0(\xi,t)-L^{-1}\left\{\frac{-1}{M(\mu)}\left(1-\mu+\frac{\mu}{s^{\mu}}\right)L\left\{\psi_n\frac{\partial\psi_n}{\partial\xi}+(1+t)^2\psi_n^2-\xi^2\right\}\right\}. \tag{59}$$

the solution using the n-term approximation is expressed as

$$\psi(\xi,t)=\psi_0+\psi_1+\psi_2+\dots\dots\dots+\psi_{n-1}. \tag{60}$$

$$\psi_0=\xi,$$

$$\begin{aligned}\psi_1=&\frac{t^3\xi\Gamma(\mu)}{3(\mu+\mu(-\Gamma(\mu))+\Gamma(\mu))}+\frac{t^2\xi^2\Gamma(\mu)}{\mu+\mu(-\Gamma(\mu))+\Gamma(\mu)}+\frac{t\xi\Gamma(\mu)}{\mu+\mu(-\Gamma(\mu))+\Gamma(\mu)}\\&-\frac{\mu t^3\xi\Gamma(\mu)}{3(\mu+\mu(-\Gamma(\mu))+\Gamma(\mu))}-\frac{\mu t\xi\,\Gamma(\mu)}{\mu+\mu(-\Gamma(\mu))+\Gamma(\mu)}-\frac{\mu t^2\xi^2\Gamma(\mu)}{\mu+\mu(-\Gamma(\mu))+\Gamma(\mu)}\\&+\frac{2\mu\xi^2\Gamma(\mu)t^{\mu+2}}{(\mu+\mu(-\Gamma(\mu))+\Gamma(\mu))\Gamma(\mu+3)}+\frac{\mu\xi\Gamma(\mu)t^{\mu+1}}{(\mu+\mu(-\Gamma(\mu))+\Gamma(\mu))\Gamma(\mu+2)},\end{aligned}$$

$$\begin{aligned}\psi_2=&\frac{1}{\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)}\left\{\frac{2t^5\xi}{15\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^2}+\frac{t^3\xi}{3\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^2}+\mu t\xi^2-t\xi^2\right.\\&+\frac{t^7\xi}{63\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^2}+\frac{5t^4\xi^2}{4\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^2}+\frac{t^3\xi^2}{3\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^2}\\&\left.+\frac{8\mu^3\xi^3\Gamma(2\mu+8)t^{\mu+2(\mu+4)}}{\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^2\Gamma(\mu+3)\Gamma(\mu+4)}+\frac{8\mu^3\xi^2\Gamma(2\mu+8)t^{\mu+2(\mu+4)}}{\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^2\Gamma(\mu+4)^2}\right\}\end{aligned}$$

$$\begin{aligned}\psi_3=&\frac{4t^{12}\xi}{945\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^6}+\frac{134t^{10}\xi}{4725\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^6}+\frac{4t^8\xi}{45\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^6}\\&+\frac{t^6\xi}{9\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^6}+\frac{t^{14}\xi}{3969\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^6}+\frac{t^6\xi^2}{3\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^6}\\&-\frac{384\mu^6\xi^5\Gamma(2\mu+8)^2t^{\mu+2(\mu+4)}}{\left(-\mu+\frac{\mu}{\Gamma(\mu)}+1\right)^6\Gamma(\mu+3)^2\Gamma(\mu+4)^2\Gamma(2(\mu+4)+1)\Gamma(\mu+2(\mu+4)+1)}+\cdots\end{aligned}$$

Remark 2: Table 2 and Fig. 2 illustrate the computed outcomes for obtaining approximate solutions of Eq. (2). This approach ensures accurate numerical solutions, even with lower-order approximations, for nonlinear gas dynamics equations involving fractional time derivatives in a non-homogeneous setting. Fig. 2 shows surfaces depicting the approximate solution of Eq. (2) and the exact solution for non-homogeneous fractional order gas dynamics equations. The surfaces indicate that the approximate and exact solutions are closely aligned.

Results and discussion

The results from applying the Laplace transform iteration method to solve time-fractional gas dynamics equations demonstrate its effectiveness and precision. For both homogeneous and non-homogeneous cases, the method yields accurate numerical solutions even with lower-order approximations. The close alignment between the approximate and exact solutions, as observed in the computed outcomes, highlights the method's reliability in addressing nonlinear gas dynamics equations involving fractional time derivatives.

Table 2. Assessing the disparity in absolute error between the accurate solution and the three-term approximations acquired through the iterative Laplace transform method (ILTM) of Eq. (2) for $\mu = 1$ and numerical results for $\mu = 0.6,\ 0.8$.

t	ξ	$\mu = 0.6$	$\mu = 0.8$	$\mu = 1$	Absolute error $\lvert\psi_{exact} - \psi_{apprx}\rvert$ for $\mu = 1$
0.04	0.2	0.285745	0.243016	0.214680	0.001186540
	0.6	0.862571	0.713500	0.631011	0.003101090
	1.0	1.461070	1.166890	1.030160	0.000315870
0.06	0.2	0.392077	0.296655	0.237210	0.001279000
	0.6	1.336260	0.914550	0.692986	0.001929860
	1.0	2.579890	1.588230	1.125760	0.000500755
0.09	0.2	0.581501	0.390823	0.276887	0.007688600
	0.6	2.356710	1.344820	0.825964	0.004285710
	1.0	5.293520	2.626570	1.379250	0.000823695

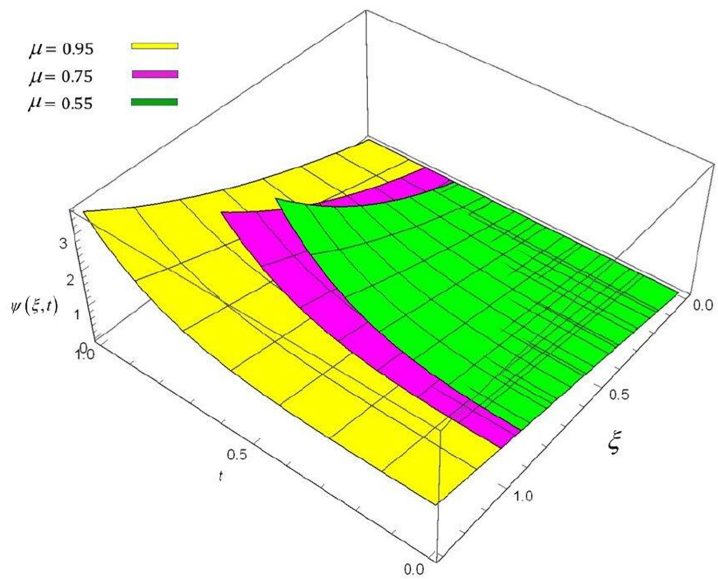

Fig. 2. An estimated solution to Eq. (2), for $\mu = 0.95$, $\mu = 0.75$, $\mu = 0.55$.

Conclusions

In the present study, the method of Laplace transform iteration has been effectively employed. To numerically solve time-fractional homogeneous and non-homogeneous gas dynamic equations. This approach yields highly realistic series solutions that exhibit rapid convergence in real-world scenarios. The theoretical analysis of the solutions includes establishing the criteria for the existence and uniqueness of solutions using the applied technique. Upon comparing absolute errors, it becomes clear that the suggested technique functions as an effective and resilient numerical method for approximating solutions. It is noteworthy that this method remains effective even in the case of nonlinear problems, without necessitating linearization or perturbation techniques. As a result, the iterative Laplace transform method proves to be easily implementable and more convenient compared to alternative approaches.

Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.

- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at KTHM College, Nashik.

Authors' contribution statement

S. W. proposed the concepts, ideas, and method. K. N. plotted the graphs of the solution of examples using Mathematica. A. S. helped in the analysis and D. P. supervised this work. All authors read and approved the final manuscript.

References

1. Zlotnik A. Conditions for L^2-dissipativity of an explicit symmetric finite-difference scheme for linearized 2D and 3D Gas dynamics equations with a regularization. Discrete Contin Dyn Syst B. 2023;28(3):1571–1589. https://doi.org/10.3934/dcdsb.2022137.
2. Kuzenov VV, Ryzhkov SV, Varaksin AY. The adaptive composite block-structured grid calculation of the Gas-dynamic characteristics of an aircraft moving in a Gas environment. Mathematics. 2022;10(12):21–30. https://doi.org/10.3390/math10122130.
3. Akgül A, Modanli M. Crank–Nicholson difference method and reproducing kernel function for third order fractional differential equations in the sense of Atangana–Baleanu Caputo derivative. Chaos Solit Fractals. 2019 Oct 1;127:10–16. https://doi.org/10.1016/j.chaos.2019.06.011.
4. Alqahtani AM. Solution of the generalized burgers equation using homotopy perturbation method with general fractional derivative. Symmetry. 2023;15(3):634. https://doi.org/10.3390/sym15030634.
5. Shaikh A, Nisar KS, Jadhav V, Elagan SK, Zakarya M. Dynamical behaviour of HIV/AIDS model using fractional derivative with Mittag-Leffler kernel. Alex Eng J. 2022;61(4):2601–2610. https://doi.org/10.1016/j.aej.2021.08.030.
6. Tarte SA, Bhadane AP, Gaikwad SB, Kshirsagar KA. Semi-analytical solutions for time-fractional fisher equations via new iterative method. Baghdad Sci J. 2024;21(7):2413–2424. https://doi.org/10.21123/bsj.2023.9137.
7. Sadaf M, Perveen Z, Akram G, Habiba U, Abbas M, Emadifar H. Solution of time-fractional Gas dynamics equation using Elzaki decomposition method with caputo-fabrizio fractional derivative. PLoS ONE. 2024 May 30;19(5):1–15. https://doi.org/10.1371/journal.pone.0300436.
8. Rasulov M, Karaguler T. Finite difference schemes for solving system equations of Gas dynamic in a class of discontinuous functions. Appl Math Comput. 2003;143(1):145–164. https://doi.org/10.1016/S0096-3003(02)00353-3.
9. Prakash A, Kumar M. Numerical method for time-fractional gas dynamic equations. Proc Natl Acad Sci, India, Sect A Phys Sci. 2019;89:559–570. https://doi.org/10.1007/s40010-018-0496-4.
10. Iqbal N, Akgul A, Shah R, Bariq A, Al-Sawalha MM, Ali A. On solutions of fractional-order Gas dynamics equation by effective techniques. J Funct Spaces. 2022;2022(1):1–14. https://doi.org/10.1155/2022/3341754.
11. Al-Luhaibi MS. New iterative method for fractional Gas dynamics and coupled burger's equations. Sci World J. 2015;2015(1):1–8. https://doi.org/10.1155/2015/153124.
12. Hussein MA. Using the elzaki decomposition method to solve nonlinear fractional differential equations with the caputo-fabrizio fractional operator. Baghdad Sci J. 2024;21(3):1044–1054. https://dx.doi.org/10.21123/bsj.2023.7310.
13. Sontakke BR, Raut SR. Analysis of fractional Kawahara and modified Kawahara equations based on Caputo-Fabrizio derivative operator. J Math Comput Sci. 2021;11(6):7105–7125.
14. Agarwal R, Airan P, Sajid M. Numerical and graphical simulation of the non-linear fractional dynamical system of bone mineralization. Math Biosci Eng. 2024 Mar 1;21(4):5138–5163. https://doi.org/10.3934/mbe.2024227.
15. Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular Kernel: Theory and application to heat transfer model. arXiv preprint arXiv:1602.03408. 2016;1–8. https://doi.org/10.48550/arXiv.1602.03408.
16. Aman S, Abdeljawad T, Al-Mdallal Q. Natural convection flow of a Fluid using Atangana and Baleanu fractional model. Adv Differ Equ. 2020;2020(1):1–15. https://doi.org/10.1186/s13662-020-02768-w.
17. Daftardar-Gejji V, Jafari H. An iterative method for solving nonlinear functional equations. J Math Anal Appl. 2006;316(2):753–763. https://doi.org/10.1016/j.jmaa.2005.05.009.
18. Atyia OM, Fadhel FS, Alobaidi MH. About the existence and uniqueness theorem of fuzzy random ordinary differential equations. Nahrain J Sci. 2023 Jul 1;26(2):30–35. https://doi.org/10.22401/ANJS.26.2.05.
19. Das S, Kumar R. Approximate analytical solutions of fractional gas dynamic equations. Appl Math Comput. 2011;217(24):9905–9917. https://doi.org/10.1016/j.amc.2011.03.144.

دراسة لمعادلة أتانجانا باليانو التفاضلية الكسرية مع تطبيق في مشكلة ديناميكية الغاز

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[2]كلية الحوسبة *MIT* ، جامعة *MIT* للفنون والتصميم والتكنولوجيا، بوني- 412201.
[3]قسم الرياضيات، كلية *K. R. T.* للفنون والتجارة والعلوم المتقدمة، ناشيك- 422401.
[4]قسم الرياضيات، كلية العلوم والإنسانيات بالخرج، جامعة الأمير سطام بن عبد العزيز، الخرج *11942* ، المملكة العربية السعودية.

الخلاصة

تقدم هذه المقالة تقريبًا تحليليًا لمعادلة ديناميكيات الغاز الغير خية ذات الرتبة الجزئية في كل من الحالات المتجانسة وغير المتجانسة، مع التركيز بشكل خاص على جبهات الصدمة. يستخدم مؤثر الاشتقاق الكسري الذي اقترحه أتانجانا وباليانو. من خلال توظيف هذا المؤثر، يتم اشتقاق حل تقريبي للمعادلات الديناميكية الغازية ذات الرتبة الكسورية في السيناريوهات التي تنطوي على جبهات الصدمة. تتضمن عملية الحل في المقام الأول إجراء تكراري يستفيد من تحويل لابلاس للحسابات العددية. إن استخدام تحويل لابلاس يقلل من الأخطاء المتتالية، مما يؤدي إلى حل دقيق ومباشر للتنفيذ. تم التحقق من دقة الطريقة وموثوقيتها باستخدام معايير محددة للوجود والوحدانية. بالإضافة إلى ذلك، تم تضمين جدول لإثبات فعالية الطريقة وقدرتها، مع توضيح الأخطاء المطلقة لقيم معينة. كما تم توفير الرسوم التوضيحية الرسومية لتصوير سلوك الحل والاختلافات عبر القيم المختلفة. تساعد هذه التصورات في فهم ديناميكيات وتعقيدات الحلول التي تم الحصول عليها باستخدام مشغل Atangana-Baleanu. بشكل عام، تثبت الطريقة أنها أداة قوية وفعالة لحل المعادلات الديناميكية لغاز الترتيب الجزئي، مما يوفر دقة وسهولة تطبيق كبيرة.

الكلمات المفتاحية: مؤثر أتانجانا-باليانو، المعادلات التفاضلية الكسرية، المعادلات الديناميكية الغازية الكسرية، طريقة تحويل لابلاس التكرارية، الحلول العددية.