On Third-Order Differential Subordination Results for Univalent Functions Associated by Differential Operator

Haneen Zaghair Hasan
College of Law / Al-Qadisiyah University

Abstract

In this research, we explore the third-order difference in membership associated with single-valued functions associated with differentiable operators. We present new information about the third-order differential membership in the unit disk.

Keywords: Analytic function , Differential Subordination , univalent function

نتائج التبعية التفاضلية من الدرجة الثالثة للدوال الأحادية المرتبطة بالمشغل التفاضلي

م.م. حنين زغير حسن محمد
كلية القانون / جامعة القادسية
haneenzaghair@gmail.com

خلاصة

في هذا البحث قمنا بدراسة التبعية التفاضلية من الدرجة الثالثة للدوال الأحادية التكافؤ المرتبطة بالمشغل التفاضلي وحصلنا على نتائج جديدة للتبعية التفاضلية من الدرجة الثالثة في قرص الوحدة.

الكلمات المفتاحية :الدالة التحليلية ، التبعية التفاضلية ، الدالة الاحادية التكافؤ

1.Introduction

Let $H(U)$ denote the class of analytic functions in the open unit disk $U=\{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, n]$ represent the subclass of functions $f \in H(U)$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (a \in \mathbb{C}) \tag{1.1}$$

Additionally, let $A(n)$ be the subclass of functions $f \in H(U)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \tag{1.2}$$

For $f, g \in H(U)$, we say that the function $f(z)$ is subordinate to $g(z)$, denoted symbolically as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) ,$$

If there is a Schwarz function $w(z)$ that is analytic in the region U, with $w(0)=0$ and $|w(z)|<1$ for $z\in U$, and if $f(z)=g(w(z))$ for all $z\in U$,

then the following equivalence holds, particularly when the function $g(z)$ is univalent in U (cf., e.g., [3]; also refer to [4, p.4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) \prec g(0) \text{ and } f(u) \subset g(u)\,.$$

The idea of differential dependence is an extension of various complex variable inequalities. To further deepen it, we introduce some additional definitions and terms from differential dependence theory.:

Definition 1.1. (refer to [1]): Consider $\Psi: \mathbb{C}^4 \times U \to \mathbb{C}$, and a univalent function h(z) defined in U. If the function p(z) is analytic in U and satisfies the following third-order differential subordination:

$$\Psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \qquad (1.3)$$

A solution of the differential subordination is called p(z).

A single-valued function that is different from zero is called the master function of a differentially subordinate solution, or the master function if p(z) is greater than or equal to q(z) for all p(z) that satisfy (1.3). For all of the q(z) in (1.3) that are dominant, the q(z) that is most dominant is said to be the best of the dominant.

Definition 1.2. (refer[2]) Let function $f(z) \in A(n)$. For $m \in N_\circ = N \cup \{0\}$, $\alpha \geq 0, \beta \geq 0$, the Wanas operator $W^{k,\delta}_{\alpha,\beta}: A \longrightarrow A$ is defined by :

$$W^{k,\delta}_{\alpha,\beta}\, f(z) = z + \sum_{n=2}^{\infty}\Big[\, \sum_{m=1}^{k}\binom{k}{m}\, (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m}\right)\Big]^{\delta}\, a^n z^n \,.. \qquad (1.4)$$

It can be confirmed from (1.4) that

$$z\,(W^{k,\delta}_{\alpha,\beta}\, f(z))' = \Big[\sum_{m=1}^{k}\binom{k}{m}\, (-1)^{m+1}\, \Big(\big(\tfrac{\alpha}{\beta}\big)^m + 1\Big)\Big]\, W^{k,\delta+1}_{\alpha,\beta}\, f(z)$$

$$-\Big[\sum_{m=1}^{k}\binom{k}{m}\, (-1)^{m+1}\, \big(\tfrac{\alpha}{\beta}\big)^m\Big]\, W^{k,\delta}_{\alpha,\beta}\, f(z). \qquad (1.5)$$

Definition 1.3. (refer to [1]): Consider the set $\mathbb{Q}$ comprising all analytic and univalent functions q defined on $\bar{U} \setminus E(q)$.

$$E(q) = \left\{\xi : \xi \in \partial U : \lim_{z \to \xi}\{q(z)\} = \infty\right\}, \qquad (1.6)$$

and are such that $\min|q'(\xi)| = \rho > 0$ for $\xi \in \partial U \setminus E(q)$. Further , let the subclass of $\mathbb{Q}$ for which $q(0) = a$ be denoted by $\mathbb{Q}(a)$ with

$\mathbb{Q}(0) = \mathbb{Q}_0$ and $\mathbb{Q}(1) = \mathbb{Q}_1$.

Subordinate techniques are used for certain classes of acceptable functions. Antonino and Miller [1] provide the following classes of acceptable functions:.

2024 آب العدد 14
No.14 Aug 2024
المجلة العراقية للبحوث الإنسانية والإجتماعية والعلمية
Iraqi Journal of Humanitarian, Social and Scientific Research
Print ISSN 2710-0952 Electronic ISSN 2790-1254

Definition 1.4. (See [1]): Assume Ω is a subset of C. Let q∈Q and n∈N\{1}, where N is a set of positive integers. The class of feasible functions ψ_n [Ω,q] includes functions that satisfy the feasibility condition ψ: C^4×U⟶C:

$$\psi(r,s,t,u;z) \notin \Omega\,,$$

whenever

$$r = q(\xi), \qquad s = k\xi q'(\xi), \qquad R\left(\frac{t}{s}+1\right) \geq\ kR\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right)$$

and

$$R\left(\frac{w}{s}\right) \geq k^2 R\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U\ ,\xi \in \partial U \setminus E(q)$ and $k \geq n$.

Lemma 1.1. (refer to [1]): If p ∈ H[a,n] with n≥2 and q ∈ ℚ (a) satisfy the prescribed conditions, then:

$$\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0\,, \qquad \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq k\,,$$

where $z \in U, \xi \in \partial U \setminus E(q)$ and $k \geq n$. If Ω is a set in $\mathbb{C}$, $\psi \in\ \psi_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z)\,, z^3p'''(z); z\,)\ \subset \Omega\,,$$

then

$$p(z) \prec q(z) \qquad\qquad (z \in U).$$

2. Results concerning the Third-Order Subordination

We first introduce the following class of admissible functions, which are essential for establishing the differential subordination theorem with the operator W_(α,β)^(k,δ) according to the equation (1.4).

Definition 2.1. Assume Ω is a subset of C and q ∈ Q_∘∩H_∘. The class of feasible functions θ_r [Ω,q] consists of functions phi:C^4×U⟶C that satisfy the specified admissibility conditions.

$$\phi(\alpha,\beta,\gamma,\delta;z) \notin \Omega$$

whenever

$$\alpha = q(\zeta)\,, \qquad \beta = \frac{k\zeta q'(\zeta) + \left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}\right) q(\zeta)\right.}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right.},$$

$$\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{2} x - \left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\alpha}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\beta - \left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\alpha} - 2\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)$$

$$\mathrm{Re}\left\{\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{2} x - \left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\alpha}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\beta - \left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\alpha} - 2\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\right\} \geq$$

$$k\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

and

$$\mathbf{Re}\left\{\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{2}\left[\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)y - 3\left(\sum_{m=1}^{k}\binom{k}{m}\right(\right.}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\beta - \left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right.}\right.$$

$$\frac{\left(2\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{3} + 3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\alpha\right.}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\beta - \left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\alpha} +$$

$$\left.\left(3\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2} + 6\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right) + 2\,).\right\} \geq$$

$$k^{2}\mathrm{Re}\left\{\frac{\zeta^{2} q'''(\zeta)}{q'(\zeta)}\right\},$$

Given $z \in U, \zeta \in \partial U \setminus E(q)$ and $k \in \mathbb{N} \setminus \{1\}$

Theorem 2.2. Let $\phi \in \Theta_r[\Omega, q]$. If the function $f \in A(n)$and $q \in Q_\circ$ satisfy the following conditions :

$$\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0\,, \qquad \left|\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{q'(\zeta)}\right| \leq k\,, \tag{2.1}$$

and

$$\left\{\phi\left(W_{\alpha,\beta}^{k,\delta} f(z), W_{\alpha,\beta}^{k,\delta+1} f(z), W_{\alpha,\beta}^{k,\delta+2} f(z), W_{\alpha,\beta}^{k,\delta+3} f(z)\,; z\right) z \in U\right\} \subset \Omega$$
(2.2)

المجلة العراقية للبحوث الإنسانية والإجتماعية والعلمية
Iraqi Journal of Humanitarian, Social and Scientific Research
Print ISSN 2710-0952 Electronic ISSN 2790-1254

then

$W_{\alpha,\beta}^{k,\delta} f(z) \prec q(z), \quad (z \in U)$

Proof. *Proof*: Define the analytic function (w(z)) in (U) by:

$$p(z) = W_{\alpha,\beta}^{k,\delta} f(z). \tag{2.3}$$

Then , differentiating (2.3) with respect to z and using (1.5) , we have

$$W_{\alpha,\beta}^{k,\delta+1} f(z) = \frac{zp'(z)+\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right]p(z)}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1} \tag{2.4}$$

Further computations show that

$$W_{\alpha,\beta}^{k,\delta+2} f(z) = \frac{z^2p''(z)+\left(2\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)zp'(z)+\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right]^2p(z)}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2}$$

(2.5)

and

$$W_{\alpha,\beta}^{k,\delta+3} f(z) = \frac{z^3p'''(z)+3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)+1\right)z^2p''(z)+}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}$$

$$\frac{3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)^2+3\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)+1\right)zp'(z)+\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)\right]^3p(z)}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}$$

(2.6)

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,w) = r, \qquad \beta(r,s,t,w) = \frac{s+\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right]r}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1},$$

$$x(r,s,t,w) = \frac{t+\left(2\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)+1\right)s+\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right]^2r}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2},$$

العدد 14 آب 2024
No.14 Aug 2024
المجلة العراقية للبحوث الإنسانية والإجتماعية والعلمية
Iraqi Journal of Humanitarian, Social and Scientific Research
Print ISSN 2710-0952 Electronic ISSN 2790-1254

$$y(r,s,t,w) = \frac{w+3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\Big)t+3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2+}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}$$

$$\frac{3\sum_{m=1}^{k}\binom{k}{m}\left((-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\Big)s+\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^3 r}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}$$

Let

$$\psi(r,s,t,w;z) = \phi(\alpha,\beta,x,y;z) =$$

$$= \emptyset\Bigg(r, \frac{s+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m r}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1},$$

$$\frac{t+\left(2\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\Big)s+\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right]^2 r}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2},$$

$$\frac{w+3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\Big)t+3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2+}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}$$

$$\frac{3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\Big)s+\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3 r}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2};z\Bigg)$$

(2.7)

The proof will make use of Lemma(1.1).Using the equations (2.3) to (2.6) ,and from the equation (2.7) ,we have

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \phi\left(W_{\alpha,\beta}^{k,\delta} f(z), W_{\alpha,\beta}^{k,\delta+1} f(z), W_{\alpha,\beta}^{k,\delta+2} f(z), W_{\alpha,\beta}^{k,\delta+3} f(z); z\right) \quad (2.8)$$

Hence (2.2) becomes

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega$$

Note that

$$\frac{t}{s}+1=\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2 x-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2\alpha}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)\beta-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\alpha}-2\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)$$

,

(2.9)

and

$$\frac{w}{s}=\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2\left[\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)y-3\left(\sum_{m=1}^{k}\binom{k}{m}\left(-\cdots\right.\right.\right.}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)\beta-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\cdots}$$

$$\frac{\left(2\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^3+3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2\alpha\right.}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)\beta-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\alpha}+$$

$$\left(3\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2+6\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+2\,). \tag{2.10}$$

Therefore, the admissibility conditions of 〚ψεθ〛 _r [Ω,q] in Definition (2.1) are clearly equivalent to the admissibility conditions of ψεψ_2 [Ω,q] in Definition (4) when n=2. Therefore, using (2.1) and Lemma 1 we have

p(z)≺q(z),

or, more generally,

W_(α,β)^(k,δ) f (z) is more significant than q(z).

The proof of (2.2) is now complete.

If the behavior of q(z) near the boundary of U is unknown, we can deduce a similar result to Theorems 2.2 from the behavior of q(z) on the boundary of U.

Corollary 2.3. Given $\Omega\subset\mathbb{C}$ and a function q that is univalent in U with $q(0)=0$, suppose $\phi\in\Theta_r[\Omega,q]$ for some $p\in(0,1)$, where $q_p(z)=q(p_z)$. If $f\in A(n)$ and q_p satisfies the specified conditions:

$$\operatorname{Re}\left\{\frac{\zeta q_p''(\zeta)}{q_p'(\zeta)}\right\}\ge 0\,,\quad\left|\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{q_p'(z)}\right|\le k\,,\quad\left(z\in U,\zeta\in\partial U\setminus E(q_p)\right), \tag{2.11}$$

and

$$\phi \quad \left\{W_{\alpha,\beta}^{k,\delta} f(z), W_{\alpha,\beta}^{k,\delta+1} f(z), W_{\alpha,\beta}^{k,\delta+2} f(z), W_{\alpha,\beta}^{k,\delta+3} f(z)\ ; z \in U\right\} \in \Omega$$
(2.12)

then

$$W_{\alpha,\beta}^{k,\delta} f(z) \prec q(z) \qquad (z \in U)$$

Proof: By utilizing Theorem (2.2), we obtain

$$W_{\alpha,\beta}^{k,\delta} f(z) \prec q_p(z)\,. \quad (z \in U)$$

The conclusion stated in Corollory (2.3) is now derived from the subsequent dependency property

q_p (z) is the approximation of the real value of z, as a result, it is also a function of z.

For a domain that is simply connected and has a conformal mapping h(z) that converts U to a domain of type Ω≠C. Here, the class of functions that are defined as h(U),q is referred to as h(U),q.

The following two results are the direct results of Theorem (2.2) and of Corollary (2.3).

Theorem 2.4. Given $\phi \in \Theta_r[h, q]$, if the function f∈A(n) and q∈Q_° satisfy condition (2.1). and

$$\phi\left(W_{\alpha,\beta}^{k,\delta} f(z), W_{\alpha,\beta}^{k,\delta+1} f(z), W_{\alpha,\beta}^{k,\delta+2} f(z), W_{\alpha,\beta}^{k,\delta+3} f(z); z\right) \prec h(z) \qquad (2.13)$$

Then

$$W_{\alpha,\beta}^{k,\delta} f(z) \prec q(z)\,, \quad (z \in U)\,.$$

Corollary 2.5. For $\Omega \in \mathbb{C}$ and a univalent function q in U with q(0)=0, and $\phi \in \Theta_r[h, q_p]$for some p∈(0,1), where $q_p(z) = q(p_z)$. If $f \in A(n)$ and q_p satisfy condition (2.12),

$$\phi\left(W_{\alpha,\beta}^{k,\delta} f(z), W_{\alpha,\beta}^{k,\delta+1} f(z), W_{\alpha,\beta}^{k,\delta+2} f(z), W_{\alpha,\beta}^{k,\delta+3} f(z); z\right) \prec h(z)\,.$$
(2.14)

Then

$$W_{\alpha,\beta}^{k,\delta} f(z) \prec q(z), \quad (z \in U).$$

The best dominant of the differential subordination (2.13) is obtained.

Theorem 2.6. Assume h is univalent in U. Let ϕ $\phi: \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and ψ and ψ be defined by (2.7). If the differential equation.

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z)$$
(2.15)

has a solution q(z) with q(0)=1 satisfying condition (2.1) and f∈A(n) satisfies condition (2.13) and

$$\phi\left(W_{\alpha,\beta}^{k,\delta} f(z), W_{\alpha,\beta}^{k,\delta+1} f(z), W_{\alpha,\beta}^{k,\delta+2} f(z), W_{\alpha,\beta}^{k,\delta+3} f(z); z\right)$$

is analytic in U,

$$W_{\alpha,\beta}^{k,\delta} f(z) \prec q(z) ,$$

then q(z) serves as the best dominant.

Proof . From Theorem 2.2, it can be deduced that q is the primary solution to equation (2.13). Since q also satisfies (2.15), it is therefore a solution of (2.14). As a result, the majority of the question will be answered by the dominants. As a result, q is determined to be the most effective dominant. This ends the verification of the theorem.

According to Definition (2.1), and specifically in the case where q(z)=M_z (M>0), the class of functions that are permitted, referred to as Θ_r [Ω,q], can be described as follows

Definition 2.7. Let Ω be a set in C,l≥0 and M≥0. The set of permissible functions Θ_r [Ω,M] consists of functions ϕ:C^4×U⟶C that satisfy the following conditions.

$$\phi\left(Me^{i\theta} , \frac{k+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1} Me^{i\theta}, \frac{L+\left[2\left(\sum_{m=1}^{k}\binom{k}{m}\left(-1\right)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\right)k+}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2}\right.$$

$$\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 Me^{i\theta}}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2}, \frac{N+3\left(\sum_{m=1}^{k}\binom{k}{m}\left(-1\right)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\right)L+}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}$$

Print ISSN 2710-0952 Electronic ISSN 2790-1254

$$\frac{\left[3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}+3\sum_{m=1}^{k}\binom{k}{m}\left(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)+1\Big)k+\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{3}\right] Me^{i\theta}}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{3}}$$

$; z) \notin \Omega,$
(2.16)

Whenever $z \in U, \operatorname{Re}(Le^{-i\theta}) \geq (k-1)kM$, and for all $\in R$ and $, k \in \mathbb{N} \setminus \{1\}$.

Corollary 2.8. Suppose $\phi \in \theta_r[\Omega, M]$, If the function $f \in A(n)$ meets the following condition:

:

$$\left|W_{\alpha,\beta}^{k,\delta+1} f(z)\right| \leq kM \quad (k \geq 2; M > 0)\,.$$

And

$$\phi\left(W_{\alpha,\beta}^{k,\delta} f(z), W_{\alpha,\beta}^{k,\delta+1} f(z), W_{\alpha,\beta}^{k,\delta+2} f(z), W_{\alpha,\beta}^{k,\delta+3} f(z); z\right) \in \Omega,$$

Then

$$\left|W_{\alpha,\beta}^{k,\delta} f(z)\right| < M\,, \qquad (z \in U, k \in \mathbb{N} \setminus \{1\})$$

In the scenario where $\Omega = q(U) = \{w: |w| < M(M > 0)\}$,let's use the notation $\Theta_r[M]$to represent the class $\Theta_r[\Omega, M]$.

Corollary 2.9. If $\phi \in \theta_r[M]$. and the function f∈A(n) meets the conditions:

$$\left|W_{\alpha,\beta}^{k,\delta+1} f(z)\right| \leq kM \qquad (k \geq 2\,; M > 0)\,,$$

and

$$\left|\phi\left(W_{\alpha,\beta}^{k,\delta} f(z), W_{\alpha,\beta}^{k,\delta+1} f(z), W_{\alpha,\beta}^{k,\delta+2} f(z), W_{\alpha,\beta}^{k,\delta+3} f(z); z\right)\right| < M\,,$$

then

$$\left|W_{\alpha,\beta}^{k,\delta} f(z)\right| < M\,.$$

Corollary 2.10. Considering $M > 0\,, K \in \mathbb{N} \setminus \{1\}$ and $\sum_{m=1}^{k}\binom{k}{m}\ (-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m} \geq 0$., and f∈A(n) satisfies the conditions:

$$\left|W_{\alpha,\beta}^{k,\delta+1} f(z) - W_{\alpha,\beta}^{k,\delta} f(z)\right| < \frac{k+1}{\sum_{m=1}^{k}\binom{k}{m}\ (-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m} + 1} M,$$

then

$$\left|W_{\alpha,\beta}^{k,\delta} f(z)\right| < M\,.$$

Proof . Defining ϕ (α,β,γ,δ;z)=β-α and Ω=h(U), where

$$h(z) = \frac{k+1}{\sum_{m=1}^{k}\binom{k}{m}\ (-1)^{m+1}\ (\frac{\alpha}{\beta})^m + 1} Mz . \qquad M > 0 .$$

By using Corollary (2.8), we aim to demonstrate that $\phi \in \Theta_r[\Omega, M]$, meaning that the admissibility condition (2.17) is fulfilled. This is evident, as observed when

$$|\ \phi\ (\ Me^{i\theta}\ ,\frac{k+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m+1} Me^{i\theta}, \frac{L+[\,2\,(\sum_{m=1}^{k}\binom{k}{m}\left(-1\right)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)+1\,)\,k+}{(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m+1)^2}$$

$$\frac{(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m)^2\ Me^{i\theta}}{(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m+1)^2}, \frac{N+3\,(\sum_{m=1}^{k}\binom{k}{m}\left(-1\right)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)+1)\,L+}{(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m+1)^3}$$

$$\frac{[\,3\,(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m)^2+3\sum_{m=1}^{k}\binom{k}{m}\left(-1\right)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)+1)k+(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1)^3\big]\ Me^{i\theta}}{(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m+1)^3}$$

$$;\,z)\ | \quad = \left|\frac{k+1}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m+1}\right| \ = \frac{k+1}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m} M,$$

for all $z \in U$, $\theta \in \mathbb{R}$ and $k \geq 2$. The desired outcomes are derived from Corollary (2.8), thus substantiating Corollary (2.10).

Definition 2.11. Assuming that Ω is a set in C and that q is in the first quadrant of H, the class of functions that are admittedible on C^4 × U and have the property:

$$\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega$$

whenever

$$\alpha = q(\zeta)\,, \qquad \beta = \frac{\left[(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)+1\right]}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m+1},$$

$$\mathrm{Re}\{\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^m+1\right)^2 x - \left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m)+1\right)^2\alpha}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)\beta - \left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)\alpha} -$$

$$2(\sum_{m=1}^{k}\binom{k}{m}\ (-1)^{m+1}\ \left(\frac{\alpha}{\beta}\right)^m + 1)\} \geq k\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}$$

and

$$\mathrm{Re}\{\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^{m}+1\right)^{2}\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)y-\left(3\sum_{m=1}^{k}\binom{k}{m}(-\right.}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\beta-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)+1)}$$

$$\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^{m}+1\right)^{2}x-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^{m}\right)+1\Big)^{2}\alpha-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^{m}\right)+1\Big)^{3}\alpha}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^{m}+1\right)\beta-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^{m}\right)+1\Big)\alpha}$$

$+$

$$\left(3\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^{m}\right)^{2}+12\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)+11\ ).\}\geq$$

$$k^{2}\mathrm{Re}\left\{\frac{\zeta^{2}q'''(\zeta)}{q'(\zeta)}\right\},$$

where $k\in\mathbb{N}\setminus\{1\},\zeta\in\partial U\setminus E(q)$ and $z\in U$.

Theorem 2.12. states: Let ϕ be an element of $\phi\in\Theta_{r,1}[\Omega,q]$ .If the function f belongs to A(n) and q belongs to Q1 and they satisfy the given conditions:

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)\geq 0\,,\qquad\left|\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{zq'(\zeta)}\right|\leq k\lambda\,,\tag{2.17}$$

and

$$\{\ \phi\ (\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z},\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{z},\frac{W_{\alpha,\beta}^{k,\delta+2}f(z)}{z},\frac{W_{\alpha,\beta}^{k,\delta+3}f(z)}{z});z\ )\ z\in U\ \}\subset\Omega$$

Then

$$\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}\prec q(z)\,,\qquad(z\in U).$$

Proof . An analytic function w(z) in U is defined as

$$p(z)=\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z}.\tag{2.18}$$

By utilizing equations (1.5) and (2.19), we derive

$$\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{z}=\frac{zp'(z)+\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^{m}\right)p(z)}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}(\frac{\alpha}{\beta})^{m}+1\right)}.\tag{2.19}$$

المجلة العراقية للبحوث الإنسانية والإجتماعية والعلمية
Iraqi Journal of Humanitarian, Social and Scientific Research
Print ISSN 2710-0952 Electronic ISSN 2790-1254

Further calculations reveal that

$$\frac{W_{\alpha,\beta}^{k,\delta+2} f(z)}{z} = \frac{z^2p''(z)+\left(2\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+3\right)zp'(z)+\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right]^2 p(z)}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2} \quad (2.20)$$

and

$$W_{\alpha,\beta}^{k,\delta+3} f(z) = \frac{z^3p'''(z)+3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+6\Big) z^2p''(z)+}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}$$

$$\frac{3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2+9\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\Big)+7\Big)zp'(z)+\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\Big]^3 p(z)}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}. \quad (2.21)$$

A transformation from $\mathbb{C}^4$ to $\mathbb{C}$ is defined as

$$\alpha(r,s,t,w) = r, \qquad \beta(r,s,t,w) = \frac{s+\left[\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\right]r}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1},$$

$$x(r,s,t,w) = \frac{t+\left(2\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+3\Big) s+\left[\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+1\right]^2 r}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2},$$

$$y(r,s,t,w) = \frac{w+3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+6\Big)t+3\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2+}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}$$

$$\frac{9\sum_{m=1}^{k}\binom{k}{m}\left(-1\right)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\Big)+7\Big)s+\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3 r}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^3}.$$

Let

$$\psi(r,s,t,w;z) = \phi(\alpha,\beta,x,y;z) =$$

$$= \emptyset\left(r, \frac{s+\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right) r}{\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1},\right.$$

$$\frac{t+\left(2\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)+3\Big)s+\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right]^2 r}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m+1\right)^2},$$

المجلة العراقية للبحوث الإنسانية والإجتماعية والعلمية
Iraqi Journal of Humanitarian, Social and Scientific Research
Print ISSN 2710-0952 Electronic ISSN 2790-1254

$$\frac{w+3\left(\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}\right)+6\Big)t+3\left(\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}+}{\left(\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{3}}$$

$$\frac{9\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)+7\Big)\,s+\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{3}r}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{2}}\,;z\,)$$

(2.22)

The proof will involve applying Lemma 1.1 along with equations (2.19) through (2.22) and using (2.23), resulting in Hence, (2.18) can be expressed as

$$\psi(p(z),zp'(z),z^{2}p''(z),z^{3}p'''(z);z)=$$

$$\phi\left(\frac{W_{\alpha,\beta}^{k,\delta}f(z)}{z},\frac{W_{\alpha,\beta}^{k,\delta+1}f(z)}{z},\frac{W_{\alpha,\beta}^{k,\delta+2}f(z)}{z},\frac{W_{\alpha,\beta}^{k,\delta+3}f(z)}{z}\right);z\,).$$

(2.23)

Hence (2.18) becomes

$$\psi(p(z),zp'(z),z^{2}p''(z),z^{3}p'''(z);z)\in\Omega\,.$$

Note that

$$\frac{t}{s}+1=\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{2}x-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)+1\Big)^{2}\alpha}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\beta-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\alpha}-$$

$$2\left(\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}+1\right)$$

and

$$\frac{w}{s}$$

$$=\frac{\left(\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{2}\left(\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}+1\right)y-\left(3\sum_{m=1}^{k}\binom{k}{m}\,(-1\right.}{\left(\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\beta-\left(\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}\right)+1\Big)}$$

$$\frac{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)^{2}x-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)+1\Big)^{2}\alpha-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)+1\Big)^{3}}{\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\beta-\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)+1\Big)\alpha}$$

$$\left(3\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}+12\left(\sum_{m=1}^{k}\binom{k}{m}\,(-1)^{m+1}\,\left(\frac{\alpha}{\beta}\right)^{m}\right)+11\,).$$

المجلة العراقية للبحوث الإنسانية والإجتماعية والعلمية
Iraqi Journal of Humanitarian, Social and Scientific Research
Print ISSN 2710-0952 Electronic ISSN 2790-1254

As a result, the condition for acceptance of aϕ in the set of parameters Θ_(r,1) [Ω,q], as defined in Definition (2.11), is the same as the condition for acceptance of aψ in the set of parameters ψ_2 [Ω,q], as specified in Definition (1.5) with n=2. Through the use of (2.17) and the first part of Lemma 1.1, we have

$$p(z) = \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \prec q(z)\,,$$

This completes the proof of Theorem 2.12.

If Ω is a simply connected domain that is not equal to C and it is represented as Ω=h(U) where h is a conformal mapping of U onto Ω, then the class $\Theta_{r,1}[h(U), q]$ can be denoted as $\Theta_{r,1}[h, q]$.

The subsequent outcome is an immediate consequence of Theorem 2.12.

Theorem 2.13. Suppose $\phi \in \Theta_{r,1}[h, q]$where $f \in A(n)$ and $q \in Q_1$ satisfy condition (2.17).

$$\phi\left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+2} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+3} f(z)}{z}\right); z\,) \prec h(z)\,, \tag{2.24}$$

then

$$\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \prec q(z)\,,$$

$(z \in U\,, \zeta \in \partial U \setminus E(q)$ and $k \in \mathbb{N} \setminus \{1\})$.

The following result extends Theorem (2.11) to a scenario where the behavior of q(z) on ∂U is unknown.

Corollary 2.14. Corollay 2.14. Let the domain of the function be equal to the unit circle and let z represent a univalent function in U with z(0) = 1. For every p in (0, 1), let the function ϕ(z)=(1-p)z + pz 2 be defined in the domain of attraction of the set. Assured that f(z) is in the domain of A(n) and that q_p fulfills the necessary requirements:

$$\operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q_p'(\zeta)}\right\} \geq 0\,, \qquad \left|\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{z q_p'(\zeta)}\right| \leq kM(\quad k \in \quad N \quad \setminus \quad \{1\};M \quad > 0, \tag{2.25}$$

and

$$\phi\left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+2} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+3} f(z)}{z}; z\right) \in \Omega,$$

(2.26)

Then

$$\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \prec q(z), \qquad \left(z \in U, \zeta \in \partial U \setminus E(q_p) \text{ and } k \in \mathbb{N} \setminus \{1\}\right).$$

Proof . Following Theorem (2.11),

$$\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \prec q_p(z).$$

As a result, the proof of Corollary (2.13) can be deduced from the following property of subordination.:

$$q_p(z) \prec q(z).$$

The proof of Corollary (2.13) is now concluded.

Corollary 2.15.states the following: If z is in the domain of attraction of the set C, then the probability of the state of affairs is: if z is in the interior of the circle, then the probability is 1/16. Let h,q_p for some p∈(0,1) where q_p (z)=q(pz) If f(z)∈A(n) and q_p have the following properties:

$$\operatorname{Re}\left\{\frac{\zeta q_p''(\zeta)}{q_p'(\zeta)}\right\} \geq 0, \qquad \left|\frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{z q_p'(\zeta)}\right| \leq kM (k \in \mathbb{N}\setminus\{1\}; M > 0,$$

(2.27)

and

$$\phi\left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+2} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+3} f(z)}{z}; z\right) \prec h(z), \tag{2.28}$$

then

$$\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \prec q(z), \qquad \left(z \in U, \zeta \in \partial U \setminus E(q_p) \text{ and } k \in \mathbb{N} \setminus \{1\}\right).$$

The next Theorem yields best dominant of the differential subordination

Theorem 2.16. Let h(z) be a function that is univalent in U and satisfies (2.23) in C, and let ψ be defined by (2.23). The equation of the differential operator

$$\psi(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z) = h(z), \tag{2.29}$$

If there exists a solution $q(z) \in Q_1 \cap \mathcal{H}$, where f∈A(n) satisfies condition (2.24) and

$$\phi\left(\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+2} f(z)}{z}, \frac{W_{\alpha,\beta}^{k,\delta+3} f(z)}{z};;z\right), \tag{2.30}$$

is analytic in U, then (2.24) implies that

$$\frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \prec q(z)$$

and $q(z)$ is the best dominant .

Proof. Through the use of Theorem (2.11), we can deduce that q(z) is the most significant term in the presence of condition (2.24). Because q(z) satisfies (2.29), it is also a solution of (2.24), thus, all of the dominants will have a chance to succeed. As a result, q(z) is the most significant dominant..

References

[1] J .A .Antonion and S. S. Miller , Third – order differential inequalities and subordinations in the complex plane , complex Var .Elliptic Equ. 56(2011),439-545.

[2] A. K. Wanas, New differential operator for holomorphic functions, Earthline

Journal of Math-ematical Sciences, 2(2) (2019), 527-537.

[3] S.S. Miller ,P.T. Mocanu ,Differential subordinations and univalent functions ,Michigan Math. J.28(1981),157-171.

[4] S.S. Miller ,P.T. Mocanu ,Differential subordinations Theory and Applications ,series on Monographs and Text books in pure and Applied Math ,No .225,Marcel Dekker Inc. Newyork ,Basel ,2000.

[5] De Branges, A proof of the Bieberbach conjecture, Acta Math, 154(1985), 137-152.

[6] T. Bulboaca, Differential subordinations and superordinations , Recent Results, House of Scientific Book publ., Cluj-Napoca , (2005) .

[7] A.W. Goodman, Univalent Functions, Vol. III, Polygonal publishing House Washington, New Jersey, (1983).

[8] S.S. Miller , P.T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J., 28 (1981), 157-171 .

[9] Ch. Pommerenke, Univalent Function. Vandenhoeck and Ruprecht, Gottingen, 1975 .

[10] S. Porwal and M. Darus, On a new subclass of bi-univalent functions, J. Egyptain Math. Soc., 21 (3) (2013), 190-193 .

[11] A.W. Goodman, Univalent Functions , Vol. I, II, Mariner, Tampa, FL, (1983)