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Gompertz Topp–Leone invers Rayleigh Distributions: Some Properties and Application

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Keywords:
Gompertz Topp–Leone G family, Invers Rayleigh distributions, Hazard rate, Moments, and Estimation.

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Abstract: This article also introduces the Gompertz-Topp-Leone inverse Rayleigh distribution and a one-parameter probability model, the statistical properties of which are thoroughly examined. Expressions for its moments as well as moment- producing, quantile, reliability, and hazard functions. Using the greatest likelihood technique, we estimate the GoTLIR distribution's parameters (MLE).

The model's hazard function has an inverted bathtub form, and we suggest the GoTLIR distribution's utility. The inverse Rayleigh distributions of Gompertz Topp–Leone have provided a generic formulation for the new distributions' density and distribution. Function Moments and hazard rate expressions have also been provided. Voda made the IR distribution suggestion in.(1972)

An application has been with a real lifetime data from engineering. The following data represent the tensile strength, of 74 carbon fibers tested under tension at gauge lengths of 20mm, available in Bader and Priest (1982) [17].

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بعض خصائص توزيع جمبيرتز توب ليون معكوس رايلي مع تطبيق

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المستخلص

في هذا البحث سنقدم توزيعا جديدا يتكون من ثلاث معالم ويسمى بتوزيع جمبيرتز توب ليون معكوس رايلي وتمتاز بمرونة عالية وملائمة جيدة لنمذجة البيانات. كما ويمكن دمج التوزيع معكوس رايلي بمعالم أكثر لتوليد العديد من التوزيعات الجديدة. ثم نقوم بتوسيع التوزيع وإعادة كتابة كل من دالة التوزيع التراكمي CDF ودالة الكثافة الاحتمالية PDF لتوزيع جديد وايضا يعطي التوزيع الجديد اشكال بيانية لدالة الكثافة الاحتمالية وتكون ممتدة أكثر او الملتوية لليسار إذا كانت الالتواء سالب او تكون ممتدة لليمين إذا كان الالتواء موجب. كما نقدم العديد من الخصائص الاحصائية للتوزيع الجديد مثل العزوم والدالة المولدة للعزوم والدالة التجزيئية والدالة البقاء ودالة المخاطرة وبعض الدوال الاخرى. وكذلك تم تطبيق طريقة الامكان الاعظم (MLE) لتقدير معالم التوزيع الجديد. ولبيان مدى ملائمة ومرونة التوزيع الجديد تم استخدام مجموعة من البيانات الحقيقية والتي تتعلق بقوة الشد لـ 74 ألياف كربون تم اختبارها تحت الشد بطول قياس 20 مم، الذي قدم به الباحثان (Bader and Priest) في عام (1982) [17].

حيث اثبت التوزيع الجديد ملائمة عالية ومرونة جيدة في نمذجة للبيانات الحقيقية بالمقارنة مع بعض التوزيعات المعروفة باستخدام بعض المعايير الاحصائية.

الكلمات المفتاحية: عائلة جمبيرتز توب ليون، العزوم، دالة الخطر، دوال مولدة للعزوم، تقدير المعالم باستخدام دالة الإمكان الأعظم، توزيع معكوس رايلي.

1. Introduction.

Nowadays, in probability theory, both the reliability function and the survival function have the same property, which is the measurement of the life span of a particular system or living organism. The flexible distribution is considered to fit different lifetime data. These distributions have more flexibility than the base line distribution because they add one or more shape parameters. Furthermore, new distribution can take many different shapes. Many families and distributions have been established and studied by researchers.

In the literature on statistical modeling, several lifespan distributions are examined.

By changing existing distributions or combining well,known distributions. researchers are producing new generalized and extended distributions. The goal of this research is to

increase the goodness-of- fit and flexibility of data modeling by augmenting

the current classical distributions.One of the distributions that has found use in disciplines
like biology and marketing science is the Gompertz distribution (Gompertz [1]).

Recently, some authors have done this. proposed modifications to the Gompertz distribution that are more adaptable and may be used to represent,a variety of real data sets.Jafari et al. [2] proposed the Beta-Gompertz distribution;

El- Dames et al. [3] the odd generalized exponential Gompertz distribution.

Alizadeh et al. [4] The Gompertz-G family of distributions and the transmuted
Gompertz-G family

$$F_{GoTLIR}(x;\gamma,b,\zeta) = \left[1 - e^{\left\{\frac{1}{\gamma}(1-(1-[1-\bar{G}^2(x,\zeta)]^b)^{-\gamma}\right\}}\right] \quad (1)$$

Where $\bar{G}(x;\zeta) = 1 - G(x;\zeta) \Longrightarrow \bar{G}^2(x;\zeta) = (1 - G(x;\zeta))^2$ and respectively, for γ, $b > 0$ and ζ is the parameter vector.

$$f_{GoTLIR}(x;\gamma,b,\zeta) = 2b\mathrm{g}(x;\zeta)\bar{\mathrm{G}}(x;\zeta)(1 - \bar{\mathrm{G}}^2(x;\zeta))^{b-1}[1 - (1 - \bar{\mathrm{G}}^2(x;\zeta)^b]^{-\gamma-1}\left(e^{\left\{\frac{1}{\gamma}(1-(1-[1-\bar{G}^2(x,\zeta)]^b)^{-\gamma}\right\}}\right) \quad (2)$$

respectively, for γ, b > 0 and ζ is the parameter vector. We are motivated by the usefulness

of the Gompertz-G and Topp-Leone-G family of distributions to propose a new family of distributions which is a combination of these two distributions. The creation of heavy tailed distributions for modeling various real data sets, the acquisition of special models that exhibit

monotonic and non-monotonic hazard rate functions, and the provision of consistently better fits than other distributions are the main goals of the new family of distributions. broader distributions based on the same fundamental principle. The inverse Rayleigh distribution, sometimes called the one-parameter inverse Rayleigh distribution, Rayleigh (IR) distribution has been named, and its many statistical characteristics have been examined. Entropy and mode expressions have also been developed. Additionally, it includes a few well-known specific instances. Additionally, by using the

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maximum likelihood approach to estimate the parameter and applying real data sets, we demonstrate the superiority of this novel distribution.

$$F(x;\tau) = e^{(-\tau x^{-2})} \quad \text{Where} \quad x > 0 \;\; , \tau > 0 \qquad (3)$$

$$f(x;\tau) = \frac{2\tau}{x^3} e^{(-\tau x^{-2})} \quad \text{Where} \quad x > 0 \, . \quad \tau > 0 \qquad (4)$$

We are inspired to suggest a distribution because:

- ❖ It may be viewed as a suitable distribution to represent the skewed data included in the literature, which may not be well suited.
- ❖ It may be trusted to construct additional unstudied distributions with better outcomes and with the same or different fact.
- ❖ It can be used to represent a wide range of real-world data sets in the fields of survival and industrial dependability.

2. Gompertz Topp - Leone Inverse Rayleigh Distribution:

We express the cdf and pdf of GoTLIR distribution as a mixture of an inverse Rayleigh distribution which is useful in presenting the mathematical properties of the GoTLIR distribution. By inserting Eq. (3) in Eq. (1) and by inserting Eq. (4) in Eq. (2) we have a cdf Eq. (5) and Eq. (6) of new distribution as follows.

$$F_{GoTLIR}(x;b,\gamma,\tau) = \left[1 - e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left[(1-e^{(-\tau x^{-2})})^2\right]\right]^b)^{-\gamma}\right\}}\right] \qquad (5)$$

$$x \;\; \geq 0 \qquad b,\gamma,\tau > 0$$

$$f_{GoTLIR}(x;b,\gamma,\tau) = 4b\frac{\tau}{x^3} e^{(-\tau x^{-2})}(1-e^{(-\tau x^{-2})})^2 *$$

$$(1-[(1-e^{(-\tau x^{-2})})^2])^{b-1} * \left[1-\left[1-\left[\left(1-e^{(-\tau x^{-2})}\right)^2\right]\right]^b\right]^{-\gamma-1}$$

$$* \left(e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left[\left(1-e^{(-\tau x^{-2})}\right)^2\right]\right]^b)^{-\gamma}\right\}}\right) \qquad (6)$$

The following is how the paper is structured: We create a new distribution family and give support for it. the density function's expansion We provide a few examples of special situations of Exponentiation via Gompertz-Topp-Leone-inverse Rayleigh GoTLIR. as well as the statistical features of the distributions. The Monte Carlo simulation results for

Gompertz inverse Rayleigh are provided. The suggested model is applied to real-world data sets, and closing thoughts are provided.

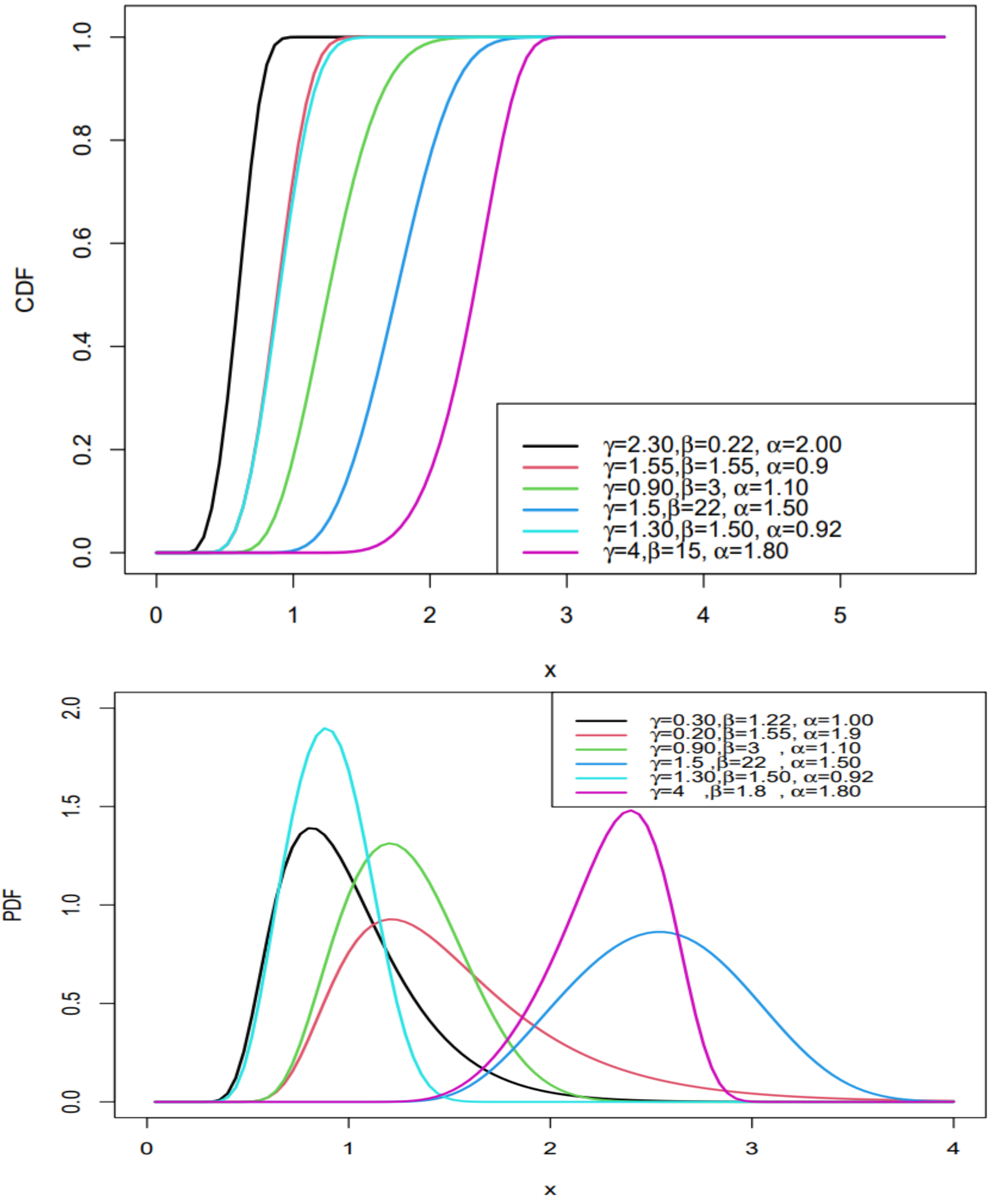

Figure 1,2: Different shapes of the cdf and pdf, with different value of parameters.

3. Expansion

Expansion of PDF

Gompertz Toppe - Leone inverse Rayleigh distribution from Eq. (6)

$$f_{GoTLIR}(x;b,\gamma,\tau) = \sum_{k,L,M,p,q=0}^{\infty} \frac{(-1)^{L+M+p+q}}{k!\,\gamma^k(q+1)} \binom{k}{L} \binom{-\gamma(L+1)-1}{M} \binom{b(M+1)-1}{p} \binom{1+2p}{q} * (q+1)\,\frac{4b\tau}{x^3} e^{(-\tau x^{-2})}.\left[e^{(-\tau x^{-2})}\right]^q$$

Where $\mho =$
$\sum_{k,L,M,p,q=0}^{\infty} \frac{(-1)^{L+M+p+q}(2b)}{k!\gamma^k(q+1)} \binom{k}{L} \binom{-\gamma(L+1)-1}{M} \binom{b(M+1)-1}{p} \binom{1+2p}{q}$

$$f_{GoTLIR}(x;b,\gamma,\tau) = \mho(q+1)\ \frac{2\tau}{x^3} e^{(-\tau x^{-2})}.\left[e^{(-\tau x^{-2})}\right]^q$$

$$f_{GoTLIR}(x;b,\gamma,\tau) = \mho(q+1)2\tau x^{-3}e^{-\tau(1+q)x^{-2}} \qquad (7)$$

The pdf and cdf function for Gompertz Topp - Leone inverse Rayleigh distribution

Expansion of CDF

Gompertz Topp - Leone inverse Rayleigh distribution from Eq.(5)

$$F_{GoTLIR}(x;b,\gamma,\tau) = \left[1 - e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left[\left(1-e^{(-\tau x^{-2})}\right)^2\right]\right)\right]^b)^{-\gamma}\right\}}\right]$$

$$F_{GoTLIR}(x;b,\gamma,\tau) = \left[1 - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{L=0}^{\infty}\sum_{Z=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^{n+m+k+L+z}}{m!\,\gamma^m q!} \binom{2L}{Z}\binom{bk}{L}\binom{-n\gamma}{k}\binom{m}{n}\tau^z x^{-2z}\right]$$

Where $\Psi =$
$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{L=0}^{\infty}\sum_{Z=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^{n+k+L+z}}{m!\gamma^m q!} \binom{2L}{Z}\binom{bk}{L}\binom{-n\gamma}{k}\frac{(-1)^n}{m!\gamma^m}\binom{m}{n}\tau^z x^{-2z}$

$$F_{GoTLIR}(x;b,\gamma,\tau) = \left[1 - \Psi x^{-2z}\right] \qquad (8)$$

The Survival and Hazard function of Gompertz Topp - Leone inverse Rayleigh distribution.

The Survival function is given by.

$$S_{GoTLIR}(x;b,\gamma,\tau) = 1 - \left[1 - e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left[\left(1-e^{(-\tau x^{-2})}\right)^2\right]\right)\right]^b)^{-\gamma}\right\}}\right]$$

$$= \left[e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left[\left(1-e^{(-\tau x^{-2})}\right)^2\right]\right)\right]^b)^{-\gamma}\right\}}\right] \qquad (9)$$

The Hazard function is given by.

$$H_{GoTLIR}(x;b,\gamma,\tau) = \frac{f_{GoTLIR}(x;b,\gamma,\tau)}{F_{GoTLIR}(x;b,\gamma,\tau)} = \left(\frac{4b\frac{\tau}{x^2}e^{\left(-\frac{\tau}{x}\right)}\left(1-e^{(-\tau x^{-2})}\right)^2}{\left[1 - e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left[\left(1-e^{(-\tau x^{-2})}\right)^2\right]\right)\right]^b)^{-\gamma}\right\}}\right]}\right) *$$

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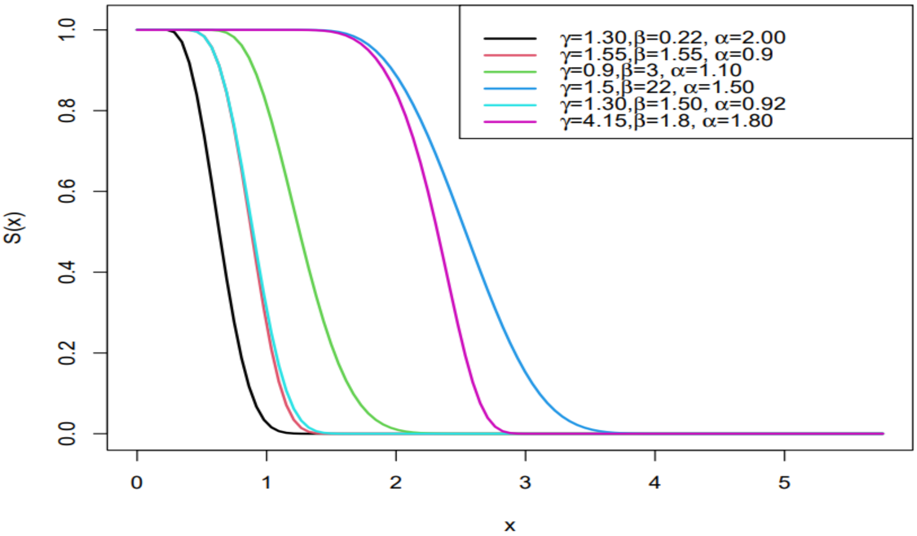
$$(1-\left[\left(1-e^{(-\tau x^{-2})}\right)^{2}\right])^{b-1}\left[1-\left[1-\left[\left(1-e^{(-\tau x^{-2})}\right)^{2}\right]^{b}\right]^{-\gamma-1}\left(e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left[\left(1-e^{(-\tau x^{-2})}\right)^{2}\right]^{b}\right)^{-\gamma}\right\}}\right)\right] \tag{10}$$



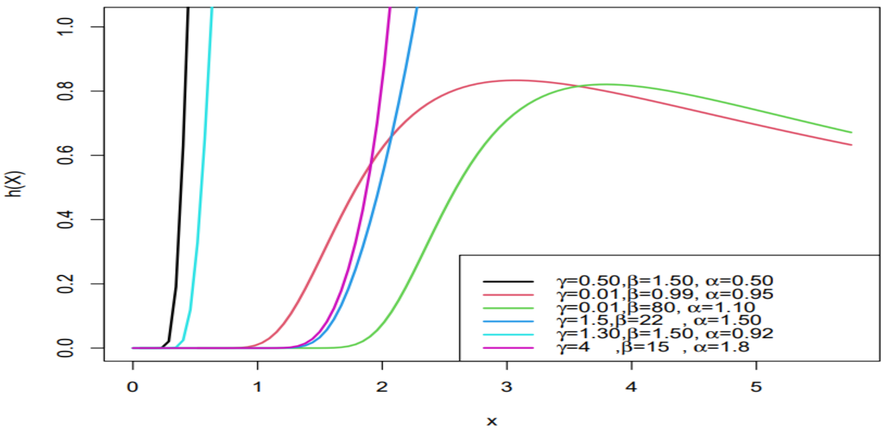


Figure 3,4: Different shapes of the Hazard and Survival function with different value of parameters.

4. Mathematical Properties.

Here, several mathematical characteristics of the inverse exponential distribution discovered by Gompertz, Topp, and Leone are derived. The quantile function (QF), Moments, order statistics, the characteristic function,

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the moment generating function, R'enyi entropy, and stress strength are some of these characteristics.

4.1 Moments

Moments play a key role in identifying and calculating a number of statistical properties, including smoothing and dispersion, coefficient of variation, standard deviation, and others. The GoTLIR distribution's rth moments may be calculated using the following connection, which is written as:

$$E(X^r) = \int_0^\infty x^r f_{GoTLIR}(x; b, \gamma, \tau)dx \tag{11}$$

Using the PDF of GoTLIR in Eq. (10), we get

$$E(X^r) = \mho \int_0^\infty 2\tau x^r \ (q+1)x^{-3} e^{-\tau(1+q)x^{-2}} \ dx$$

Let $y = \tau(1+q)x^{-2} \Longrightarrow x = \left(\tau(1+q)\right)^{\frac{1}{2}} y^{-\frac{1}{2}} \Longrightarrow dy = -2\tau(1+q)x^{-3}dx$: then after some mathematical ways we can cut back Eq. (14) to

$$E(X^r) = \mu_r = \mho \int_0^\infty \left[\left(\tau(1+q)\right)^{\frac{1}{2}} y^{-\frac{1}{2}}\right]^r e^{-y} dy$$

$$\Longrightarrow \mho \left[\left(\tau(1+q)\right)^{\frac{1}{2}}\right]^r \int_0^\infty y^{-\frac{r}{2}} e^{-y} dy$$

The end result as.

$$\mu_r = E(X^r) = \mho \left[\left(\tau(1+q)\right)^{\frac{1}{2}}\right]^r \ \Gamma(1 - \frac{r}{2}) \tag{12}$$

Many statistical concepts, including mean, the fourth moments, variance, CV, moment generating function (MGF), and others, may be found using equation (12).

4.2 Moment Generating Function.

In this subsection we derive the moment generating function for the Gompertz

Topp - Leone Inverse Exponential distribution [13],[6]. The MGF is defined as:

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$$M_x(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!} \int_0^{\infty} x^s f(x) dx \tag{13}$$

For the given distribution, we have

Were $\sum_{\mathbf{s=0}}^{\infty} \frac{\mathbf{t^s}}{\mathbf{s!}} = Å$

$$\mu_r = E(X^r)_{GOTLIE}$$

$$= Å℧ \left[\left(\tau(1+q) \right)^{\frac{1}{2}} \right]^s \Gamma\left(1 - \frac{s}{2}\right) \tag{14}$$

4.3 Quantile Function.

The quantitative function is the inverse of the cumulative distribution function, and it is one of the approaches for finding the probability function. It is used to compute the median, skewness, and oblateness of distributions in the absence of torque or a large deviation number. In order to examine simulations, data must be created. This is a mathematical representation of it. [22],[23]

$$u = F^{-1}(x) \tag{15}$$

$$u = 1 - e^{\left\{ \frac{1}{\gamma} \left[(1-(1-\left[1-\left(1-e^{-\tau(x)^{-2}}\right)^2\right]^b)^{-\gamma}) \right] \right\}}$$

$$1 - e^{-\tau(x)^{-2}} = \left[1 - \left(1 - ((1-\gamma\log(1-u)))^{\frac{1}{-\gamma}} \right)^{\frac{1}{b}} \right]^{\frac{1}{2}}$$

$$1 - \left[1 - \left(1 - ((1-\gamma\log(1-u)))^{\frac{1}{-\gamma}} \right)^{\frac{1}{b}} \right]^{\frac{1}{2}} = e^{-\tau(x)^{-2}}$$

$$(x)^{-2} = \frac{-1}{\tau} \log 1 - \left[1 - \left(1 - ((1-\gamma\log(1-u)))^{\frac{1}{-\gamma}} \right)^{\frac{1}{b}} \right]^{\frac{1}{2}}$$

which implies

$$Q(x)_{\text{GOTLIE}} = \left(\tau^{-1} \log 1 - \left[1 - \left(1 - ((1-\gamma\log(1-u)))^{\frac{1}{-\gamma}} \right)^{\frac{1}{b}} \right]^{\frac{1}{2}} \right)^{-2} \quad ; 0 < u < 1 \tag{16}$$

Equation (16) very important to study simulation of the GoTLIR distribution where U is $Uniform$ (0,1).

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Hence, the median of the distribution is derived by substituting $u = 0.5$ in Equation (16)

$$\text{Median}(x)_{\text{GoTLIR}} = \left(a^{-1} \log 1 - \left[1 - \left(1 - \left((1 - \gamma \log(1 - 0.5)) \right)^{\frac{1}{-\gamma}} \right)^{\frac{1}{b}} \right]^{\frac{1}{2}} \right)^{\frac{1}{\beta}} \quad (17)$$

4.4 Order Statistics

If we have a sample of n symmetrically distributed independent random variables (X1 X2,... Xn) from the GoTLIR distribution with a probability density function according to equation (6) and a cumulative distribution function according to equation (5), then ordered statistics Xi:n arranges the sample from smallest to largest value Xi:n. The order statistics are defined as follows.[11]

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x).[F(x)]^{i-1}.[1 - F(x)]^{n-i} \quad (18)$$

$$g_{i:n}(x) = \sum_{j=0}^{n-i} W(-1)^j \binom{n-i}{j} [F(x)]^{j+i-1} f(x)$$

Where $\mathrm{W} = \frac{\mathrm{n!}}{\mathrm{(i-1)!(n-i)!}}$

$$g_{i:n}(x)_{\text{GoTLIR}} = \sum_{j=0}^{n-i} W(-1)^j \binom{n-i}{j} * \left[1 - e^{\left\{ \frac{1}{\gamma} \left(1-\left(1-\left[1-\left(\left(1-\mathrm{e}^{-\tau(\mathrm{x})^{-2}}\right)^2\right)\right]^b\right)^{-\gamma}\right) \right\}} \right]^{j+i-1} * \left(4\tau b \mathrm{e}^{-\tau(\mathrm{x})^{-2}} (\mathrm{x})^{-3}\right)\left[1 - \mathrm{e}^{-\tau(\mathrm{x})^{-2}}\right] * \left[1 - \left(1 - \mathrm{e}^{-\tau(\mathrm{x})^{-2}}\right)^2\right]^{b-1} * \left[1 - \left(1 - \left(1 - \mathrm{e}^{-\tau(\mathrm{x})^{-2}}\right)^2\right)^b\right]^{-\gamma-1} * \left(e^{\left\{ \frac{1}{\gamma} \left(1-\left(1-\left[1-\left(1-\mathrm{e}^{-\tau(\mathrm{x})^{-2}}\right)^2\right]^b\right)^{-\gamma}\right) \right\}} \right) \quad (19)$$

It is quite helpful to utilize Equation (19) to get the first order when s=1 and the huge order when s=n.

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4.5 Characteristic Function.

The Characteristic Function is defined by $Q_x(t) = \int_0^\infty e^{itx} f(x)dx$. For the given Gompertz Topp - Leone inverse Rayleigh distribution, we have.

$$Q_{(X)}(t)_{\text{GoTLIR}} = E(e^{itx})$$

$$= \mho \int_0^\infty e^{itx}\ f_{GoTLIR}(x; b, \gamma, \tau)dx \qquad (20)$$

Where

$$\mho = \sum_{k,L,M,p,q=0}^{\infty} \frac{(-1)^{L+M+p+q}(2b)}{k!\,\gamma^k(q+1)} \binom{k}{L} \binom{-\gamma(L+1)-1}{M} \binom{b(M+1)-1}{p} \binom{1+2p}{q}$$

$$e^{it.x} = \sum_{z=0}^{\infty} \frac{i^z.t^z}{z!} x^z$$

Where $\sum_{z=0}^{\infty} \frac{i^z.t^z}{z!} x^z = \mathcal{H}$

$$Q_{(X)}(t)_{\text{GoTLIR}} = E(e^{itx}) = \mho\mathcal{H} \int_0^\infty x^z\ f_{GoTLIR}(x; b, \gamma, \tau)dx$$

$$Q_{(X)}(t)_{\text{GoTLIR}} = \mho\mathcal{H} \left[(\tau(1+q))^{\frac{1}{2}}\right]^z \Gamma(1 - \frac{z}{2}) \qquad (21)$$

4.6- R'enyi Entropy.

The amount of uncertainty for a random variable and the oscillation of any researched event is defined as entropy. It is employed in the field of statistics, for example, in statistical inference [14], [15]. The R'enyi Entropy is defined by $I_R(v) = \frac{1}{1-s} log \int_0^\infty f(x)^s dx$. For the Gompertz Topp - Leone Inverse Rayleigh distribution given distribution, we have

$$I_R(S) = \frac{1}{1-s} \log \int_0^\infty [f_{GoTLIR}(x; b, \gamma, \tau)]^s dx \qquad (22)$$

$$I_R(S) = \frac{1}{1-s} \log \int_0^\infty \Bigg[4b\frac{\tau}{x^3} e^{(-\tau x^{-2})}(1 - e^{(-\tau x^{-2})})^2 * (1 - [(1 - e^{(-\tau x^{-2})})^2])^{b-1} * \left[1 - \left[1 - [(1 - e^{(-\tau x^{-2})})^2]\right]^b\right]^{-\gamma-1} * \left(e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left[(1-e^{(-\tau x^{-2})})^2\right]\right]^b)^{-\gamma}\right\}}\right)\Bigg]^s dx$$

Following a few algebraic procedures to lower the R'enyi Entropy, we obtain.

$$I_R(S) = \frac{1}{1-s}\log\int_0^\infty (2b)^s \sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty\sum_{L=0}^\infty\sum_{p=0}^\infty \frac{s^i(-1)^{j+k+L+p}}{i!\,\gamma^i}\binom{i}{j}\binom{-\gamma(s+j)-s}{k}\binom{b(s+k)-s}{L}\binom{2L+s}{p} * (2\tau)^s(x)^{-3s}e^{-\tau(s+p)(x)^{-2}}dx$$

Where $\psi = (2b)^s \sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty\sum_{L=0}^\infty\sum_{p=0}^\infty \frac{s^i(-1)^{j+k+L+p}}{i!\gamma^i}\binom{i}{j}\binom{-\gamma(s+j)-s}{k}\binom{b(s+k)-s}{L}\binom{2L+s}{p}$

$$I_R(S) = \psi\frac{(2)^{s-1}(\tau)^s)[(\tau(s+p)]^{-3s+\frac{1}{2}}}{(1-s)}\log\int_0^\infty y^{(\frac{3s}{2}-\frac{1}{2})-1}(e^{-y})dy$$

$$= \psi\frac{(2\tau)^{s-1})[(\tau(s+p)]^{-3s+\frac{1}{2}}}{1-s}\log\Gamma\left(\frac{3s}{2}-\frac{1}{2}\right) \quad (23)$$

4.7 Stress - Strength

Stress strength can be obtained. For the Gompertz Topp - Leone inverse Rayleigh distribution given distribution, we have.[19],[10]

$$R = \int_0^\infty f_1(x)F_2(x)dx \quad (24)$$

From (7) and (8)

$$R = \int_0^\infty \mho(q+1)2\tau x^{-3}e^{-\tau(1+q)x^{-2}}[1-\Psi x^{-2z}]\,dx \quad (25)$$

$$R = \int_0^\infty \mho(q+1)2\tau x^{-3}e^{-\tau(1+q)x^{-2}} - \int_0^\infty \Psi\mho(q+1)2\tau x^{-3}x^{-2z}e^{-\tau(1+q)x^{-2}}\,dx$$

$$R = \mho[1-\Psi[\tau(q+1)]^Z\Gamma(z+1)] \quad (26)$$

4.8 Maximum Likelihood Estimation

The maximum likelihood approach is one of the best methods for estimating distribution parameters. In this stage of the study, we will calculate the partial derivative of the distribution's maximum likelihood function in order to estimate the parameters of the new distribution GoTLIR [16].

$$EP = \prod_{i=1}^n f_{\text{GoTLIR}}(x) \quad (27)$$

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$$EP = \prod_{i=0}^{n}\left[2bg(x;\zeta)\bar{G}(x;\zeta)(1 - \bar{G}^2(x;\zeta))^{b-1}[1-(1-\bar{G}^2(x;\zeta)^b]^{-\gamma-1}\left(e^{\left\{\frac{1}{\gamma}(1-(1-[1-\bar{G}^2(x,\zeta)]^b)^{-\gamma})\right\}}\right)\right]$$

$$\ln(EP) = n\log(2b) + \sum_{i=0}^{n}\ln g(x;\zeta) + \sum_{i=0}^{n}\ln\bar{G}(x;\zeta) + (b-1)\sum_{i=0}^{n}\ln\left(1-\bar{G}^2(x;\zeta)\right) -(\gamma+1)\sum_{i=0}^{n}\ln(1-(1-\bar{G}^2(x;\zeta)^b) + \sum_{i=0}^{n}\frac{1}{\gamma}(1-(1-(1-\bar{G}^2(x;\zeta)^b)^{-\gamma}))$$

$$EP = \prod_{i=0}^{n}\left[2bg(x;\zeta)\bar{G}(x;\zeta)(1 - \bar{G}^2(x;\zeta))^{b-1}[1-(1-\bar{G}^2(x;\zeta)^b]^{-\gamma-1}\left(e^{\left\{\frac{1}{\gamma}(1-(1-[1-\bar{G}^2(x,\zeta)]^b)^{-\gamma})\right\}}\right)\right]$$

$$\ln(EP) = n\log(2b) + \sum_{i=0}^{n}\ln(\frac{2\tau}{{x_i}^3}e^{(-\tau x^{-2})}) + \sum_{i=0}^{n}\ln(1-e^{(-\tau x_i{}^{-2})})^2 + (b-1)\sum_{i=0}^{n}\ln\left(1-(1-e^{(-\tau x_i{}^{-2})})^2\right) -(\gamma+1)\sum_{i=0}^{n}\ln\left(1-(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b\right) + \sum_{i=0}^{n}\frac{1}{\gamma}\left(1-(1-(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b)^{-\gamma}\right)$$

$$\frac{\partial(EP)}{\partial b} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln((1-e^{-\tau x_i^{-2}})^2) + \sum_{i=1}^{n}\frac{ln\left(1-(1-e^{(-\tau x_i{}^{-2})})^2\right)\left(1-(1-e^{(-\tau x_i{}^{-2})})^2\right)^b(-\gamma-1)}{1-(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b} - \sum_{i=1}^{n}\left(1-\left(1-(1-e^{(-\tau x_i{}^{-2})})^2\right)^b\right)^{-\gamma-1}(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b ln(1-(1 - e^{(-\tau x_i{}^{-2})})^2 \quad (28)$$

$$\frac{\partial(EP)}{\partial\tau} = \sum_{i=1}^{n}\frac{2}{x_i{}^2(e^{-\tau x_i{}^{-2}}-1)} - \sum_{i=1}^{n}\frac{2(b-1)}{x_i{}^2(e^{-\tau x_i{}^{-2}}+1)} + \sum_{i=1}^{n}\frac{2b(-\gamma-1)\left(1-(1-e^{(-\tau x_i{}^{-2})})^2\right)^{b-1}(1-e^{(-\tau x_i{}^{-2})})e^{(-\tau x_i{}^{-2})}}{x_i{}^2(1-(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b} - \sum_{i=1}^{n}2bx_i{}^{-2}((1-(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b)^{-\gamma-1}))\left(1-(1-e^{(-\tau x_i{}^{-2})})^2\right)^{b-1}(1 - e^{(-\tau x_i{}^{-2})})(e^{(-\tau x_i{}^{-2})}) \quad (29)$$

$$\frac{\partial(EP)}{\partial\gamma} = \sum_{i=1}^{n} ln(1-e^{(-\tau x_i{}^{-2})})^2 {}_{-} \sum_{i=1}^{n}\ln(1-(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b + \sum_{i=1}^{n}\frac{ln\,(1-(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b\gamma-(1-(1-(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b)^\gamma+1}{(1-(1-(1-(1-e^{(-\tau x_i{}^{-2})})^2)^b)^\gamma\gamma^2} \quad (30)$$

The Maximum Likelihood Estimation is obtained by equating equation numbers, (28), (29), and (29) with zero. The aforementioned equations have a convoluted form and are difficult to solve algebraically. As a result, we use numerical methods to solve them. R program Mathematical is a good candidate in most circumstances.

5 Application

In this part, we provide a real-world phenomenon for the GoTLIR distribution, which also fits better than other distributions. Their (NLL) negative log-likelihood is included in the comparison, (HQIC) Hanan and Quinn Information Criteria, (BIC) Bayesian Information Criteria, (CAIC) Consistent Akaike Information Criteria, (AIC) Akaike Information Criteria values,

The data fitting comparison between the Gompertz Toppe – Leone inverses Rayleigh distributions (GoTLIR). and other distributions such as, Kumaraswamy Inverse Rayleigh distributions (KuIR), Exponentiated Generalized Inverse Rayleigh distributions (EGIR), Beta Inverse Rayleigh distributions (BeIR), Gompertz Inverse Rayleigh distributions (GoIR),

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Truncated-Exponential Skew-Symmetric Distribution (TESSIR)[18], and Inverse Rayleigh distributions (IR).

6 The data.

An application and the goodness of fit of the PAD have been explained with a real lifetime data from engineering. The following data represent the tensile strength, measured in GPa, of 74 carbon fibers tested under tension at gauge lengths of 20mm, available in Bader and Priest (1982). [17]

(1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585).

Table (1): MLE of GOTLIR for 74 carbon fibers tested under tension at gauge lengths of 20 mm.

Distribution	**Parameters estimation**		
	$\hat{\alpha}$	$\hat{b}$	$\hat{\gamma}$
GoTLIR	2.1236123	0.7817069	3.4980449
BeIR	0.2700802	84.144088	6.6775424
KuIR	0.7971387	11.029227	4.5370669
EGIR	74.998221	0.3049921	6.2920777
GoIR	0.3132611		4.2490645
TESSIR	13.53278		4.17200
IR	2.370785		

Table (2): -LL, AIC, CAIC, BIC. for (74) carbon fibers tested under tension at gauge lengths of 20mm

Distribution	**-LL**	**AIC**	**CAIC**	**BIC**	**HQIC**
GoTLIR	51.48	108.96	109.30	115.87	111.72
BeIR	51.86	109.72	110.06	116.63	112.48
KuIR	55.57	117.14	117.49	124.06	119.90
EGIR	51.94	109.89	110.23	116.80	112.64
GamIR	89.32	182.65	182.82	187.26	184.49
TESSIR	56.59	117.19	117.36	121.80	119.03
IR	141.3	284.61	284.66	286.91	285.53

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Table (3): Statistics

Statistics		
DATA		
N	Valid	74
Percentiles	25	2.1295
	50	2.5125
	75	2.8180

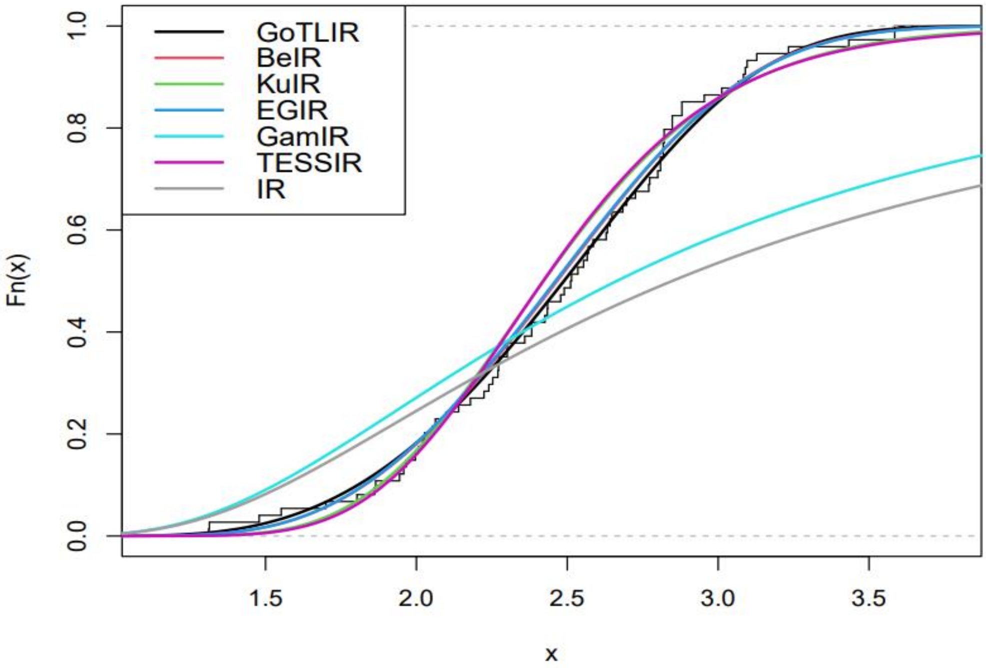
Table (4): Statistics

Statistics		
Data		
N	Valid	74
Mean		2.4773
Median		2.5125
Skewness		-0.157
Std. Error of Skewness		0.279
Kurtosis		0.033
Std. Error of Kurtosis		0.552
Maximum		3.59
Percentiles	25	2.1295
	50	2.5125
	75	2.8180

Table (5): Tests of Normality

Tests of Normality						
	Kolmogorov-Smirnov[a]			**Shapiro-Wilk**		
		df	**Sig.**		**df**	**Sig.**
DATA	0.056	74	0.200*	0.988	74	0.728

The tables numbered (1) (2) (3) (4) and (5) were created by the researchers using the R statistical program.



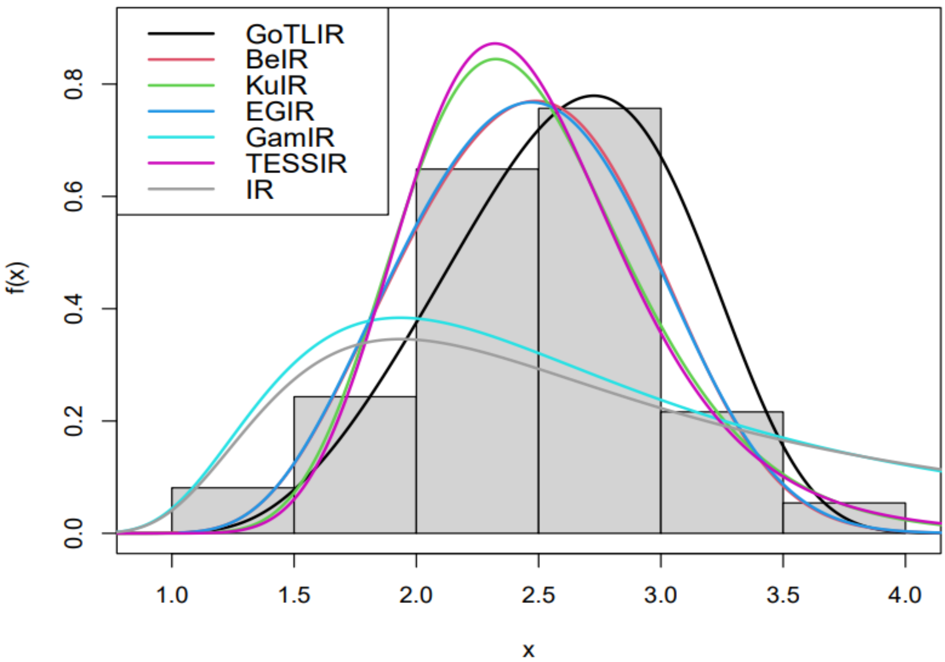


Fig. 1: fitted densities. Fig. 2: fitted densities of eight distribution.

The researchers provided both figures, which were created using program R

7 Concluding Remarks

The Gompertz Topp Leone Inverse Rayleigh GoTLIR distribution was successfully generated in this work, and its many statistical characteristics were examined. The model's shape might be ascending, falling, or unimodal. It is suggested to use the greatest likelihood method to estimate unidentified model parameters. Failure rates can be used to simulate and explain actual occurrences such bathtubs, inverted bathtubs, and bathtub increases and decreases. The GoTLIR distribution is discovered to be an advancement and a superior choice than other distributions when used with actual data sets.

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