Tikrit Journal of Administrative and Economic Sciences, Vol. 19, No. Special Issue, Part (5): 160-174
Doi: www.doi.org/10.25130/tjaes.19.SP3.5.11

Tikrit Journal of Administrative and Economics Sciences
مجلة تكريت للعلوم الإدارية والاقتصادية
ISSN: 1813-1719 (Print)

Gompertz Inverse Weibull Distribution, some statistical properties with Application Real Dataset

Amany F. Khubaz [A], **Moudher Kh. Abdal-Hameed** [B],
Noor H. Mohamood [C], **Mundher A. Khaleel** [C]

[A] Salah al-Din Education Directorate, Salah al-Din, Tikrit, Iraq
[B] College of Administration and Economics. Tikrit University
[C] Collage of Computer Sciences and Mathematics. Tikrit University

Keywords:
Generalized family, Gompertz family of distributions, Inverse Weibull, Maximum likelihood estimation.

Article history:

Received	18 Apr. 2023
Accepted	27 Apr. 2023
Available online	30 Aug. 2023

©2023 College of Administration and Economy, Tikrit University. THIS IS AN OPEN ACCESS ARTICLE UNDER THE CC BY LICENSE
http://creativecommons.org/licenses/by/4.0/

*Corresponding author:
Amany F. Khubaz
Salah al-Din Education Directorate, Salah al-Din, Tikrit, Iraq

Abstract: In this paper, we proposed a new distribution named Gompertz Inverse Weibull (GoIW) by using the Gompertz generalized family of distributions. The study extensively examined the significant properties of the GoIW distribution and utilized the maximum likelihood estimation technique to estimate its model parameters. A real data on rainfall was used to demonstrate the distribution's importance and compare it to commonly used lifetime distributions. The research results indicate that the GoIW distribution outperforms other popular distributions like Weibull Inverse Weibull (WeIW) and many others, in terms of fitting performance.

Doi: www.doi.org/10.25130/tjaes.19.SP3.5.11

توزيع كومبيرتز معكوس ويبل: بعض الخصائص الإحصائية مع تطبيق بيانات حقيقية

اماني فائق خباز	مظهر خالد عبد الحميد	نور حسن محمود	منذر عبدالله خليل
مديرية تربية صلاح الدين	كلية الادارة والاقتصاد جامعة تكريت	كلية علوم الحاسوب والرياضيات جامعة تكريت	كلية علوم الحاسوب والرياضيات جامعة تكريت

المستخلص

في هذا البحث، تم تقديم توزيع جديد يسمى توزيع كومبيرتز معكوس ويبل من خلال استنتاجه من عائلة توزيعات كومبيرتز العامة. تمت مناقشة الخصائص الإحصائية الأساسية لتوزيع كومبيرتز معكوس ويبل بشكل موسع في البحث، وتم تقدير معلمات النموذج باستخدام طريقة الامكان الاعظم. تم توضيح أهمية التوزيع المقترح من خلال تحليل مجموعة بيانات حقيقية للأمطار ومقارنتها مع توزيعات بقاء مختلفة مدروسة سابقا. علاوة على ذلك، قدم التوزيع الجديد تفوق عالي في الاداء والمطابقة مقارنة مع توزيعات ويبل مثل معكوس ويبل والعديد من التوزيعات.

الكلمات المفتاحية: العوائل المعممة، عائلة توزيع كومبيرتز، معكوس ويبل، طريقة الامكان الاعظم.

1. Introduction

Keller et al. (1985) introduced the Inverse Weibull (IW) distribution. To model datasets with an inverted bathtub failure rate. Unlike the Weibull distribution, the IW distribution does not have a constant failure rate, which makes it suitable for a range of real-world events in engineering, medicine, and biology, particularly those exhibiting an inverted bathtub failure rate. Researchers, such as Calabria and Pulcini (1989), Jiang et al. (2001), De Gusmao et al. (2011), Hanook et al. (2013), and numerous others, have explored the Inverse Weibull distribution's usefulness in detail.

In recent years, there has been a trend in research to enhance the modeling capability of existing probability distributions. One specific area of focus has been the extension of the IW distribution. Various studies have attempted to achieve this, including the works of Basheer (2019), Almarashi et al. (2020), Sindhu and Atangana (2021), and Afify et al. (2021). These studies have used family to modify the (IW) distribution. A lot of families study the IW, and can be found in Eugene et al. (2002), Oguntunde et al. (2018), Oguntunde et al. (2019), Khaleel et al. (2020), Ahmed et al. (2020), and Khaleel et al. (2022). Notably, the Beta generalized family of distribution, which was the first family of dist. developed from the logit of a random variable, is included in this list.

The Gompertz (Go) family is the main focus of this paper due to its recent development and high level of flexibility. This family of distributions is simpler to work with compared to the Beta (Be) generalized family of dist., as it does not involve complex functions. Previous studies by Alizadeh et al. (2017), Abdal-hameed et al. (2018), Oguntunde et al. (2018), Oguntunde et al. (2019), ADEMOLA et al. (2021), Bodhisuwan and Aryuyuen (2021), Al-Noor and Khaleel (2021), Al-Noor et al. (2022), and Khalaf and Khaleel (2022) have also utilized this family of distributions. The Gompertz family has a cumulative distribution function (cdf) and probability density function (pdf) given by:

$$F_{Go-G}(x) = 1 - e^{\frac{s}{\tau}\{1-[1-G(x)]^{-\tau}\}}; \qquad x > 0, s > 0, \tau > 0 \tag{1}$$

$$f_{Go-G}(x) = sg(x)[1 - G(x)]^{-\tau-1} e^{\frac{s}{\tau}\{1-[1-G(x)]^{-\tau}\}} \tag{2}$$

The Gompertz family of distributions includes supplementary shape parameters s and τ and is formed using the subsequent transformation:

$$F(x) = \int_0^{-log[1-G(x)]} w(t) \tag{3}$$

The Gompertz distribution is characterized by its probability density function $w(t)$, with T being a random variable. The cdf and pdf of the baseline distribution are represented by $G(x)$ and $g(x)$, respectively, and are associated with the inverse Weibull (IW) distribution, which can be expressed as follows:

$$G_{IW}(x) = e^{-\psi x^{-\varphi}} \tag{4}$$

and (5)

$$g_{IW}(x) = \psi\varphi\, x^{-\varphi-1}\; e^{-\psi x^{-\varphi}}$$

The development of new distributions serves two primary purposes:

- ❖ To achieve greater flexibility than existing sub-models.
- ❖ To achieve a better fit than sub-models such as the Beta inverse Weibull (BeIW) and Exponentiated generalized inverse Weibull (EGIW).

The objectives of this paper are:

- ❖ To demonstrate that the new model is more flexible than its sub-models by conducting a goodness-of-fit test on real data.

- ❖ To introduce a new distribution, the Gompertz IW distribution.
- ❖ To examine and analyze various mathematical properties of the Gompertz IW distribution.

This paper is organized as: In Section 2, the cdf and pdf of the GoIW distribution are defined. Basic statistical properties, such as the reversed hazard function (rhf), hazard function (hf), survival function (S), and odds (O) function, are presented in Section 3. Mathematical properties of distribution, such as the Quantile Function (QF) and Median, are discussed in Section 4, and order statistics are covered in Section 5. The maximum likelihood estimation (MLE) method for estimating the parameters is presented in Section 6. In Section 7, the practicality of the new model is demonstrated using a real dataset. Finally, Section 8 concludes the paper with some remarks.

2. The new distribution Gompertz Inverse Weibull

To obtain the cdf of the Gompertz Inverse Weibull (GoIW) distribution, Eq(4) is substituted into Eq(1), yielding:

$$F(x) = 1 - e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}} \quad , s, \psi, \tau, \varphi > 0 \qquad (6)$$

The pdf of the Gompertz Inverse Weibull (GoIW) distribution is obtained by plugging Eqs (4) and (5) into Eq (2), yielding:

$$f(x) = s\psi\varphi\, x^{-\varphi-1}\; e^{-\psi x^{-\varphi}}\left[1 - e^{-\psi x^{-\varphi}}\right]^{-\tau-1} e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}; \; s, \psi, \tau, \varphi > 0 \qquad (7)$$

The GoIW distribution has four parameters: s, ψ, τ and φ, the parameters s and τ is the shape of the Go distribution, while ψ and φ are the parameters of the baseline IW distribution. Figures 1 and 2 showcase the range of shapes that the pdf and cdf of the GoIW distribution can take, depending on the values of parameters s, ψ, τ, and φ. The pdf of GoIW can exhibit reversed-J, right-skewed, symmetric, or left-skewed shapes. Similarly, Figure 2 highlights the flexibility of the GoIW distribution's cdf, which can take on a wide range of shapes.

Doi: www.doi.org/10.25130/tjaes.19.SP3.5.11

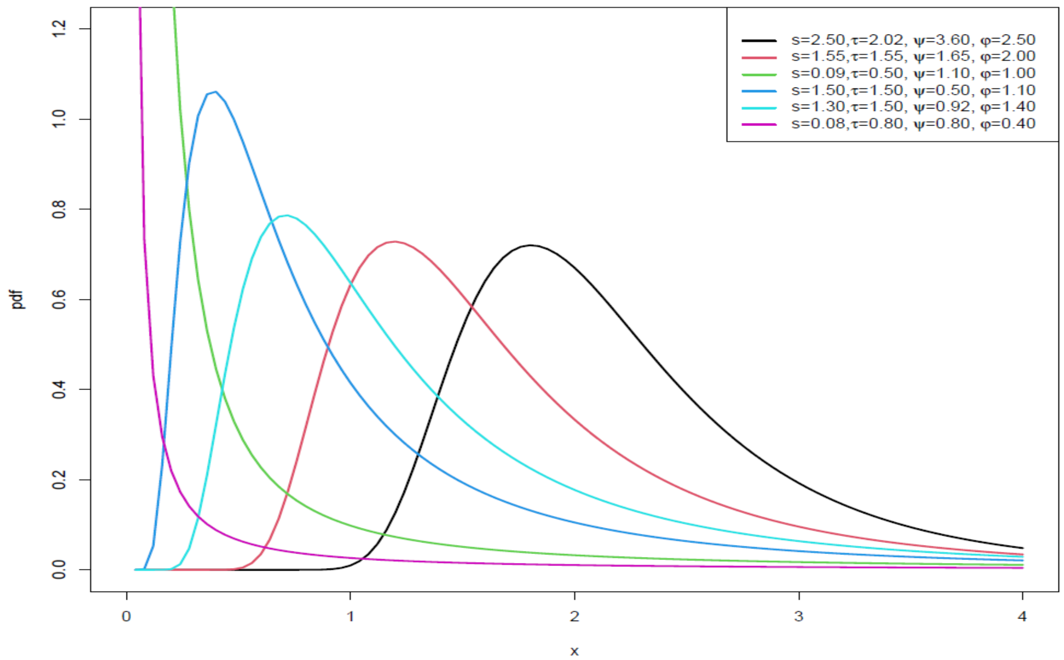

Figure (1): Some shapes of PDF of the GoIW

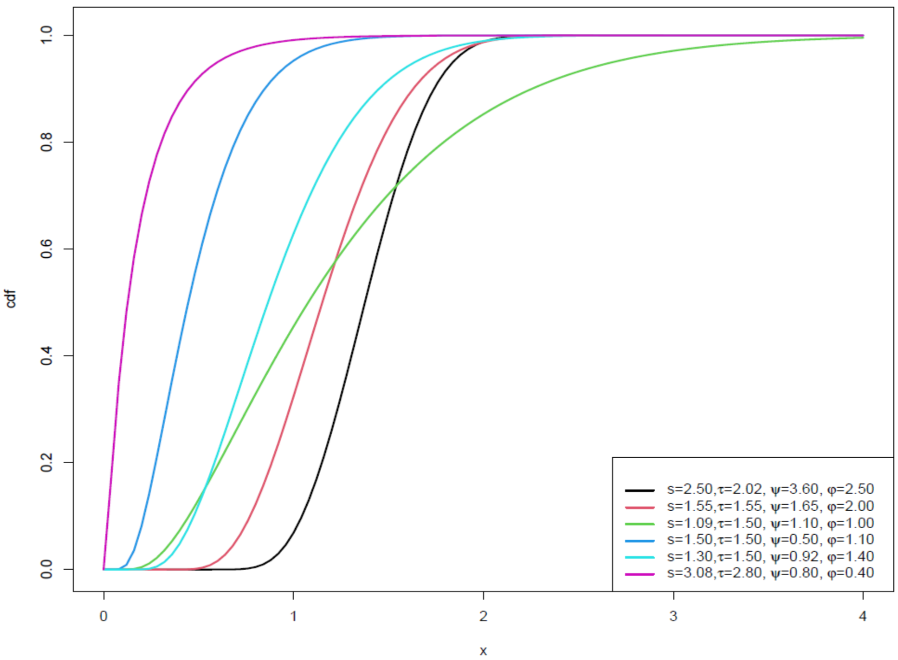

Figure (2): Displays some cdf shapes of the GoIW distribution

3. Several Fundamental Properties of GoIW

To establish some fundamental statistical properties of the GoIW distribution, we start by calculating the Survival (S) function using the formula provided by Abdal (2020): $S(x) = 1 - F(x)$

The S function of the GoIW distribution can be derived as follows:

$$S(x) = e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}} \tag{8}$$

Dividing the pdf presented in Eq(7) by the S function given in Eq(8) yields the failure rate of the GoIW distribution, as follows:

$$h(x) = \frac{s\psi\varphi\, x^{-\varphi-1}\;\; e^{-\psi x^{-\varphi}}\left[1 - e^{-\psi x^{-\varphi}}\right]^{-\tau-1} e^{\frac{s}{\beta}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}}{e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}}$$

The hf can be expressed as follows after performing some algebraic manipulations:

$$h(x) = s\psi\varphi\, x^{-\varphi-1}\;\; e^{-\psi x^{-\varphi}}\left[1 - e^{-\psi x^{-\varphi}}\right]^{-\tau-1} \quad (9)$$

The rhf for the GoIW can be derived by dividing the pdf from Eq(7) by the cdf from Eq(6), as presented by Abdal-hameed et al. (2018), resulting in:

$$r(x) = \frac{s\psi\varphi\, x^{-\varphi-1}\;\; e^{-\psi x^{-\varphi}}\left[1 - e^{-\psi x^{-\varphi}}\right]^{-\tau-1} e^{\frac{s}{\beta}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}}{1 - e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}} \quad (10)$$

The O function can be derived by dividing the cdf in Eq(6) by the S function in Eq(8), resulting in:

$$O(x) = \frac{1 - e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}}{s\psi\varphi\, x^{-\varphi-1}\;\; e^{-\psi x^{-\varphi}}\left[1 - e^{-\psi x^{-\varphi}}\right]^{-\tau-1}} \quad (11)$$

The GoIW distribution can be used to derive various distributions by setting one parameter equal to 1 or 2. Figure 4 displays some of the related distributions that are possible.

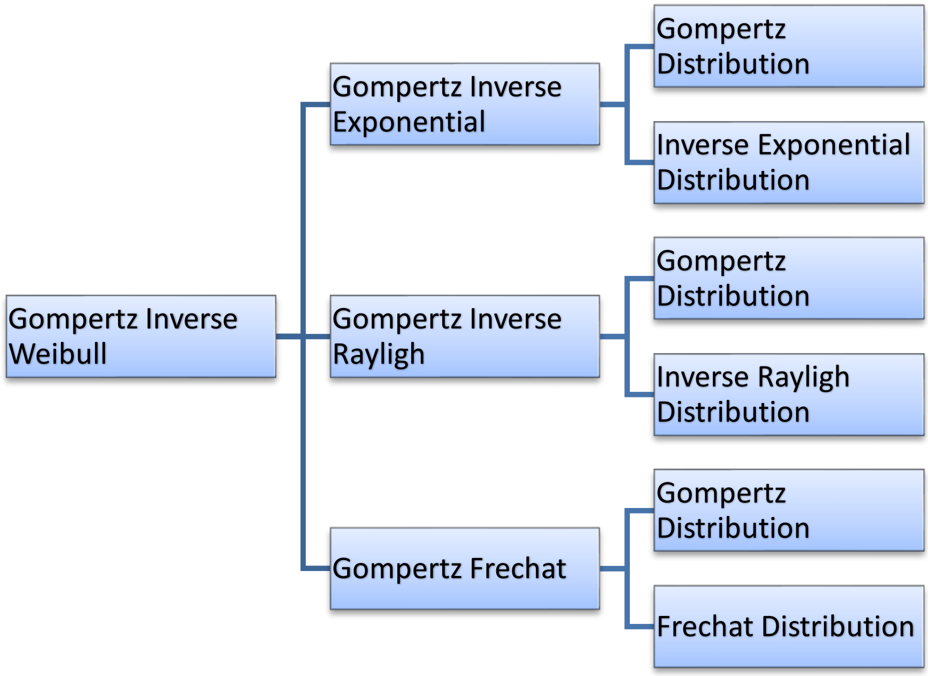

Figure (3): Some distribution found form GoIW

Doi: www.doi.org/10.25130/tjaes.19.SP3.5.11

4. Quantile Function and Related Concept

The quantile function (QF) $Q(u)$ is a statistical tool used to determine the value that divides a probability distribution into equal or specified portions. Specifically, the QF determines the inverse of the cdf Abdullah et al. (2019). The QF of the GoIW distribution, denoted by $Q(u)$, is obtained using the relation $Q(u) = F^{-1}(u)$ where F^{-1} represents the inverse of the cdf. Thus, the quantile function of the GoIW distribution can be expressed as follows:

$$Q(u) = \left(\frac{1}{-\psi} ln\left\{1 - \left(1 - \frac{\tau}{s}\ln(1-u)\right)^{\frac{-1}{\tau}}\right\}\right)^{\frac{1}{-\varphi}} \quad (12)$$

The random samples (RS) from the GoIW distribution can be found by using:

$$x = \left(\frac{1}{-\psi} ln\left\{1 - \left(1 - \frac{\tau}{s}\ln(1-U)\right)^{\frac{-1}{\tau}}\right\}\right)^{\frac{1}{-\varphi}}$$

where $U \sim Uniform(0,1)$

The median (*Me*) is a specific quantile that divides the probability distribution in half, with 50% of the distribution below the median and 50% above it AbuJarad et al. (2020). The *Me* of the GoIW distribution can be obtained simply by substituting $u = 0.5$ in Equation (12), which gives:

$$Median = \left(\frac{1}{-\psi} ln\left\{1 - \left(1 - \frac{\tau}{s}\ln(0.5)\right)^{\frac{-1}{\tau}}\right\}\right)^{\frac{1}{-\varphi}} \quad (13)$$

The first quartile and third quartile of the GoIW distribution can be obtained by substituting $u = 0.25$ and $u = 0.75$, respectively, into Eq(12). The quantile function and median are important tools for understanding the shape and central tendency of a probability distribution, and are commonly used in statistics and data analysis.

5. Order Statistics

Order statistics (*OS*) refer to the statistical analysis of the sorted values of a random sample drawn from a probability distribution Ahmedd et al., (2020), Ahmed et al., (2021). The pdf of the *k-th OS* for n sample size drawn from a distribution function *F(x)* with pdf *f(x)* can be as:

$$f_{k:r}(x) = \frac{r!}{(k-1)!\,(r-k)!} f(x)[F(x)]^{k-1}[1-F(x)]^{r-k} \quad (14)$$

Doi: www.doi.org/10.25130/tjaes.19.SP3.5.11

Where the pdf of the *k-th* *OS* for *r* sample size from a distribution function *F(x)* and its pdf *f(x)* can be expressed as shown in Eq(15), where *f(x)* and *F(x)* are the pdf and cdf of the GoIW distribution, respectively. Therefore, for the GoIW distribution, the pdf of the *k-th* order statistic for *r* sample size can be given as:

$$f_{k:r}(x) = \frac{r!}{(k-1)!(r-k)!}\left\{s\psi\varphi\, x^{-\varphi-1}\ e^{-\psi x^{-\varphi}}\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau-1} e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}\right\}$$
$$\times\left[1-e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}\right]^{k-1}\left[e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}\right]^{r-k} \quad (15)$$

So, to obtain the pdf of the min *OS*, we can substitute *k=1* in Eq(15), resulting in:

$$f_{1:r}(x) = r\left\{s\psi\varphi\, x^{-\varphi-1}\ e^{-\psi x^{-\varphi}}\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau-1} e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}\right\} \quad (16)$$

The pdf of the max *OS* can be obtained by substitute *k*=*r* in Eq(15), resulting in:

$$f_{r:r}(x) = r\left\{s\psi\varphi\, x^{-\varphi-1}\ e^{-\psi x^{-\varphi}}\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau-1} e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}\right\}$$
$$\times\left[1-e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}\right]^{r-1} \quad (17)$$

The *OS* is useful in a variety of statistical applications, including hypothesis testing, estimation, and modeling of complex data structures.

6. Estimation

The maximum likelihood estimation (*MLE*) is a statistical method that uses the observed data to estimate the parameters of a probability distribution. Maximum likelihood estimation is widely used in fields such as finance, engineering, and the natural sciences to model and analyze data. The log-likelihood function is utilized for estimating the parameters of the GoIW distribution using the method of *MLE*, as described in Al Abbasi et al. (2019). Given a random sample $x_1, x_2, \ldots, x_n$ that follows the GoIW distribution pdf, the log-likelihood function is derived as follows:

$$f(x_i; s,\tau,\varphi,\psi) = \prod_{i=1}^{n}\left[s\psi\varphi\, x^{-\varphi-1}\ e^{-\psi x^{-\varphi}}\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau-1} e^{\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x^{-\varphi}}\right]^{-\tau}\right\}}\right] \quad (18)$$

The log-likelihood (L) function is;

$$L = n\,ln(s) + n\,ln(\varphi) + n\,ln(\psi) - (\varphi+1)\sum_{i=1}^{n} ln(x_i) - \sum_{i=1}^{n}\psi {x_i}^{-\varphi}$$

Doi: www.doi.org/10.25130/tjaes.19.SP3.5.11

$$-(\tau+1)\sum_{i=1}^{n} ln\left(1-e^{-\psi x_i{}^{-\varphi}}\right)+\frac{s}{\tau}\left\{1-\left[1-e^{-\psi x_i{}^{-\varphi}}\right]^{-\tau}\right\} \quad (19)$$

To estimate the parameters, differentiate L w.r.t s, τ, ψ, and φ, and equate the results to zero. Since a closed-form solution may not exist, numerical software like R can be used to obtain the estimates, as suggested by Faris and Khaleel (2022).

One can obtain estimates of the parameters of the GoIW distribution using the method of MLE as described by Al Abbasi et al. (2019). To do this, we begin by obtaining the log-likelihood function based on a random sample $x_1, x_2, \ldots, x_n$ distributed according to the GoIW distribution. Then, we differentiate the log-likelihood function with respect to the parameters s, τ, ψ, and φ, equate the results to zero, and solve the resulting simultaneous equations to obtain the parameter estimates. It is important to note that closed-form solutions may not always be available, so numerical methods may be necessary. Software such as R (Faris and Khaleel, 2022) can be used for this purpose.

7. Application

The performance of the GoIW distribution was evaluated by comparing it with several other distributions including the Kumaraswamy inverse Weibull (KuIW), Exponentiated generalized inverse Weibull (EGIW), Marshall Olkin inverse Weibull (MOIW), Weibull inverse Weibull (WeIW), beta inverse Weibull (BeIW), and IW distributions. The analysis was conducted using R software and model selection was based on different criteria such as Akaike Information Criteria (G1), Bayesian Information Criteria (G3), Consistent Akaike Information Criteria (G2), Negative Log-likelihood (-LL) and Hannan and Quinn Information Criteria (G4). Goodness-of-fit tests were also performed using the Anderson-Darling (AD) test and Kolmogorov Smirnov (KS) test. The selection of the best fit distribution was based on the lowest values of these criteria. The study was conducted by Mohammad et al. (2020) and Ibrahim and Khaleel (2018) and Khaleel et al. (2020, 2022).

Rainfall Data: The data consist the average of rainfall from 1975 to 2005 for 35 stations in peninsular Malaysia. This data used for the show the flexibility in application part, and it was used by Khaleel et al., (2017) and Ahmed et al., (2021). The data are: (0.933, 0.937, 0.989, 1.047, 1.187, 1.092, 1.249, 1.110, 1.106, 1.003, 1.018, 0.953, 0.703, 0.955, 0.955, 0.991, 0.842, 1.298, 1.027, 1.007, 1.068, 1.141, 0.914, 1.164, 1.234, 1.056, 0.880, 1.163, 0.835, 1.077, 1.048, 1.178, 1.181, 1.196, and 1.134). Table 1 summarizes the results of the analysis.

Table (1): Table of result

Models	Estimates	-LL	G1	G2	G3	G4	Rank
GoIW (new)	$\hat{s} = 30.30$ $\hat{\tau} = 21.77$ $\hat{\psi} = 4.430$ $\hat{\varphi} = 1.839$	-22.534	-37.068	-35.734	-30.846	-34.290	1
KuIW	$\hat{s} = 2.528$ $\hat{\tau} = 108.4$ $\hat{\psi} = 2.182$ $\hat{\varphi} = 1.839$	-21.931	-35.862	-34.529	-29.461	-33.715	4
EGIW	$\hat{s} = 32.38$ $\hat{\tau} = 0.468$ $\hat{\psi} = 5.511$ $\hat{\varphi} = 3.102$	-21.773	-35.547	-34.216	-29.326	-33.400	5
BeIW	$\hat{s} = 0.338$ $\hat{\tau} = 42.19$ $\hat{\psi} = 7.016$ $\hat{\varphi} = 3.242$	-21.944	-35.888	-34.555	-29.667	-33.741	3
WeIW	$\hat{s} = 7.053$ $\hat{\tau} = 0.329$ $\hat{\psi} = 1.398$ $\hat{\varphi} = 0.860$	-22.454	-36.909	-35.567	-30.688	-34.001	2
MOIW	$\hat{s} = 0.003$ $\hat{\psi} = 6.344$ $\hat{\varphi} = 2.065$	-17.756	-29.586	-28.816	-24.924	-27.975	6
IW	$\hat{\psi} = 7.016$ $\hat{\varphi} = 3.242$	-14.120	-24.240	-23.864	-21.130	-23.166	7

Based on the values of the criteria used for distribution selection, the GoIW distribution is considered to have the best fitness since has the lowest G1, G2, G3, -LL and G4 values. The goodness of fit tests conducted using the Anderson-Darling (AD) and Kolmogorov-Smirnov (KS) tests also support this conclusion as indicated by the p-values presented in Table 2.

Table (2): Test of goodness of fit using rainfall data

Distributions	AD statistic	KS statistic	KS p-value
GoIW (Proposed)	**0.1333**	**0.0598**	**0.9998**
KuIW	0.1576	0.0758	0.9876
EGIW	0.1800	0.0705	0.9949
BeIW	0.1636	0.0697	0.9956
WeIW	0.1308	0.0681	0.9968
MOIW	0.6465	0.1038	0.8447
IW	1.1679	0.1413	0.4869

The histogram of the rainfall data is shown in Figure 4, along with the competing distributions.

Figure 5 based on the dataset used the empirical cumulative distribution functions (Ecdfs) of competing distributions is shown.

Based on the values presented in Tables 1 and 2, it can be observed that the GoIW distribution has the minimum values for all criteria used. Therefore, it is deemed to be the most suitable distribution to fit the dataset compared to the different distributions analyzed. This conclusion is also supported by the plots in Figures 4 and 5.

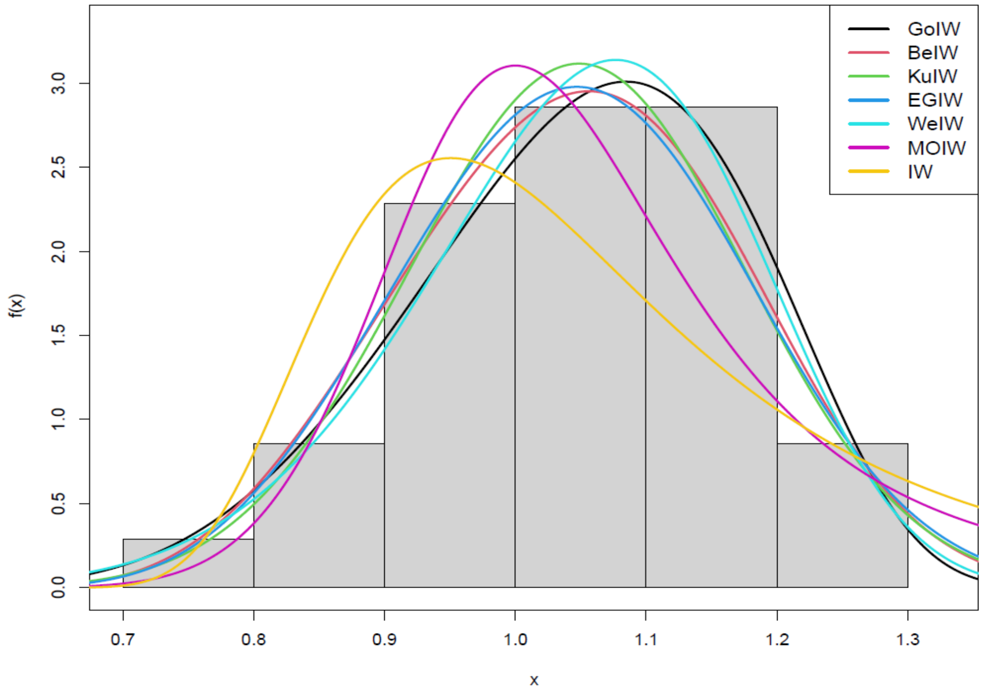

Figure (4): Histogram of the rainfall data utilizing various distributions

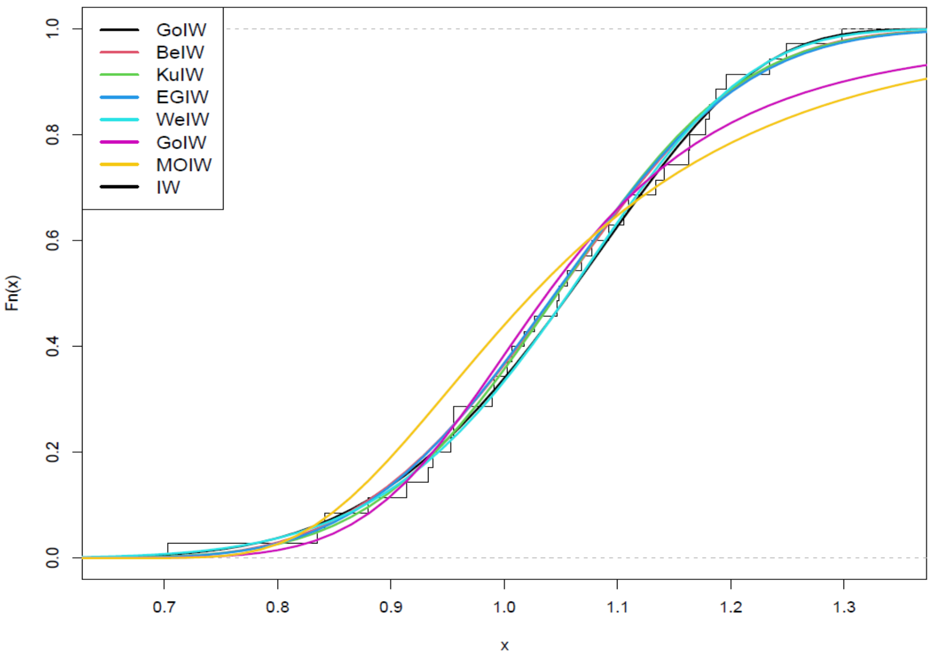

Figure (5): Displays the Ecdfs of the rainfall dataset along with the competing distributions.

8. Conclusion

This study has introduced the GoIW distribution and investigated some of its properties. The distribution's pdf and failure rate have been found to have unimodel shapes, making it a useful model for events with unimodel hfr. The model is tractable, capability, and high modeling flexible, outperforming other distributions such as the KuIW, WeIW, MOIW, BeIW, EGIW and IW distributions. This is evident from the lower G1, G2, G3, NLL, and G4 values obtained for the GoIW distribution. Hence, it can be considered a competitive model with potential applications in fields such as engineering, biology, and medicine. Further studies can explore other statistical properties of the distribution and conduct simulation studies to validate its performance.

References

1. Abdal, A. P. D. M. K., (2020), The statistical distribution for crisis consumption of drugs containing narcotic substances. Tikrit Journal of Administration and Economics Sciences, 16(Special Issue part 2).
2. Abdal-hameed, M. K., Khaleel, M. A., Abdullah, Z. M., Oguntunde, P. E., & Adejumo, A. O., (2018), Parameter Estimation and Reliability, Hazard Functions of Gompertz Burr Type XII Distribution. Tikrit Journal of Administration and Economics Sciences, 14(41 part 2).

3. Abdullah, Z. M., Hameed, M., Hisham, M., & Khaleel, M. A., (2019), Modified new conjugate gradient method for Unconstrained Optimization. Tikrit Journal of Pure Science, 24(5), 86-90.
4. AbuJarad, M. H., Khan, A. A., Khaleel, M. A., AbuJarad, E. S., AbuJarad, A. H., & Oguntunde, P. E., (2020), Bayesian reliability analysis of Marshall and Olkin model. Annals of Data Science, 7, 461-489.
5. Ademola, A., Adeyeye, J., Khaleel, M., & Olubisi, A. A. K. O., (2021), Exponentiated Gompertz exponential (Egoe) distribution: Derivation, properties and applications. Istatistik Journal of the Turkish Statistical Association, 13(1), 12-28.
6. Afify, A. Z., Ahmed, S., & Nassar, M., (2021), A new inverse Weibull distribution: properties, classical and Bayesian estimation with applications. Kuwait Journal of Science, 48(3).
7. Ahmed, M. T., Khaleel, M. A., & Khalaf, E. K., (2020), The new distribution (Topp Leone Marshall Olkin-Weibull) properties with an application. Periodicals of Engineering and Natural Sciences, 8(2), 684-692.
8. Ahmed, M. T., Khaleel, M. A., Oguntunde, P. E., & Abdal-Hammed, M. K., (2021, March). A new version of the exponentiated Burr X distribution. In Journal of Physics: Conference Series (Vol. 1818, No. 1, p. 012116). IOP Publishing.
9. Ahmedd, M. T., (2020), The Probability Distribution Marshall-Olkin-Generalized Exponential Weibull Properties with Application. Tikrit Journal of Administration and Economics Sciences, 16(51 part 1).
10. Al Abbasi, J. N., Khaleel, M. A., Abdal-hammed, M. K., Loh, Y. F., & Ozel, G., (2019), A new uniform distribution with bathtub-shaped failure rate with simulation and application. Mathematical Sciences, 13, 105-114.
11. Alizadeh, M., Cordeiro, G. M., Pinho, L. G. B., & Ghosh, I., (2017), The Gompertz-G family of distributions. Journal of statistical theory and practice, 11, 179-207.
12. Almarashi, A. M., Elgarhy, M., Jamal, F., & Chesneau, C. (2020). The Exponentiated truncated inverse Weibull-generated family of distributions with applications. Symmetry, 12(4), 650.
13. Al-Noor, N. H., & Khaleel, M. A., (2021, March). Marshal Olkin Marshal Olkin gompertz distribution. In AIP Conference Proceedings (Vol. 2334, No. 1, p. 090001). AIP Publishing LLC.
14. Al-Noor, N. H., Khaleel, M. A., & Assi, N. K., (2022), The Rayleigh Gompertz distribution: Theory and real applications. International Journal of Nonlinear Analysis and Applications, 13(1), 3505-3516.
15. Basheer, A. M. (2019). Alpha power inverse Weibull distribution with reliability application. Journal of Taibah University for Science, 13(1), 423-432.
16. Bodhisuwan, W., & Aryuyuen, S., (2021), The Gompertz Weibull Frechet Distribution: ´ Properties and Applications. Thailand Statistician, 19(4).

17. Calabria, R., & Pulcini, G., (1989), Confidence limits for reliability and tolerance limits in the inverse Weibull distribution. Reliability Engineering & System Safety, 24(1), 77-85.
18. De Gusmao, F. R., Ortega, E. M., & Cordeiro, G. M., (2011), The generalized inverse Weibull distribution. Statistical Papers, 52, 591-619.
19. Eugene, N., Lee, C., and Famoye, F., (2002), Beta-normal distribution and its applications. Communications in Statistics-Theory and methods, 31(4):497–512.
20. Faris, R. W., & Khaleel, M. A. (2022, November). Theory and application of the truncated exponential Marshall olkin gamma distribution: A model bounded on [0, 1] support. In AIP Conference Proceedings (Vol. 2394, No. 1, p. 070010). AIP Publishing LLC.
21. Hanook, S., Shahbaz, M. Q., Mohsin, M., & Golam Kibria, B. M. (2013). A note on beta inverse-Weibull distribution. Communications in Statistics-Theory and Methods, 42(2), 320-335.
22. Ibrahim, N. A., & Khaleel, M. A., (2018), Exponentiated Kumaraswamy Exponentiated Weibull Distribution with Application. Jurnal Karya Asli Lorekan Ahli Matematik Vol, 11(1), 015-022.
23. Jiang, R., Murthy, D. N. P., & Ji, P., (2001), Models involving two inverse Weibull distributions. Reliability Engineering & System Safety, 73(1), 73-81.
24. Keller, A. Z., Goblin, M. T., & Farnworth, N. R. (1985). Reliability analysis of commercial vehicle engines. Reliability Engineering, 10(1), 15-25.
25. Khalaf, A. A., & khaleel, M. A., (2022, November), [0, 1] Truncated exponentiated exponential gompertz distribution: Properties and applications. In AIP Conference Proceedings (Vol. 2394, No. 1, p. 070035). AIP Publishing LLC.
26. Khaleel, M. A., Abdulwahab, A. M., Gaftan, A. M., & Abdal-hammed, M. K., (2022), A new [0, 1] truncated inverse Weibull rayleigh distribution properties with application to COVID-19. International Journal of Nonlinear Analysis and Applications, 13(1), 2933-2946.
27. Khaleel, M. A., Ibrahim, N. A., Shitan, M., Merovci, F., & Rehman, E., (2017), Beta burr type x with application to rainfall data. Malaysian Journal of Mathematical Sciences, 11, 73-86.
28. Khaleel, M. A., Oguntunde, P. E., Al Abbasi, J. N., Ibrahim, N. A., & AbuJarad, M. H., (2020), The Marshall-Olkin Topp Leone-G family of distributions: A family for generalizing probability models. Scientific African, 8, e00470.
29. Oguntunde, P. E., Adejumo, A. O., Khaleel, M. A., Okagbue, H. I., & Odetunmibi, O. A. (2018, October). The logistic inverse exponential distribution: Basic structural properties and application. World Congress on Engineering.
30. Oguntunde, P. E., Khaleel, M. A., Adejumo, A. O., Okagbue, H. I., Opanuga, A. A., & Owolabi, F. O., (2018), The Gompertz inverse exponential (GoIE) distribution with applications. Cogent mathematics & statistics, 5(1), 1507122.

31. Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., & Okagbue, H. I., (2019), The Gompertz fréchet distribution: properties and applications. Cogent Mathematics & Statistics, 6(1), 1568662.
32. Oguntunde, P. E., Khaleel, M. A., Okagbue, H. I., & Odetunmibi, O. A., (2019), The Topp–Leone Lomax (TLLo) distribution with applications to airbone communication transceiver dataset. Wireless Personal Communications, 109, 349-360.
33. Sindhu, T. N., & Atangana, A., (2021), Reliability analysis incorporating exponentiated inverse Weibull distribution and inverse power law. Quality and Reliability Engineering International, 37(6), 2399-2422.