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The Dynamics of Solutions for a Broad Range of Difference Equations

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الحلول الديناميكية لمجال واسع للمعادلات الفرقية

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Abstract. This study examines the long-term behavior of solutions to a general category of difference equations, the solution's periodicity and local and global stability is specifically taken into consideration. Furthermore, some intriguing counterexamples to validate our robust findings are provided.

Keywords: Periodic solutions; Asymptotically of difference equation; Difference Equations

الخلاصة

في هذا الدراسة، سنختبر الحلول ذات السلوك طويل المدى لبعض الأنماط المعممة للمعادلات الفرقية، متمثلة بالحلول الدورية وبشكل ادق الاستقرارية المحلية والكلية لتلك الحلول. بالأضافة، أثبتنا ببعض الأمثلة العكسية المميزة لدعم نتائجنا.

Introduction

Differentiation equations are found in numerous areas of mathematics and other sciences. They are crucial to a wide range of applications, including the discrete approximation of a continuous process, ecology, mathematical models of different biological systems (like the population model and discrete delay logistic model) [4], optics (p. 182 of [14]), and the inherent model of a discrete process in combinatory, for instance. Therefore, a number of mathematicians, engineers, and other experts from all around the world have taken notice of these qualities. Difference equations (DEs) serve as mathematical models that explain real-life phenomena in areas such as theory of probability, theory of queuing, problems of statistics, stochastic times series, counting methods analysis, theory of number, electric networks, quantum radiation, psychology, and more. The global asymptotics of solutions to rational difference equations have been extensively

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studied in various works, including [1, 2], [6-18], and the references cited within them. It is essential to examine the long-term behavior of solutions to a nonlinear-DEs system, focusing of the equilibrium-points (EP) on the boundedness, periodicity, and stability (local and global). This study aims to explore the long-term behavior of solutions to a broad category of DEs.

$$x_{h+1}=ax_{h-\mu}+\mathbb{f}(x_{h-\mu}+x_{h-\beta}), \qquad (1.1)$$

In which a is a nonnegative real number and the function $\mathbb{f}(t_1,t_2):(0,\infty)^2\to(0,\infty)$ is continuous real valued- mapping that is homogenous of degree equal 0. Next, we introduce the key definitions and theorems of our model, covering EP, local and global stability, boundedness, periodicity, and solution oscillation.

Key words and phrases: Rational DEs, Periodic solution.

Definition 1. (Equilibrium point) Examine the following DEs:

$$x_{h+1}=\Theta\left(x_{h-\mu,}x_{h-\beta}\right),\ h=0,1,\ \ldots$$
(1,2)

where the positive integers are μ and β and Θ as a continuous function. In case a point $\hat{x}$ is to be a fixed point of Θ, that is, $\hat{x}=\Theta\,(\hat{x},\hat{x})$, then it is considered an EP of equation (1.2).

Definition 2. (Stability) Consider the EP of equation (1.2) to be $\hat{x}\epsilon$ (0,∞). Next, we have

(a) (Local stability) A locally stable EP is one that is located at $\hat{x}$ in equation (1.2), if for each ε > 0 there is σ>0 such that, if $x_{-v}\epsilon$ (0,∞) for v=0,1,… ,r, r= $\{\beta,\mu\}$ with

$$\sum_{i=0}^{r}|x_{-i}-\hat{x}|<\sigma$$

Then $|x_h-\hat{x}|<\varepsilon$ for all h≥ −r.

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(b) (Local asymptotic stability) If in equation (1.2), the EP x is stable in the local region, it is considered locally asymptotically stable and there is ϒ> 0 such that, if $x_{-v} \in (0,\infty)$ for v=0,1,… ,r with

$$\sum_{i=0}^{r} |x_{-i} - \hat{x}| < \delta$$

(c) (Global stability) A global attractor is a point of equilibrium $\hat{x}$ of equation (1.2) if for every $x_{-v} \in (0,\infty)$ v=0,1,… ,r we have

$$x_h = \hat{x}$$

(d) (Unstability) In equation (1.2), a point of equilibrium $\hat{x}$ is considered non-stable in case it lacks local stability.

Definition (3). (Periodicity) A sequence $\{x_h\}_{h=-r}^{\infty}$ is considered periodic, having a period of t if $x_{h+t} = x_h$ for all $h \geq -r$. A sequence $\{x_h\}_{h=-r}^{\infty}$ is considered periodic with prime period t if t is the smallest positive integer that exhibits this characteristic.

Definition (4). (BoundednessEquation (1.2) is referred to be permanent and bounded in the event that both g and G exist with 0<g<G<∞ in that for any primary conditions $x_{-v} \in (0,\infty)$ for v=0,1,… ,r There occurs a positive integer N that relies on the primary conditions in which 0<g<G<∞ for all n≥ N.

Definition 5. The linear-DEs defends The linearized form of equation (1.2) around the EP $\hat{x}$.

$$z_{h+1} = \sum_{i=0}^{r} K_i z_{h-i},$$

Where

$$K_i = \frac{\partial F(\hat{x},\hat{x},\ldots,\hat{x})}{\partial x_{h-i}}, \ i=0, 1, \ldots, r$$

Theorem 1.1. [5] Assume that $K_i \in R$, i=0, 1 , … , r. Then

$$\sum_{i=0}^{r} |K_i| < 1,$$

is a necessary condition for equation (1.2)'s asymptotic stability.

Theorem 1.2. [10] Let I denote an interval of real numbers, supposing that

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$$\theta: I \times I \rightarrow T$$

is the continuous function that meets the mentioned requirements listed below:

$\theta(u, v)$ is nondecreasing in u for each $v \epsilon I$, and is nonincreasing in v for each $u \epsilon I$.

If $(g, G) \in I \times I$ is a solution of the system

$$G = \theta(G, g)$$
$$g = \theta(g, G),$$

then g = G.

Then, any solution to (1.2) converges to $\hat{x}$, and there is only one equilibrium for (1.2), which is $\hat{x}$.

Dynamics of Equation (1.1)

Here, we look into a few of the qualitative characteristics of the equation (1.1) solutions.
The Local stability. This section examines the local stability of the EP in equation (1.1). Equation (1.1)'s EP is provided by

$$x = a\hat{x} + \mathbb{f}(\hat{x}, \hat{x}),$$

Then

$$x = \frac{1}{1-a}\mathbb{f}(1, 1),\ a<1.$$

Now, let $\theta\ (t_1, t_2) : (0, \infty)^2 \rightarrow (0, \infty)$ is continuous real function and

$$\theta(t_1, t_2) = at_1 + \mathbb{f}\ (t_1, t_2)$$

and hence,

$$\frac{\partial}{\partial t_1}\theta(t_1, t_2) = a + \mathbb{f}_{t_1}(t_1, t_2) \qquad (2.1)$$

$$\frac{\partial}{\partial t_2}\theta(t_1, t_2) = \mathbb{f}_{t_2}(t_1, t_2) \qquad (2.2)$$

Theorem 2.1. If

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$$|a\mathbb{f}(1,1) + (1-a)\mathbb{f}_{t_1}(1,1)| + |(1-a)\mathbb{f}_{t_1}(1,1)| < \mathbb{f}(1,1) \qquad (2.3)$$

the point of equilibrium of equation (1.1) x̲ is locally stable asymptotically.

Proof: From (2.1) and (2.2), it is observed that

$$\frac{\partial}{\partial t_1}\theta(\hat{x},\hat{x}) = a + \mathbb{f}_{t_1}(\hat{x},\hat{x})$$

$$\frac{\partial}{\partial t_2}\theta(\hat{x},\hat{x}) = \mathbb{f}_{t_2}(\hat{x},\hat{x}),$$

The linear-DEs is the linear equation of (1.1) regarding x.

$$Z_{h+1} - \left[\frac{\partial}{\partial t_1}\theta(\hat{x},\hat{x})\right]Z_{h-\mu} - \left[\frac{\partial}{\partial t_2}\theta(t_1,t_2)\right]Z_{h-\beta} = 0$$

Theorem 1.1 implies equation (1.1) is locally stable if

$$|a + \mathbb{f}_{t_1}(\hat{x},\hat{x}) + \mathbb{f}_{t_2}(\hat{x},\hat{x})| < 1. \qquad (2,4)$$

The homogeneous function theorem of Euler gives us that $t_1\mathbb{f}_{t_1} = -t_2\mathbb{f}_{t_2}$ and hence $\mathbb{f}_{t_2}(\hat{x},\hat{x}) = -\mathbb{f}_{t_1}(\hat{x},\hat{x})$. As result we obtain:

$$|a + \mathbb{f}_{t_1}(\hat{x},\hat{x})| + |\mathbb{f}_{t_1}(\hat{x},\hat{x})| < 1.$$

Form [[3], Corollary 2], we get that

$$\left|a + \frac{1}{\hat{x}}\mathbb{f}_{t_1}(1,1)\right| + \frac{1}{\hat{x}}|\mathbb{f}_{t_1}(1,1)| < 1.$$

Thus, we find

$$\left|\frac{a}{1-a}\mathbb{f}(1,1) + \mathbb{f}_{t_1}(1,1)\right| + |\mathbb{f}_{t_1}(1,1)| < \frac{1}{1-a}\mathbb{f}(1,1).$$

This implies

$$|a\mathbb{f}(1,1) + (1-a)\mathbb{f}_{t_1}(1,1)| + |(1-a)\mathbb{f}_{t_1}(1,1)| < \mathbb{f}(1,1).$$

Thus, the proof is achieved.

Remark 2.1. If $\mathbb{f}_{t_1} > 0$, then the condition (2.3) becomes

$$\mathbb{f}_{t_1}(1,1) < \frac{1}{2}\mathbb{f}(1,1).$$

Example 2.1. Consider the recurrence relation

$$x_{h+1} = ax_{h-\mu} + \frac{bx_{h-\mu}}{cx_{h-\mu} + dx_{n-\beta}},$$

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as a, b, c and d are positive real numbers. It is noted that

$$\mathbb{f}(u,v) = \frac{bt_1}{ct_1 + dt_2}$$

and so,

$$\mathbb{f}_{t_1}(t_1,t_2)=\frac{bdt_2}{(ct_1+dt_2)^2}>0$$

Regarding Theorem 2.1, the EP of Equation (2.5) is

$$\underline{x} = \frac{b}{(1-a)(c+d)}$$

This is locally asymptotically stable if d < c. For numerical example, $\mu = 1$, β = 0,
a = 0.5, b = 1, c = 2, d = 1, $x_{-\mu}$ = 1 and x_0 = 2, (see Fig. 1).

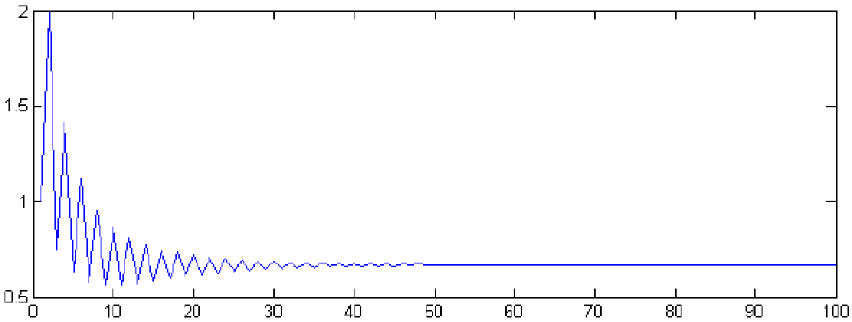

Figure 1: The stable solution corresponding to differences equation (2.5).

2.2. Global attractively of EP. In this part, the global asymptotic stability of equation (1.1) is analyzed. Note that, from Euler's homogeneous function Theorem, $t_1\mathbb{f}_{t_1} = t_2\mathbb{f}_{t_2}$ and hence $\mathbb{f}_{t_1}\ \mathbb{f}_{t_2} < 0$ for all $t_1,t_2 \in (0,\infty)$ are obtained.

Theorem 2.2. Let there occurs a positive a constant μ and L in which $\mu < \mathbb{f}(t_1,t_2) < L$ for all for all $t_1,t_2 \in (0,\infty)$. If $\mathbb{f}_{t_1} > 0$; then the EP $\underline{x}$ is a global attractor of equation (1.1).

Proof. First, we demonstrate that every solution to equation (1.1) is bounded. From equation (1.1), we obtain:

$$x_{h+1} < ax_{h+\mu} + L$$

Employing the Comparison theorem, it follows that

$$\sup x_h \leq \frac{L}{1-a} \qquad (2.6)$$

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Now, assume that $\mathbb{f}_{t_1} > 0$ and from (2.1), (2.2), we have $\theta_{t_1} > 0$ and $\theta_v < 0$.
Now, we let (g, G) is a solution of the system

$$G = \theta(G, g)$$
$$g = \theta(g, G),$$

Thus, we obtain

$$(1 - a)g = \mathbb{f}(g, G)$$
$$(1 - a)G = \mathbb{f}(G, g)$$

then,

$$(1 - a)(g - G) = \mathbb{f}(g, G) - \mathbb{f}(G, g)$$

and hence,

$$(g - G)\left[(1 - a) + \frac{\mathbb{f}(g,G) - \mathbb{f}(G,g)}{g - G}\right] \qquad (2.7)$$

From (2.6), we have

$$0 < g, G < \frac{L}{1 - a}$$

then,

$$(1 - a)\frac{\mathbb{f}\left(1,\frac{g}{G}\right) - \mathbb{f}\left(\frac{g}{G},1\right)}{g - G} > (1 - a) + (1 - a)\frac{\mu - L}{L}$$
$$= (1 - a)\frac{\mu}{L} > 0$$

which with (2.7) gives g=G. It is shown through Theorem 1.2 that $\underline{x}$ is a global attractor of equation (1.1). Thus, the proof is achieved.

Existence of Periodic Solutions

The presence of periodic solutions of (1.1) is examined in this section. The conditions that are both necessary and sufficient for this equation to have periodic solutions with a prime period of two are given by the following theorems.
Theorem 3.1. If μ, β odd or μ, β even, as equation (1.1) has no prime period two solution.
Proof. Assume that μ, β even and equation (1.1) possesses a prime period two solution

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$$\ldots, \varphi, \omega, \varphi, \omega, \ldots .$$

Then $x_{h-\mu} = x_{h-\beta} = \omega$. From equation (1.1), we find

$$\varphi = a\,\omega + \mathbb{f}(\omega, \omega).$$
$$\omega = a\,\varphi + \mathbb{f}(\varphi, \varphi).$$

Thus, we get

$$\varphi = \omega = \frac{1}{1-a}\mathbb{f}(1.1)$$

which contradicts itself. In the same way, we can demonstrate the second scenario, which is left out for ease of reading. Thus, the evidence is finished.

Theorem 3.2. Assume that μ odd, β even. Equation (1.1) has a solution with a prime period of two.

$$\ldots, \varphi, \omega, \varphi, \omega, \ldots .$$

If and only if

$$\mathbb{f}(\tau, 1) = \tau\mathbb{f}(1, \tau) \qquad (3.1)$$

Where $\tau = \frac{\varphi}{\omega.}$

Proof. We can presume that $\mu > \beta$ without losing generality. Let us now consider a prime period two solution to equation (1.1).

$$\ldots, \varphi, \omega, \varphi, \omega, \ldots .$$

Since l odd and β even, we have $x_{h-\mu} = \varphi$ and $x_{h-\beta} = \omega$. From equation (1.1), we get

$$\varphi = a\,\varphi + \mathbb{f}(\varphi, \omega).$$
$$\omega = a\,\omega + \mathbb{f}(\omega, \varphi).$$

This implies

$$(1-a)\varphi = \mathbb{f}(\tau, 1)$$
$$(1-a)\omega = \mathbb{f}(1, \tau)$$

Where $= \frac{\varphi}{\omega.}$. Since $\varphi = \tau\omega$, we find

$$\mathbb{f}(\tau, 1) = \tau\mathbb{f}(1, \tau)$$

Conversely, suppose that (3.1) holds. Right now, we select

$$x_{-\mu+2r+1} = \frac{1}{1-a}\mathbb{f}(1, \tau) \text{ and } x_{-\mu+2r} = \frac{1}{1-a}\mathbb{f}(\tau, 1), r = 0,1, \ldots, \frac{(\mu-1)}{2}$$

and $\tau \in R^{+}$. Hence, we see that

$$x_1 = ax_{-\mu} + \mathbb{f}(x_{-\mu}, x_{-\beta})$$

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$$= a\frac{1}{1-a}\, \mathbb{f}(\tau,1) + \mathbb{f}\left(\frac{1}{1-a}\, \mathbb{f}(\tau,1), \frac{1}{1-a}\, \mathbb{f}(1,\tau)\right)$$

(3.2)

From (3.1), we have

$$x_1 = \frac{1}{1-a}\, \mathbb{f}(\tau,1) + \mathbb{f}\left(\frac{1}{1-a}\mathbb{f}(\tau,1), \frac{1}{1-a}\, \mathbb{f}(1,\tau)\right)$$

and so,

$$x_1 = \frac{1}{1-a}\, \mathbb{f}(\tau,1) + \mathbb{f}(\tau,1)$$

$$= \frac{1}{1-a}\, \mathbb{f}(\tau,1)$$

Similarly, we can proof that $x_2 = \frac{1}{1-a}\, \mathbb{f}(1,\tau)$. Hence, follows by induction that $x_{2h-1} = \frac{1}{1-a}\, \mathbb{f}(\tau,1)$ and $x_{2h} = \frac{1}{1-a}\, \mathbb{f}(1,\tau)$ for all h > 0.

Therefore, Equation (1.1) possesses a prime period-two solution.

In the fowolloing theorem, assume that μ is even with k being odd. Then, Equation (1.1) has a prime period-two solution.

Theorem 3.3. Assume that μ even, k odd. Equation (1.1) has a prime period two solution

$$\ldots, \varphi, \omega, \varphi, \omega, \ldots .$$

This occurs only if

$$(1 - a\tau)\mathbb{f}(1,\tau) = (\tau - a)\mathbb{f}(\tau,1)$$

(3.3)

Where $\tau = \frac{\varphi}{\omega}.$

Proof. We can assume this without affecting the generality, that $\mu > \beta$. Now, let equation (1.1) have a prime period two solution

$$\ldots, \varphi, \omega, \varphi, \omega, \ldots .$$

Since μ even and β odd, we have $x_{h-\mu} = \omega$ and $x_{h-\beta} = \varphi$. From equation (1.1), we get

$$\varphi = a\,\omega + \mathbb{f}(\omega, \varphi).$$

$$\omega = a\,\varphi + \mathbb{f}(\varphi, \omega).$$

This implies

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$$(1-a^2)\varphi = a\mathbb{f}(\varphi,\omega) + \mathbb{f}(\omega,\varphi)$$

$$(1-a^2)\omega = a\mathbb{f}(\omega,\varphi) + \mathbb{f}(\varphi,\omega)$$

And so,

$$\varphi = \frac{a}{1-a^2}\,\mathbb{f}(\tau,1) + \frac{a}{1-a^2}\,\mathbb{f}(1,\tau)$$

$$\omega = \frac{a}{1-a^2}\,\mathbb{f}(1,\tau) + \frac{a}{1-a^2}\,\mathbb{f}(\tau,1),$$

Where $= \frac{\varphi}{\omega.}$. Since $\varphi = \tau\omega$, we find

$$\frac{a}{1-a^2}\,\mathbb{f}(\tau,1) + \frac{a}{1-a^2}\mathbb{f}(1,\tau) - \frac{a\tau}{1-a^2}\,\mathbb{f}(1,\tau) + \frac{\tau}{1-a^2}\,\mathbb{f}(\tau,1) = 0$$

Then,

$$(1-a\tau)\mathbb{f}(1,\tau) = (\tau-a)\mathbb{f}(\tau,1)$$

Alternatively, assuming that (3.1) is true, we select:

$$x_{-\mu+2r+1} = a\lambda\,\mathbb{f}(\tau,1) + \lambda\mathbb{f}(1,\tau) \text{ and } x_{-\mu+2r} = a\lambda\mathbb{f}(1,\tau) + \lambda\mathbb{f}(\tau,1),$$

$$\text{for all } r = 0,1,\dots,\frac{(\mu-1)}{2}$$

Where,

$$\lambda = \frac{1}{1-a^2}$$

and $\tau \in R^+$. Hence, we see that

$$x_1 = ax_{-\mu} + \mathbb{f}\left(x_{-\mu}, x_{-\beta}\right)$$

$$= a\left(a\lambda\,\mathbb{f}(1,\tau) + \lambda\mathbb{f}(\tau,1)\right)$$

$$+\mathbb{f}(a\lambda\,\mathbb{f}(1,\tau) + \lambda\mathbb{f}(\tau,1), a\lambda\mathbb{f}(\tau,1) + \lambda\,\mathbb{f}(1,\tau))$$

(3.4)

From (3.1), we have

$$a\mathbb{f}(\tau,1) + \mathbb{f}(1,\tau) = \tau\left(a\,\mathbb{f}(1,\tau) + \mathbb{f}(\tau,1)\right)$$

Which with (3.4) yields

$$x_1 = a\left(a\lambda\,\mathbb{f}(1,\tau) + \lambda\mathbb{f}(\tau,1)\right)$$

$$+\mathbb{f}(\lambda(a\,\mathbb{f}(1,\tau) + \mathbb{f}(\tau,1)), \lambda\tau(a\,\mathbb{f}(1,\tau) + \mathbb{f}(\tau,1))$$

And so,

$$x_1 = a^2\lambda\,\mathbb{f}(1,\tau) + a\,\lambda\mathbb{f}(\tau,1) + \mathbb{f}(1,\tau)$$

$$= a\lambda\,\mathbb{f}(\tau,1) + \left(a^2\,\frac{1}{1-a^2} + 1\right)\mathbb{f}(1,\tau)$$

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$$= a\lambda f(\tau, 1) + \lambda f(1, \tau)$$

Similarly, we can proof that $x_2 = a\lambda f(1, \tau) + \lambda f(\tau, 1)$. Therefore, it follows by induction that:
$x_{2h-1} = a\lambda f(\tau, 1) + \lambda f(1, \tau)$ and $x_{2h} = a\lambda f(1, \tau) + \lambda f(\tau, 1)$ for all $n > 0$.
Thus, Equation (1.1) has a solution with a prime period of two, completing the proof.

Example 3.1. Consider the DEs (2.5). Let μ be odd and β be even, by using Theorem 3.2 states that equation (2.5) has solutions with a prime period of two.

$$\ldots, \varphi, \omega, \varphi, \omega, \ldots .$$

Only in the case that:

$$\frac{b\tau}{c\tau + d} = \frac{b\tau}{c + d\tau}$$

Since φ is positive, we obtain $\tau \neq 0$ and hence,

$$c(\tau - 1) + d(1 - \tau) = 0$$

Also $\tau \neq 1$, then we get $c = d$. For numerical example, $\mu = 1, \beta = 0$, $a = 0.5, b = 2$,
$c = d = 1,\ x_{-1} = \frac{8}{3}$ and $x_0 = \frac{4}{3}$, (see fig. 2)

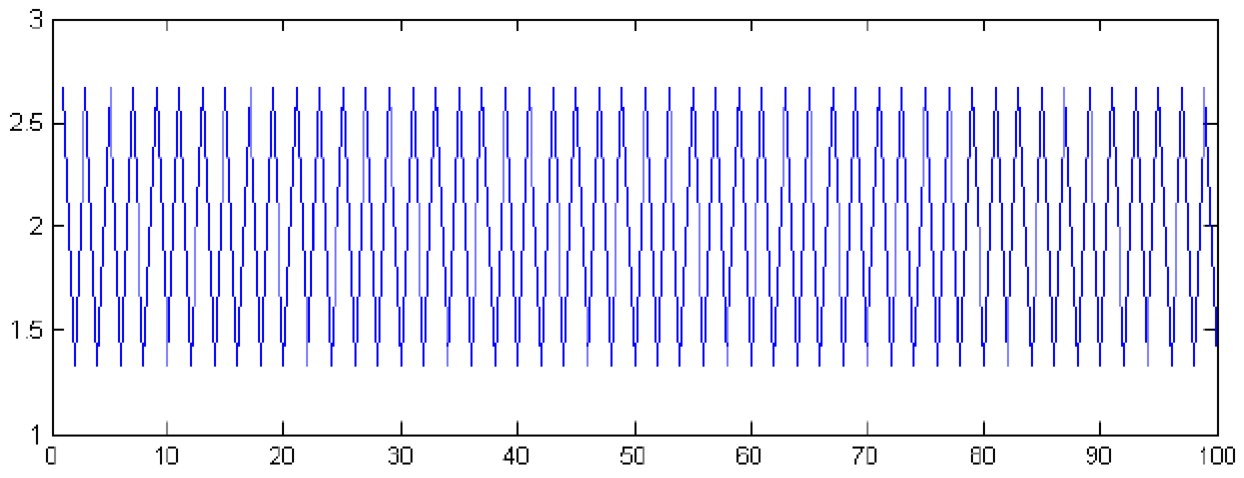

Figure 2: Prime period two solution of equation (2.5).

Example 3.2. Consider the DEs

$$x_{h+1} = ax_h + be^{\frac{x_h}{x_{h-1}}} \tag{3.5}$$

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As b is a positive real number. It is important to note that...

$$f(u,v) = e^{\frac{u}{v}}$$

By using Theorem 3.3, equation (3.5) has a prime period two solution

$$\ldots, \varphi, \omega, \varphi, \omega, \ldots .$$

If and only if

$$(1 - a\tau)e^{\frac{1}{\tau}} = (\tau - a)e^{\tau},$$

And so,

$$a = \frac{\tau e^{\tau} - e^{\frac{1}{\tau}}}{e^{\tau} - \tau e^{\frac{1}{\tau}}}$$

Now, we see that

$H(\tau) = \frac{\tau e^{\tau} - e^{\frac{1}{\tau}}}{e^{\tau} - \tau e^{\frac{1}{\tau}}} > H(\tau) = 3$ for $\frac{\tau \in R^{+}}{\{1\}}$ (see fig. 3).

Then, we get a > 3. For numerical example, a = 3.2089, b = 1, x_{-1} = -2.7277 and x_0 = -1.3638 (see Fig. 4).

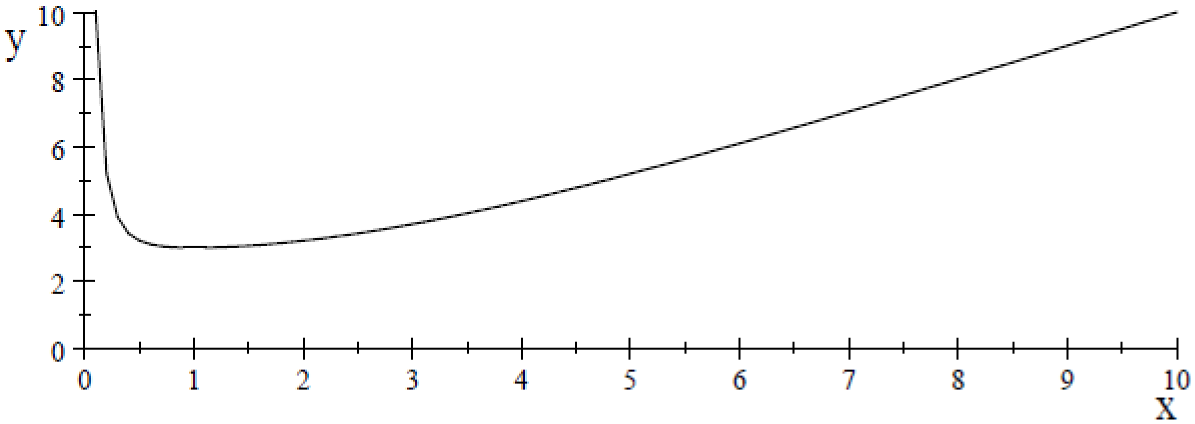

Figure 3: Graph of the function $H(\tau)$

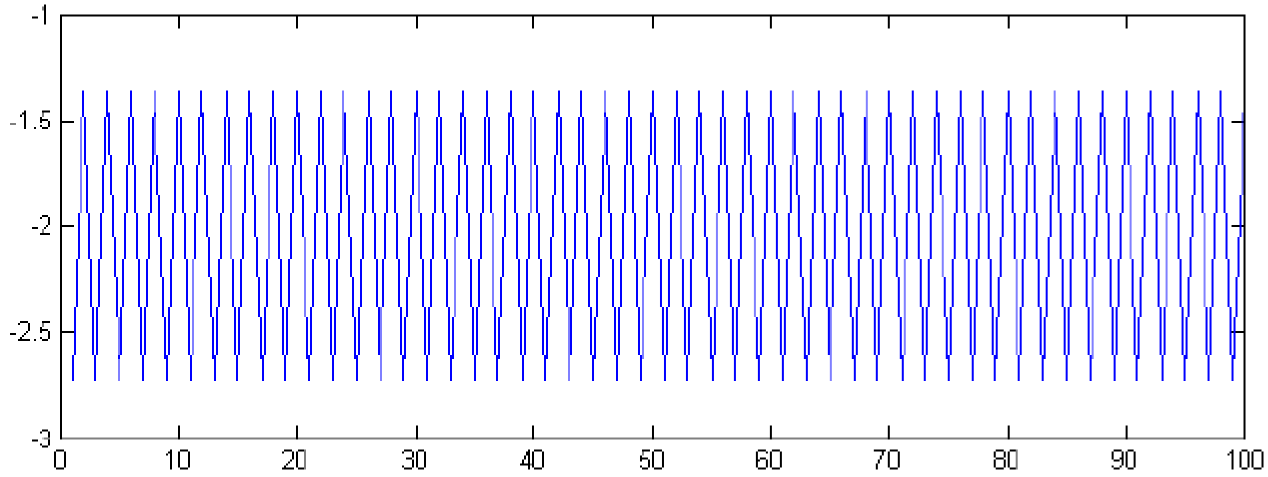

Figure 4: Prime period two solution of equation (3.5).

References

[1] M.A.E. Abdelrahman and O. Moaaz, Investigation of the new class of the nonlinear rational difference equations, Fundamental Research and Development International, 7(1) (2017), 59-72.

[2] M.A.E. Abdelrahman and O. Moaaz, On the new class of the nonlinear difference equations, Electronic Journal of Mathematical Analysis and Applications, 6(1) (2018), 117 - 125.

[3] K. C. Border, Eulers theorem for homogeneous functions. http://www.hss.caltech. edu/kcb/Ec121a/Notes/EulerHomogeneity.pdf, 23 July 2009.

[4] H. El- Metwally, E. A. Grove, G. Ladas, R. Levins, M. Radin, On the difference equation $x_{h+1} = \alpha + \beta x_{h-1} e^{-x_h}$, Nonlinear Analysis, Theory, Methods and Applications 47 (2001), 4623-4634.

[5] M. R. S. Kulenovic, G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall/CRC, Florida, 2001.

[6] E.A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, vol. 4, Chapman & Hall / CRC, (2005).

[7] A.E. Hamza, On the recursive sequence $x_{h+1} = \alpha + \frac{x_{h-1}}{x_h}$, J. Math. Anal. Appl., 322(006), 668-674.

[8] V.V. Khuong, On the positive no oscillatory solution of the difference equations

$x_{h+1} = (\alpha + \frac{x_{h-k}}{x_{h-m}})^p$, p, Appl. Math. J. Chinese Univ. 24 (2008) 45-48.
[9] V.L. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic, Dordrecht (1993).
[10] M. R. Kulenovic, G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall/CRC Press, Florida, (2002).
[11] O. Moaaz and M.A.E. Abdelrahman, Behaviour of the New Class of the Rational Difference Equations, Electronic Journal of Mathematical Analysis and Applications, 4(2) (2016), 129 - 138.
[12] O. Moaaz, Comment on "New method to obtain periodic solutions of period two and three of a rational difference equation" [Nonlinear Dyn. 79:241,250], Nonlinear Dyn., (2016)
[13] O. Ocalan, Dynamics of the difference equation $x_{h+1} = p + \frac{x_{h-k}}{x_h}$ with a period two coefficient, Appl. Math. Comput., 228, (2014), 31-37.
[14] T. L. Saaty, Modern nonlinear equations McGraw-Hill, New York, (1967).
[15] M. Saleh and M. Aloqeili, On the rational difference equation $x_{h+1} = A + \frac{x_{h-k}}{x_h}$, Appl. Math. Comput., 171(2), (2005), 862-869.
[16] S. Stevic, On the recursive sequance $x_{h+1} = \alpha + \frac{x_{h-1}^p}{x_h^p}$, , J. Appl. Math. & computing,18 (2005), 229-234.
[17] T. Sun and H. Xi, On convergence of the solutions of the difference equation $x_{h+1} = 1 + \frac{x_{h-k}}{x_h}$, J. Math. Anal. Appl., 325(2) (2007), 1491-1494.
[18] Elsayed, E.M.: New method to obtain periodic solutions of period two and three of a rational difference equation. Nonlinear Dyn. 79, 241–250 (2015)