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وضع العلامات الرشيقة لبعض الرسوم البيانية

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On Edge Graceful Labeling for Some Graph

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الخلاصة

تُعد تسمية الرؤوس (Vertex labeling) هو عملية إسناد أرقام إلى رؤوس الرسم البياني. ويُقال إن الرسم البياني يمتلك تسمية "رشيقة" (Graceful) إذا وُجد تطبيق واحدي من مجموعة رؤوسه إلى المجموعة $\{0,1,\ldots,e\}$ حيث e هو عدد الحواف، بحيث تُعطى كل حافة تسمية (ترقيم) تساوي القيمة المطلقة للفرق بين رقمي طرفيها، وتكون جميع أرقام الحواف الناتجة مميزة وغير مكررة. إذا تحقق هذا الشرط ، فإن الرسم البياني يُسمى "رشيقًا". في هذا البحث، نستعرض ستة رسومات بيانية وجدناها مثيرة للاهتمام، ونُثبت أن كلاً منها يمثل توسيم رشيق.

الكلمات المفتاحية: ترقيم الرسوم البيانية، ترقيم الرؤوس، الترقيم الرشيق، الترقيم الرشيق للحواف.

Abstract:

A vertex labeling is a function that assigns labels to the vertices of a graph for representation purposes the nodes by distinct objects a graceful-labeling of such a plot can always be guaranteed in case that an injective mapping is exist to the graph vertices to the set from 0,1 to e where e indicates the edges number such that the difference in labels of the pairs of adjacent vertices yields unique values across the edges graphs that possess this are said to be graceful the paper recognizes and examines six remarkable graphs proving that each one has a graceful labeling.

Keywords: graph labeling, vertex labeling, graceful labeling, edge graceful labeling.

Introduction:

In this study, all graphs are supposed to be finite, undirected, and simple. We consider a plane graph (G), where $E(G)$ represent its group of edges and $V(G)$ is groups of vertices.

A graph-labeling is the assigning of labels to a graph's vertices, edges, or both. These labels are specifically positive integers or non-negative integers [1], in this

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research paper, we will assign only labels to the vertices positioned on me the vertices, and the relevant edges are close to the induced edge labels for some graphs.

The theory of labeling is a significant subject in the theory of graphs, Alex Rosa first presented the labeling idea in 1967, In his research entitled "On certain valuations of the vertices of a graph," The idea of labeling is one of many issues in graph theory. Rosa's β-valuation is the labeling that has received the most attention[2]. And who Solomon W. Golomb afterward referred to as a graceful labeling[3].

There are sundry different types of graph labeling; nonetheless, the following definitions are crucial for this paper present.

Consider a graph G whose vertex set contains p elements and whose edge set contains q elements.

Graceful Labeling: Function $V(G) \to \{0,1,\dots,q\}$ is said to be a *graceful labeling* if it is injective, and the corresponding edge labeling $f'(e) = |f(u) - f(v)|$ or each edge $e = uv$ is also injective, assigning all integers from 1 to q exactly once to the edges. graphs that include such a labeling type is known as *graceful* [4].
Edge-Graceful Labeling: Graph G is considered to be *"edge-graceful"* when there exists a 1 to 1 correspondence $f: E(G) \to \{1,2,\dots,q\}$ that assigns labels to edges in such a way that induced vertex-labeling $f^*(x) \equiv \sum_{xy\in E} f(xy)$ mod p is also one-to-one. This labeling method, which acts as the dual of graceful labeling, was introduced by Lo in 1985 [5].
In this paper, we will prove several theorems a number of graphs where edge graceful vertex labeling can be found by assigning a positive integer to the vertex so that each vertex is distinct from other vertices in accordance with the graph's graceful labeling conditions.

In our research, we take a number of the graph that have not previously been proven as edge graceful labeling, and from them $(K_1 + K_{1,n})^*$, $(C_4 + nK_1)^*$, $F^*_{n,2}$, $(k_2 + K_{1,n})^*$, the Fibonacci cordial C^*_n, and finally $(nC_4 \circ P_3)^*$.

Theorem 1: Each integer $n >= 2$, graph $(K_1 + K_{1,n})^*$ possesses an edge-graceful labeling.

Proof. Let us examine the graph$(K_1 + K_{1,n})^*$, which has:

Vertex set: $\{v1, v2, \dots, vn + 2\}$

Edge set: $\{v_{n+1}v_i \mid 1 \le i \le n\} \cup \{v_{n+2}v_i \mid 1 \le i \le n\} \cup \{v_{n+1}v_{n+2}\}$

This construction yields $p = n + 2$ vertices, and $q = 2n + 1$ edges. edge-gracefulness is demonstrated by assign labels via a mapping $\varphi: E(G) \to \{1,2,\dots,q\}$ such that the resulting vertex-labeling is induced accordingly:

$$\varphi^*(x) \equiv \sum_{xy \in E(G)} \varphi(xy)\ (mod\ p)$$

yields a bijection to $\{0,1,\dots,p-1\}$. For any $n \ge 2$, we construct the labeling as follows:

$$\gamma_1: V(K_1 + K_{1,n})^* \to \{0,1,\dots,2n+1\}$$

Such that:

$$\gamma_1(v_i) = \begin{cases} 2i+1 & \text{to } i \in \{1,2,\dots n\}, \\ i-n-1 & i \in \{n+1, n+2\}. \end{cases}$$

For $n >= 2$, the edge-graceful labeling to graph $(K_1 + K_{1,n})^*$ is obtained using a vertex labeling γ_1 and the corresponding induced edge labeling γ_1^*, where:

$\gamma_1(v_i) = 2i + 1$ to $i \in \{1,2,\dots n\}$ (pendant vertices)

$\gamma_1(v_{n+1}) = 0$ (star center)

$\gamma_1(v_{n+2}) = 1$ (join vertex)

The induced edge labels are computed as follows:

Edges connected to the central vertex of the star $(v_{n+1}v_i)$:

For each $i \in \{1,\dots,n\}$, the induced edge label is computed as

$$\gamma_1^*(v_{n+1}v_i) = |\gamma_1(v_{n+1}) - \gamma_1(v_i)| = |0 - (2i+1)| = 2i + 1$$

Edges linked to the join vertex $(v_{n+2}v_i)$, For each $i \in \{1,\dots,n\}$, the induced edge label is determined by:

$$\gamma_1^*(v_{n+2}v_i) = |\gamma_1(v_{n+2}) - \gamma_1(v_i)| = |1 - (2i+1)| = 2i$$

Edge connecting v_{n+1} and v_{n+2}, for this edge, the induced label is calculated as:

$$\gamma_1^*(v_{n+1}v_{n+2}) = |\gamma_1(v_{n+1}) - \gamma_1(v_{n+2})| = |0 - 1| = 1.$$

The complete collection of edge labels can be expressed as:

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$$\{2i+1: i \in \{1,\dots,n\}\} \cup \{2i: i \in \{1,\dots,n\}\} \cup \{1\}$$
$$= \{1,2,\dots,2n+1\}.$$

This set forms a sequence of consecutive integers from 1 to $2n+1$, where $2n+1$ corresponds to the total number of edges in the graph. Therefore, $(K_1+K_{1,n})^*$ supports an edge-graceful labeling for every integer $n >= 2$. The labeling arrangement for the specific case $n=6$ is shown in **Figure 1**.

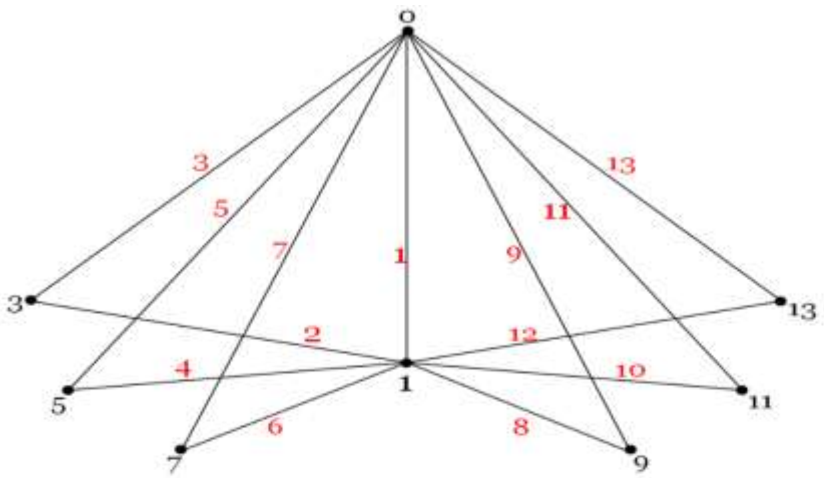

Figure 1: Constructing Edge-Graceful Labelings of $(K_1+K_{1,6})^*$

Theorem 2: The graph $(C_4+nK_1)^*$ admits an edge-graceful labeling for all $n \geq 1$.

Proof: Consider the graph $(C_4+nK_1)^*$ have:

Vertex group: $V((C_4+nK_1)^*) = \{v_i: i \in 1,2,\dots 4n\}$
Edge group:
$$E((C_4+nK_1)^*) = \{v_iv_{n+1}, v_iv_{n+2}, v_iv_{n+3}, v_iv_{n+4} : i \in 1,2,\dots n\}$$
$$\cup \{v_{n+i}v_{n+i+1}: i = 1,2,3\} \cup \{v_{n+4}v_{n+1}\}$$

Vertex labeling is as follows for every $n \geq 1$:

$\gamma_2: V((C_4+nK_1)^*) \to \{0,1,2,\dots,4n+4\}$

Such that:

$$\gamma_2(v_i) = \begin{cases} i & i = 1,2,\dots n, \\ 4in+4i-4n^2-8n-4 \quad \text{for} & i = n+1, n+2, \\ ni-2n+i-n^2-1 & i = n+3, n+4. \end{cases}$$

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To the graceful-labeling, we will check the induced edge labeling for the graph $C_4 + nK_1$ as follows:

1 $\gamma_2^*(v_iv_{n+1}) = |\gamma_2(v_i) - \gamma_2(v_{n+1})| = |i - [4(n+1)n + 4(n+1) - 4n^2 - 8n - 4]| = i$ for $i \in 1,$

2 $\gamma_2^*(v_iv_{n+2}) = |\gamma_2(v_i) - \gamma_2(v_{n+2})| = |i - [4(n+2)n + 4(n+2) - 4n^2 - 8n - 4]| = 4n - i + 4$ for $i \in 1,$

3 $\gamma_2^*(v_iv_{n+3}) = |\gamma_2(v_i) - \gamma_2(v_{n+3})| = |i - [n(n+3) - 2n + (n+3) - n^2 - 1]| = 2n - i + 2$ for $i \in 1,$

4 $\gamma_2^*(v_iv_{n+4}) = |\gamma_2(v_i) - \gamma_2(v_{n+4})| = |i - [n(n+4) - 2n + (n+4) - n^2 - 1]| = 3n - i + 3$ for $i \in 1,$

5 $\gamma_2^*(v_{n+1}v_{n+2}) = |\gamma_2(v_{n+1}) - \gamma_2(v_{n+2})| = |[4(n+1)n + 4(n+1) - 4n^2 - 8n - 4] - [4(n+2)n + 4(n+2) - 4n^2 - 8n - 4]| = 4n + 4$

6 $\gamma_2^*(v_{n+2}v_{n+3}) = |\gamma_2(v_{n+2}) - \gamma_2(v_{n+3})| = |[4(n+2)n + 4(n+2) - 4n^2 - 8n - 4] - [n(n+3) - 2n + (n+3) - 1]| = 2n + 2$

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$$|\gamma_2(v_{n+3}) - \gamma_2(v_{n+4})|$$

$$\gamma_2^*(v_{n+3}v_{n+4}) = \quad |[n(n+3) - 2n + (n+3) - n^2 - 1] - [n(n+4) - 2n + (n+4) -$$

$$n+1$$

$$|\gamma_2(v_{n+4}) - \gamma_2(v_{n+1})|$$

8 $\gamma_2^*(v_{n+4}v_{n+1}) = \quad |[n(n+4) - 2n + (n+4) - n^2 - 1] - [4(n+1)n + 4(n+1) - 4n^2 - 4]|$

$$3n+3$$

By combining 1, 2, 3, 4, 5, 6, 7, and 8, we get that the edges received the numbers $4n+5-i$ for $i = 1,2,\dots,4n+4$.

Hence, the $C_4 + nK_1$ graph is edge graceful labeling for every $n \geq 1$.

Figure 2 presents an example of an edge-graceful labeling applied to the graph $(3K_1 + C_4)$

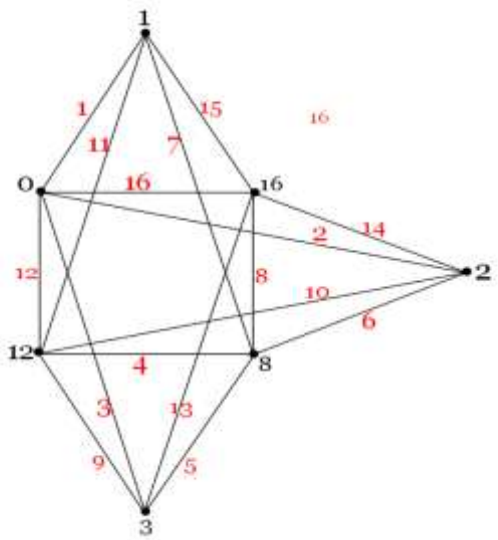

Figure 2: An illustration of the edge-graceful labeling for the graph $(\overline{K_3} + C_4)$

Theorem 3:

For all integers $n \geq 3$, the complement of the double fan graph $F_{n,2}^*$ admits an edge-graceful labeling

Proof. Let G= $F_{n,2}^*$ represent the complement of the double fan graph. The vertex set is given by:

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$V(G) = \{v_k \mid 1 \leq k \leq n+2\}$

The edge set can be described as:

$E(G) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup$
$\{v_{n+1} v_i \mid 1 \leq i \leq n\} \cup \{v_{n+2} v_i \mid 1 \leq i \leq n\}$

Here v_{n+1} and v_{n+2} are the apex vertices. The graph has $\mid V(G) \mid = n+2$ vertices and $\mid E(G) \mid = (n-1) + 2n = 3n - 1$ edges.

For every $n \geq 3$, the vertex labeling is defined as follows:

$\gamma_3 : V(F^*_{n,2}) \to \{0,1,2,\ldots,3n-1\}$ Such that:

$$\gamma_3(v_i) = \begin{cases} \dfrac{6n-3i+1}{2} & \begin{cases} if\ n\ is\ even\ and\ i\ \ i \in \{1,3,\ldots,n-1\} \\ or\ if\ n\ is\ odd\ and\ i\ \ i \in \{1,3,\ldots,n\} \end{cases} \\ \dfrac{3i-2}{2} & \begin{cases} if\ n\ is\ even\ and\ i \in \{2,4,\ldots,n-1\} \\ or\ if\ n\ is\ odd\ and\ i \in \{2,4,\ldots,n\} \end{cases} \\ i-n-1 & \text{when } i \in \{n+1, n+2\}. \end{cases}$$

Now, to graceful labeling, we will check the induced edge labeling for the double fan graph $F^*_{n,2}$, as follows:

1 $\gamma_3^*(v_i v_{i+1}) = \left|\gamma_3(v_i) - \gamma_3(v_{i+1})\right|$ to $i \in 1,2,\ldots n$-1

$$\begin{cases} \left|\dfrac{6n-3i+1}{2} - \dfrac{3(i+1)-2}{2}\right| & \begin{cases} i \in 1,3,\ldots,n-1, & n\ is\ even \\ i \in 1,3,\ldots,n-2, & n\ is\ odd \end{cases} \\ \left|\dfrac{3i-2}{2} - \dfrac{6n-3(i+1)+1}{2}\right| & \begin{cases} i \in 2,4,\ldots,n-1, & n\ is\ odd \\ i \in 2,4,\ldots,n-2, & n\ is\ even \end{cases} \end{cases}$$

$3n - 3i$ $\quad i \in 1,2,\ldots n$-1

2 $\gamma_3^*(v_{n+1} v_i) = \left|\gamma_3(v_{n+1}) - \gamma_3(v_i)\right|$ to $i \in 1,2,\ldots n$

$$\begin{cases} \left|(n+1)-n-1-\dfrac{6n-3i+1}{2}\right| & \begin{cases} i \in 1,3,\ldots,n-1, & n\ is\ even \\ i \in 1,3,\ldots,n, & n\ is\ odd \end{cases} \\ \left|(n+1)-n-1-\left(\dfrac{3i-2}{2}\right)\right| & \begin{cases} i \in 2,4,\ldots,n-1, & n\ is\ odd \\ i \in 2,4,\ldots,n-2, & n\ is\ even \end{cases} \end{cases}$$

$$\begin{cases} \dfrac{6n-3i+1}{2} & \begin{cases} i \in 1,3,\ldots,n-1, & n\ is\ even \\ i \in 1,3,\ldots,n, & n\ is\ odd \end{cases} \\ \dfrac{3i-2}{2} & \begin{cases} i \in 2,4,\ldots,n-1, & n\ is\ odd \\ i \in 2,4,\ldots,n-2, & n\ is\ even \end{cases} \end{cases}$$

3 $\gamma_3^*(v_{n+2} v_i) = \left|\gamma_3(v_{n+2}) - \gamma_3(v_i)\right|$ to $i \in 1,2,\ldots n$

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$$\begin{cases} \left|(n+2)-n-1-\dfrac{6n-3i+1}{2}\right| & \begin{cases} i \in 1,3,\dots,n-1, & n \text{ is even} \\ i \in 1,3,\dots,n, & n \text{ is odd} \end{cases} \\ \left|(n+2)-n-1-(\dfrac{3i-2}{2})\right| & \begin{cases} i \in 2,4,\dots,n-1, & n \text{ is odd} \\ i \in 2,4,\dots,n, & n \text{ is even} \end{cases} \end{cases}$$

$$\begin{cases} \dfrac{6n-3i-1}{2} & \begin{cases} i \in 1,3,\dots,n-1, & n \text{ is even} \\ i \in 1,3,\dots,n, & n \text{ is odd} \end{cases} \\ \dfrac{3i-4}{2} & \begin{cases} i \in 2,4,\dots,n-1, & n \text{ is odd} \\ i \in 2,4,\dots,n, & n \text{ is even} \end{cases} \end{cases}$$

By combining 1, 2, and 3, we get that the edges received the numbers $3n-i$ for $i=1,2,\dots,3n-1$.

Hence, The double fan graph $F^*_{n,2}$ is edge graceful labeling for every $n \geq 3$.

Figure 3 illustrates the edge-graceful labeling for the double-fan graph $F_{5,2}$

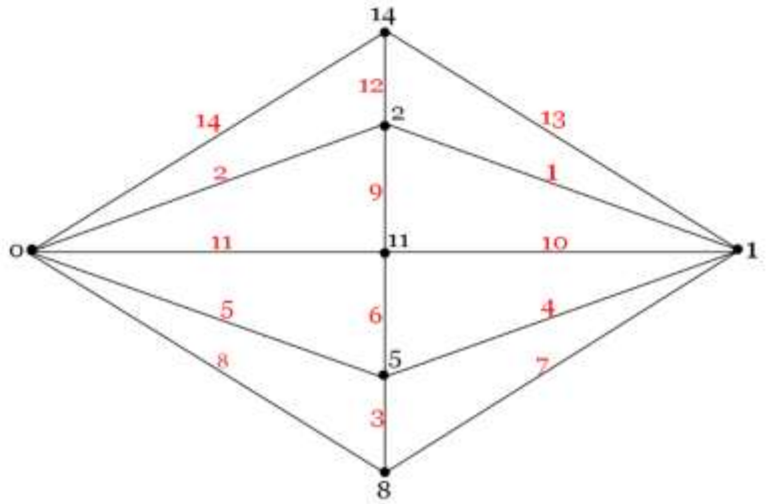

Figure 3: edge graceful Labeling of $F_{5,2}$

Theorem 4:

the graph $(k_2+K_{1,n})^*$ is edge graceful labeling, For every $n >= 2$.

Proof: Consider the graph $(k_2+K_{1,n})^*$, defined by its vertex set and edge set then:

$$V\left((k_2+K_{1,n})^*\right) = \{v_i : i = 1,2,\dots n+3\}$$

$$E\left((k_2+K_{1,n})^*\right) = \{v_i v_{n+1}, v_i v_{n+2}, v_i v_{n+3} : i = 1,2,\dots n\} \cup \{v_{n+1}v_{n+2}, v_{n+1}v_{n+3}\}$$

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For every $n \geq 2$, vertex-labeling is specified as follows:

$$\gamma_4 : V\left((k_2 + K_{1,n})^*\right) \rightarrow \{0,1,2,\dots,3n+2\}$$

Such that:

$$\gamma_4(v_i) = \begin{cases} i-1 & \text{for } i \in \{1,2,\dots n+1\} \\ (n+1)(i-n) \ -1 & \text{for } i \in \{n+2, n+3\} \end{cases}$$

To graceful-labeling, induced edge labeling for the graph$(k_2 + K_{1,n})^*$ is as follows:

1- $\gamma_4^*(v_i v_{n+1}) = \left|\gamma_4(v_i) - \gamma_4(v_{n+1})\right|$ for $i \in \{1,2,\dots n\}$

$= | \ (n-i)+1 \ |$ for $i \in \{1,2,\dots n\}$

$= n - i + 1$ for $i \in \{1,2,\dots n\}$

2- $\gamma_4^*(v_i v_{n+2}) = \left|\gamma_4(v_i) - \gamma_4(v_{n+2})\right.$ for $i \in \{1,2,\dots n\}$

$= \left|(i-1) - [n(n+2) - n^2 - n + (n+2) - 1]\right.$ for $i \in \{1,2,\dots n\}$

$= 2n - i + 2$ for $i \in \{1,2,\dots n\}$

3- $\gamma_4^*(v_i v_{n+3}) = \left|\gamma_4(v_i) - \gamma_4(v_{n+3})\right|$ for $i = \{1,2,\dots n\}$

$= \left|(i-1) - [(n-i) + (n+2)]\right|$ for $i = \{1,2,\dots n\}$

$= 3n - i + 3$ for $i = \{1,2,\dots n\}$

4- $\gamma_4^*(v_{n+1} v_{n+2}) = \left|\gamma_4(v_{n+1}) - \gamma_4(v_{n+2})\right|$

$= \left|[(n+1)-1] - [n(n+2) - n^2 - n + (n+2) - 1]\right|$

$= n + 1$

5- $\gamma_4^*(v_{n+1} v_{n+3}) = \left|\gamma_4(v_{n+1}) - \gamma_4(v_{n+3})\right|$

$= \left|[(n+1)-1] - [n(n+3) - n^2 - n + (n+3) - 1]\right|$

$= 2n + 2$

By combining 1, 2, 3, 4, and 5, we get that the edges received the numbers $3n + 2 - i$ to $i \in \{1,2,\dots,3n+2\}$.

While $(k_2 + K_{1,n})^*$ plot is edge-graceful labeling to every $n >= 2$.

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Figure 4 illustrates edge-graceful-labeling or $(k_2 + K_{1,4})^*$ plot.

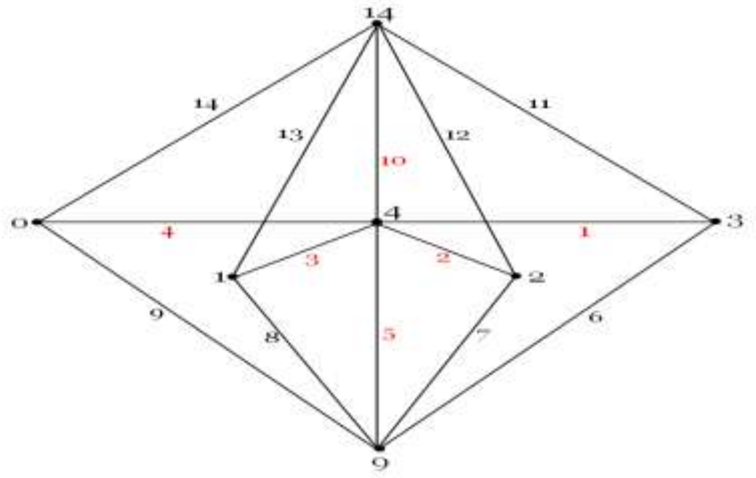

Figure 4: Edge Graceful Labeling of $(k_2 + K_{1,4})^*$

Theorem 5:

For every $n >= 5$, shell-graph C_n^* is "edge graceful labeling".

Proof: Consider a Fibonacci cordial graph, defined by its vertex set and edge set, where:

$V(C_n^*) = \{v_i : i \in 1,2, \dots n\}$

$E(C_n^*) = \{v_i v_{i+1} : i = 1,2, \dots n-1\} \cup \{v_n v_1, v_1 v_i : i = 1,2, \dots, n-1\}$

For every $n \geq 5$, we define the vertex labeling as follows:

$\gamma_5 : V(C_n^*) \rightarrow \{0,1,2, \dots, 2n-3\}$ Such that:

$$\gamma_5(v_i) = \begin{cases} i-1 & \textbf{To}: i = 1 \\ i-2 & \begin{cases} i \in \{3,5, \dots, n-1\}, & n \text{ is "even"} \\ i \in \{3,5, \dots, n\}, & n \text{ is "odd"} \end{cases} \\ 2n-i-1 & \begin{cases} i \in \{2,4, \dots, n-1\}, n \text{ is "odd"} \\ i \in \{2,4, \dots, n\}, n \text{ is "even"} \end{cases} \end{cases}$$

Now, for the graceful labeling, we will check the induced edge labeling for the Fibonacci cordial graph as follows:

1- $\gamma_5^*(v_1 v_2) = |\gamma_5(v_1) - \gamma_5(v_2)|$

$= |0 - (2n-2-1)|$

$= 2n-3$

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2-

$\gamma_5^*(v_i v_{i+1}) = |\gamma_5(v_i) - \gamma_5(v_{i+1})| \quad To: i \in \{2,3,\dots n-1\}$

$$= \begin{cases} |(2n-i-1)-(i+1-2)| & \begin{cases} i \in \{2,4,\dots,n-2\}, & n \text{ is "even"} \\ i \in \{2,4,\dots,n-1\}, & n \text{ is "odd"} \end{cases} \\ |(i-2)-[2n-(i+1)-1]| & \begin{cases} i \in \{3,5,\dots.,n-2\}, & n \text{ is "odd"} \\ i \in \{3,5,\dots,n-1\} & n \text{ is "even"} \end{cases} \end{cases}$$

$$= 2n-2i \qquad \text{To } i \in \{2,3,\dots n-1\}$$

3- $\gamma_5^*(v_n v_1) = |\gamma_5(v_n) - \gamma_5(v_1)|$

$$= \begin{cases} |(n-2)-0| & \text{to } n \text{ is "odd"} \\ |(2n-n-1)-0| & \text{to } n \text{ is "even"} \end{cases}$$

$$= \begin{cases} n-2 & \text{for } n \text{ is odd }. \\ n-1 & \text{for } n \text{ is even }. \end{cases}$$

4-$\delta_5^*(v_1 v_i) = |\delta_5(v_1) - \delta_5(v_i)| \quad \text{To: } i \in \{3,4,\dots n-1\}$

$$= \begin{cases} |0-(i-2)| & \begin{cases} i \in \{3,5,\dots,n-2\}, & n \text{ is "odd"} \\ i \in \{3,5,\dots,n-1\}, & n \text{ is "even"} \end{cases} \\ |0-(2n-i-1)| & \begin{cases} i \in \{2,4,\dots.,n-2\}, & n \text{ is "even"} \\ i \in \{2,4,\dots,n-1\}, & n \text{ is "odd"} \end{cases} \end{cases}$$

$$= \begin{cases} i-2 & \begin{cases} i \in \{3,5,\dots,n-2\}, & n \text{ is "odd"} \\ i \in \{3,5,\dots,n-1\}, & n \text{ is "even"} \end{cases} \\ 2n-i-1 & \begin{cases} i \in \{2,4,\dots.,n-2\}, & n \text{ is "even"} \\ i \in \{2,4,\dots,n-1\}, & n \text{ is "odd"} \end{cases} \end{cases}$$

By combining 1, 2, 3, and 4, we get that the edges received the numbers $2n-2-i$ for $i = 1,2,\dots,2n-3$.

Thus, the Fibonacci cordial C_n^* is edge graceful labeling for every $n \geq 5$.

Theorem 6:

For every $n \geq 2$, the graph $(nC_4 \circ P_3)^*$ is edge-gracefully labeling.

Proof: Let the vertex set and the edge set of the graph $(nC_4 \circ P_3)^*$ be:

$$V((nC_4 \circ P_3)^*) = \{v_i : i = 1,2,\dots,3n+1\} \cup \{u_i : i = 1,2,\dots,2n-2\}$$

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$E((nC_4 \circ P_3)^*) = \{v_{2n+i}v_i, v_{2n+i}v_{n+i}, v_{2n+i+1}v_i, v_{2n+i+1}v_{n+i} : i = 1,2, \dots n\}$

$\cup \{v_{2n+i+1}u_i, v_{2n+i+1}u_{n+i-1} : i = 1,2, \dots, n-1\}$

For every $n \geq 2$, we define the vertex labeling as follows:

$\gamma_6 : V(nC_4 \circ P_3)^* \to \{0,1,2, \dots ,6n-2\}$ Such that:

$$\gamma_6(v_i) = \begin{cases} 6n-4i+2 & \text{for } i = 1,2, \dots, n, \\ 10n-4i+1 & \text{for } i = n+1, n+2, \dots ,2n, \\ 2i-4n-2 & \text{for } i = 2n+1, 2n+2, \dots ,3n+1. \end{cases}$$

$$\gamma_6(u_i) = \begin{cases} 6n-4i & \text{for } i = 1,2, \dots, n-1, \\ 9n-4i & \text{for } i = n, n+1, \dots ,2n-2. \end{cases}$$

For the graceful labeling, we will check the induced edge labeling for the graph $(nC_4 \circ P_3)^*$ when $n \geq 2$, as follows:

1- $\gamma_6^*(v_{2n+i}v_i) = |\gamma_6(v_{2n+i}) - \gamma_6(v_i)|$ for $i = 1,2, \dots n$

$= |[2(2n+i) - 4n - 2] - [6n - 4i + 2]|$ for $i = 1,2, \dots n$

$= 6n - 6i + 4$ for $i = 1,2, \dots n$

2- $\gamma_6^*(v_{2n+i+1}v_i) = |\gamma_6(v_{2n+i+1}) - \gamma_6(v_i)|$ for $i = 1,2, \dots n$

$= |[2(2n+i+1) - 4n - 2] - [6n - 4i + 2]|$ for $i = 1,2, \dots n$

$= 6n - 6i + 2$ for $i = 1,2, \dots n$

3- $\gamma_6^*(v_{2n+i}v_{n+i}) = |\gamma_6(v_{2n+i}) - \gamma_6(v_{n+i})|$ for $i = 1,2, \dots n$

$= |[2(2n+i) - 4n - 2] - [10n - 4(n+i) + 1]|$ for $i = 1,2, \dots n$

$= 6n - 6i + 3$ for $i = 1,2, \dots n$

4- $\gamma_6^*(v_{2n+i+1}v_{n+i}) = |\gamma_6(v_{2n+i+1}) - \gamma_6(v_{n+i})|$ for $i = 1,2, \dots n$

$= |[2(2n+i+1) - 4n - 2] - [10n - 4(n+i) + 1]|$ for $i = 1,2, \dots n$

$= 6n - 6i + 1$ for $i = 1,2, \dots n$

5- $\gamma_6^*(v_{2n+i+1}u_i) = |\gamma_6(v_{2n+i+1}) - \gamma_6(u_i)|$ for $i = 1,2, \dots n-1$

$= |[2(2n+i+1) - 4n - 2] - [6n - 4i]|$ for $i = 1,2, \dots n-1$

$= 6n - 6i$ for $i = 1,2, \dots n-1$

6- $\gamma_6^*(v_{2n+i}v_{n+i}) = |\gamma_6(v_{2n+i+1}) - \gamma_6(u_{n+i-1})|$ for $i = 1,2, \dots n-1$

$= |[2(2n+i+1)-4n-2]-[9n-4(n+i-1)]|$ for $i = 1,2,\dots n-1$

$= 5n-6i+4$ for $i = 1,2,\dots n-1$

By combining 1, 2,3,4,5, and 6, we get that the edges received the numbers $6n-i-1$ for $i = 1,2,\dots,6n-2$.

Thus, the graph $(nC_4 \circ P_3)^*$ is edge graceful labeling for every $n \geq 2$.

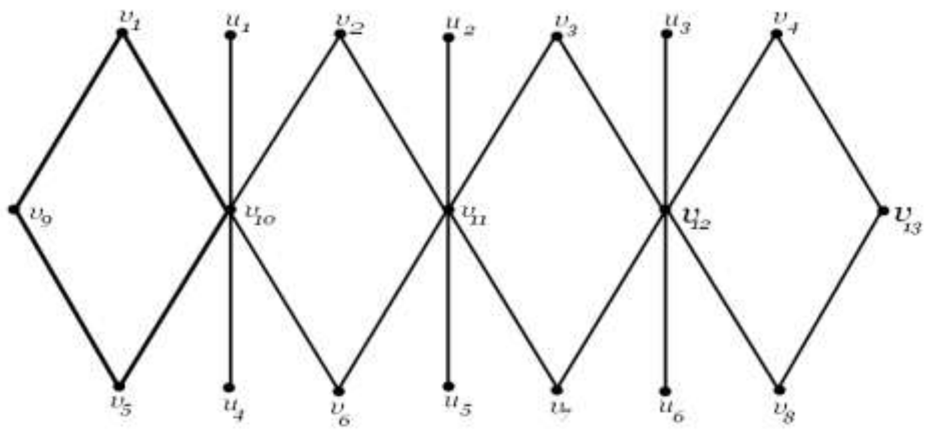

Figure 5: The graph $(nC_4 \circ P_3)^*$ when $n = 4$.

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