Special Chebyshev Polynomials for Solving Quadratic Optimal Control Problems

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Abstract

This paper aims to present a special family of Chebyshev polynomial (SFCP). The SFCP have been defined and their important properties are discussed. Then, the derivative of the state variable in the dynamic constraint of quadratic optimal control problem is approximated by SFCP with unknown coefficients. The operational matrix of derivative together with the dynamical constraints is used to approximate the control variable directly as a function of the state variable. Finally, these approximations are substituted in the performance index and necessary conditions for optimality transform the original quadratic optimal control problem to a quadratic programming problem. The resulting performance index optimal value shows that the proposed method provides a good treatment with fast convergence. The effectiveness of the presented method is illustrated by solving some numerical examples. The obtained results reveal that utilizing SFCP gives an efficient solution and it may exactly converge to the analytical one with minimum number of SFCP.

Keywords: Special family of Chebyshev polynomials, quadratic programming problem, quadratic optimal control problem, convergence.

متعددات حدود شفشف الخاصة لحل مسائل السيطرة المثلى التربيعية

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الخلاصة

يهدف هذا البحث الى تقديم عائلة خاصة من متعددات حدود شفشف (SFCP). تم تعريف SFCP ومنقشة خواصها المهمة. بعد ذلك ، يتم تقريب مشتقة متغير الحالة في القيد الديناميكي لمشكلة التحكم الأمثل التربيعي بواسطة SFCP مع معاملات غير معروفة. يتم استخدام المصفوفة التشغيلية للمشتقات مع القيود الديناميكية لتقريب متغير السيطرة مباشرة كدالة لمتغير الحالة. أخيرًا ، يتم تعويض هذه التقريبات في معامل الأداء والشروط اللازمة لتحقيق الأمثلية وتحويل مشكلة التحكم الأمثل التربيعي الأصلي إلى مشكلة البرمجة التربيعية. تظهر القيمة المثلى لمعامل الأداء الناتج أن الطريقة المقترحة توفر معالجة جيدة مع تقارب سريع. يتم توضيح فعالية الطريقة المقدمة من خلال حل بعض الأمثلة العددية. تكشف النتائج التي تم الحصول عليها أن استخدام SFCP يعطي حلاً فعالاً وقد يتقارب تمامًا مع الحل التحليلي بأقل عدد من SFCP.

الكلمات المفتاحية: عائلة خاصة من كثيرات حدود Chebyshev ، مشكلة البرمجة التربيعية ، مشكلة التحكم الأمثل التربيعي ، التقارب.

1. Introduction

Approximate methods based on orthogonal functions have been applied to find the solution for calculus of varaitional problems [1-6], fractional Optimal Control and fractional variation problem [7], fractional Emden-Fowler equation [8], nonlinear fractional optimal control problem [9], some problems arising in engineering [10] and in astrophyscics [11].The characteristic of most methods is that they transform the original problem under consideration conditions to a system of algebraic equations. It is known that orthogonal polynomials allow approximation of smooth functions where truncation error approaches zero faster than any negative power of the number of basic functions utilized in the approximation technique.

The optimal control of a system, which is minimization of a performance index subject to dynamical system, is one of the most practical subjects in science and engineering [12-14]. The numerical solution of optimal control problems have been investigated by many researchers. For example, Chebyshev cardinal functions [15], Hermite polynomials [16], Gegenbauer Polynomials [17], Hermite Wavelet method [18]. The main aim of this paper is to present an efficient direct approach based on special family of Chebyshev polynomials for solving quadratic optimal control problems.

The outline of this paper is as follows: In Section 2, special family of Chebyshev polynomials and their important properties are presented. Section 3 deals with numerical solution of quadratic optimal control problem. In Section 4, three illustrative test examples are included to show the accuracy and efficiency of the special family Chebyshev polynomials. Finally, concluding remarks are listed in Section 5.

2. Special Family of Chebyshev polynomials and Their Properties

The special family of Chebyshev polynomials (SFCP) can be defined recursively as below:

$$SF_m(t) = tSF_{m-1}(t) - SF_{m-2}(t),\ m \in N \quad (1)$$

with initial conditions $SF_0(t) = 2,\ SF_1(t) = t$.

In other words

$$SF_m(t) = \begin{cases} 2 & if\ m = 0, \\ t & if\ m = 1, \\ tSF_{m-1}(t) - SF_{m-2}(t) & if\ m > 1. \end{cases}$$

The eight SFCP are: $M_0(t) = 2$, $M_1(t) = t$, $M_2(t) = t^2 - 2$, $M_3(t) = t^3 - 3t$, $M_4(t) = t^4 - 4t^2 + 2$, $M_5(t) = t^5 - 5t^3 + 5t$, $M_6(t) = t^6 - 6t^4 + 9t^2 - 2$

$M_7(t) = t^7 - 7t^5 + 14t^3 - 7t.$

Note that the general matrix form of SFCP can be written as below:

$$SF(t) = H\tau(t)^T \tag{2}$$

where $SF(t) = [SF_0(t)\, SF_1(t)\ \ SF_2(t) \cdots SF_m(t)]$, $\tau(t) = [1 \quad t \quad t^2 \quad \cdots \quad t^m]$ and H is the lower triangle matrix constructed as

For odd n

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ -2 & 0 & 1 & 0 & \cdots & 0 \\ 0 & g_{3,1} & 0 & 1 & \cdots & 0 \\ 2 & 0 & g_{4,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2cos\frac{m}{2}\pi & h_{m,1} & h_{m,2} & h_{m,3} & \cdots & 1 \end{pmatrix}$$

For even n, the last row in matrix Hcan be defined as

$(0 \quad h_{m,1} \quad h_{m,2} \quad h_{m,3} \quad \cdots \quad h_{m,m-1} \quad 1).$

The entries h_{ij} in matrix H can be constructed as

$$h_{ij} = \begin{cases} \left(h_{i-1,j-1)} - h_{i-2,j}\right) & i - j = odd, \\ 0 & \text{otherwise.} \end{cases}$$

Another important property for SFCP is the derivative matrix of SFCP. By differentiating SFCP, one can obtain

$\dot{SF}_0(t) = 0, \dot{SF}_1(t) = 1, \dot{SF}_2(t) = 2t, \dot{SF}_3(t) = 3t^2 - 3, \dot{SF}_4(t) = 4t^3 - 8t,$
$\dot{SF}_5(t) = 5t^4 - 15t^2 + 5, \dot{SF}_6(t) = 6t^5 - 24t^3 + 18t, \dot{SF}_7(t) = 7t^6 - 35t^4 + 42t^2 - 7.$

Now, rewrite the above equation in matrix from as

$$\dot{SF}_m(t) = H\dot{\tau}(t) = H \begin{pmatrix} 0 \\ 1 \\ 2t \\ \vdots \\ mt^{m-1} \end{pmatrix}$$

or $\dot{SF}_m(t) = Hy_{(m+1)\times m}\tau(t)$

where $y_{(m+1)\times m} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{pmatrix}$ and $\tau(t) = \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{m-1} \end{pmatrix}$

3. The SFCP Algorithm for Solving Optimal Control Problems

Suppose that the process of certain optimal control problem described by the system of nonlinear differential equations on $[-1,1]$ as below

$$u(t) = f\big(x(t), \dot{x}(t)\big) \tag{3}$$

with initial conditions

$$x(-1) = \alpha,\ x(1) = \beta \tag{4}$$

where: $x(\cdot): [-1,1] \to \Re$ is the state variable,
$u(\cdot): [-1,1] \to \Re$, is the control variable and f is a real valued continuously differentiable function yielding the performance index J which is given by

$$J[x(t), u(t)] = \int_{-1}^{1} F\big(x^2(t), u^2(t)\big)dt \tag{5}$$

The proposed algorithm can be summarized by the following steps:

Step 1: Approximate the state variable $x(t)$ using SFCP, gives

$$x(t) = a^T SF(t), \tag{6}$$

where $a = [a_1, a_2, \dots, a_m]^T$, is $(m+1) \times 1$ vector of unknown parameters,

Step 2: Approximate $\dot{x}(t)$ to get

$$\dot{x}(t) = a^T \dot{SF}(t) \tag{7}$$

where $\dot{SF}(t)$ is the derivative vector of $SF(t)$.

Step 3: Obtain the approximation for the control variable by substituting Eq. 6 and Eq. 7 into Eq. 3 to obtain

$$u(t) = f\left(a^T SF(t), a^T \dot{SF}(t)\right) \tag{8}$$

Step 4: Determine the performance index value J as a function of the unknown $\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{m}$ as below

$$J(\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{m}) = \int_{-1}^{1} F\left(\left(a^T SF(t)\right)^2, \left(a^T \dot{SF}(t)\right)^2\right) dt$$

The functional J represents a nonlinear mathematical programming problem of unknown parameters $\mathrm{a}_0, a_1, a_2, \dots, a_m$.

Step 5: Approximate the boundary conditions $a^T SF(-1) = \alpha$, $a^T SF(1) = \beta$.

The resulting quadratic mathematical programming problem can be simplified as below:

$$J(\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{m}) = \frac{1}{2} a^T \mathcal{H} a$$

where $\mathcal{H} = 2\int_{-1}^{1} F\left(\left(SF(t)\right)^2, \left(\dot{SF}(t)\right)^2\right) dt$,

subject to $Fa - b = 0$

where $\mathcal{F} = \begin{bmatrix} SF^T(-1) \\ SF^T(1) \end{bmatrix}$, $b = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

Using Lagrange multiplier technique to obtain the optimal values of the unknown parameters a^*,

$$a^* = \mathcal{H}^{-1}\mathcal{F}^T(\mathcal{F}\mathcal{H}^{-1}\mathcal{F}^T)^{-1}b.$$

4. Numerical Results

All problems considered in the present paper have analytical solution to allow the validation of the algorithm comparing with exact solution results. Three quadratic optimal control problems have been solved by using the presented method for various values of m.

Example 1: This problem is concerned with minimization of

$J = \frac{1}{2}\int_{-1}^{1}(u(t)^2 + x(t)^2)d\tau,\ t\epsilon[-1,1]$

subject to

$u(t) = 2\dot{x}(t)$

with the conditions:

$x(-1) = 0,\ x(1) = 0.5.$ The exact performance index value for this problem is given by $J_{exact} = 0.328258821379.$

In Table 1, we list the optimal values of the unknown parameters a^* in case $m = 3, 4, 5$ using the proposed SFCP algorithm. The absolute error E_J of $|J_{exact} - J_{appr.}|$ is also listed in Table 1.

Table 1: Optimal values of the unknown parameters a^* and E_J for Example 1.

a_m	$m = 3$	$m = 4$	$m = 5$
a_0	0.13920454545454	0.13920454545454	0.14031835414808
a_1	0.27034883720930	0.27034883720930	0.27034883720930
a_2	0.01017441860465	0.02840909090909	0.03005129348795
a_3		0.01017441860465	0.01017441860465
a_4			0.0005.8541480820
E_J	−0.0003396634694	−0.0000005161826	−0.0000000093300

Example 2: Consider the following optimal control problem

$min\, J = \frac{1}{4}\int_{-1}^{1}(u^2\,(t) + x^2(t))dt,\ t \in [-1,1]$

when $u(t) = 2\dot{x}(t) + x(t),$

and $x(-1) = 1,\ x(1) = 0.2819695348$ are satisfied.

In Table 2, we list the optimal values of the unknown parameters a^* and E_J in case $m = 3, 4, 5$ using the proposed SFCP algorithm.

Table 2: Optimal values of the unknown parameters a^* and E_J for Example 2.

a_m	$m = 3$	$m = 4$	$m = 5$
a_0	0.387261630304167	0.387261630304167	0.397633552106756

a_1	0.359015232600000	−0.416131292331818	−0.41613129233181
a_2	0.133538493208333	0.133538493208333	0.148837077867152
a_3		−0.028558029865909	−0.02855802986590
a_4			0.005445258946359
E_J	−0.00138934373504	−0.0000.2230803705	−0.000000.14722411

Example 3: Consider the following optimal control problem
$min\, J = \frac{1}{2}\int_{-1}^{1}(u^2\,(t) + 3x^2(t))dt,\ \ t \in [-1,1]$
when $u(t) = 2\dot{x}(t) - x(t)$,
and $x(-1) = 1,\ x(1) = 0.51314538$.
The exact value $J_{exact} = 2.791658875$.

In Table 3, we list the optimal values of the unknown parameters a^* and $E_J$in case $m = 3, 4, 5$ using the proposed SFCP algorithm.

Table 3: The optimal values of the unknown parameters a^* and E_J for Example 3.

$\boldsymbol{a_m}$	$\boldsymbol{m = 3}$	$\boldsymbol{m = 4}$	$\boldsymbol{m = 5}$
a_0	0.513388611071429	0.513388611071429	0.554580117194293
a_1	−0.243427310000000	−0.317513882608696	−0.317513882608696
a_2	0.270204532142857	0.270204532142857	0.331000551875000
a_3		−0.037043286304348	−0.037043286304348
a_4			0.021586992513587
E_J	−0.005522360397764	−0.000713108377162	−0.000002049699817

5. Conclusion

The suggested modification in the direct parameterization method based on SFCP is applied to solve quadratic optimal control problems. Such technique gives an approximation to the state variable $x(t)$ in terms of SFCP which satisfy the given boundary conditions. The improvement in the suggested algorithm has succeeded to reach the solution with less number of SFCP terms. Three numerical examples are tested and the obtained results illustrate that the presented method is efficient and only small numbers of SFCP terms are needed to get satisfactory convergence.

References

[1] Rayal A., Verma SR., Numerical study of variational problems of moving or fixed boundary conditions by Muntz wavelets, Journal of Vibration and Control. vol. 28, pp. 214-229, 2022.

[2] Ouda E, Ibraheem S. Shihab S., Boubaker Wavelets Functions: Properties and Applications, Baghdad Science Journal 18 (4), 1226-1226, 2021.

[3] Keshavarz E., Ordokhani Y. and Razzaghi M., The Bernoulli wavelets operational matrix of integration and its applications for the solution of linear and nonlinear problems in calculus of variations. Applied Mathematics and Computation 351: 83–98, 2019.

[4] Mohammadi R., Shamsyeh M. and Bayat Z., Numerical Solution of Calculus of Variation Problems via Exponential Spline Method, Math. Sci. Lett. 4, No. 2, 101-108, (2015).

[5] Dolgopolik M.V, New Direct Numerical Methods for Some Multidimensional Problems of the Calculus of Variations, 2019.

[6] Bokhari A., Amir A. and Bahri, A numerical approach to solve quadratic calculus of variation problems. Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications and Algorithms 25: pp. 427-440, 2018.

[7] Rabiei L. and Ordokhani Y., Boubaker Hybrid Functions and their Application to Solve Fractional Optimal Control and Fractional Variational Problems,. Applications of Mathematics. vol. 63, pp. 1-27, 2018.

[8] Bencheikh A. and Chiter L., A new operational matrix based on Boubaker polynomials for solving fractional Emden-Fowler problem, no. x, pp. 1-14, 2022.

[9] Noori Dalawi A., Lakestani M. and Ashpazzadeh E., An Efficient Algorithm for the Multi-Scale Solution of Nonlinear Fractional Optimal Control Problems, Mathematics, vol. 10, no. 20, pp. 1-16, 2022.

[10] Alabdali O., Shihab S., Rasheed M., Orthogonal Bouboker-Turki Polynomials Algorithm for Problems Arising in Engineering, AIP Conference Proceedings 2386, 050019, 2022.

[11] Sarhan M., Shihab S., Kashem B., Rasheed M., New Exact Operational Shifted Pell Matrices and Their Application in Astrophysics, Journal of Physics: Conference Series 1879 (2), 022122, 2021.

[12] Jalal R., Shihab S., AlhadI M., Rasheed M., Spectral Numerical Algorithm for Solving Optimal Control Using Boubaker-Turki Operational Matrices, Journal of Physics: Conference Series 1660 (1), 012090, 2021.

[13] Casey R., Michael J., and Michael J., Generalized Regularization of Constrained Optimal Control Problems, Journal of Spacecraft and Rockets, 4 Jan 2022.

[14] Mohammad A., A modified pseudospectral method for indirect solving a class of switching optimal control problems, Volume: 234 issue: 9, page(s): 1531-1542, 2020.

[15] Mohammad H., A new direct method based on the Chebyshev cardinal functions for variable-order fractional optimal control problems, Journal of the Franklin Institute**,** Volume 355, Issue 12, Pages 4970-4995, 2018.

[16] Ayat O., Mirkamal M., Solving optimal control problems by using Hermite polynomials, Computational Methods for Differential Equations, Vol. 8, No. 2, pp. 314-329, 2020.

[17] Farzaneh S., Fahimeh S. and Kamal M., An Operational Matrix Method Based on the Gegenbauer Polynomials for Solving a Class of Fractional Optimal Control Problems, International Journal of Industrial Electronics, Control and Optimization ,Vol. 4, No. 4, pp. 475-484, 2021.

[18] Akram K., Asadollah M. and Sohrab E., Solving Optimal Control Problem Using Hermite Wavelet, Numerical Algebra , Control and Optimization Vol. 9, No. 1, 2019.

[19] Haniye D., Yadollah O. and Mohsen R., Fractional-order Bessel wavelet functions for solving variable order fractional optimal control problems with estimation error, International Journal of Systems Science, Vol. 51, Issue 6, 2020.