Al-Qadisiyah Journal of Pure Science

Manuscript 1247

On i-Totally Continuity and Some Relationships

Ruqayah N. Balo

Sabih W. Askandar

N. A. Abdul-Razaq

Follow this and additional works at: https://qjps.researchcommons.org/home

Part of the Biology Commons, Chemistry Commons, Computer Sciences Commons, Environmental Sciences Commons, Geology Commons, Mathematics Commons, and the Nanotechnology Commons

ARTICLE

On i-totally Continuity and Some Relationships

Ruqayah N. Balo, Sabih W. Askandar*, Nadia A. Abdul-Razaq

Department of Mathematics, College of Education for Pure Science, University of Mosul, Mosul, Iraq

Abstract

This study introduces i-totally continuity, a new generalization of strong continuity that is more robust than totally continuity. Additionally, several of these functions' characteristics are looked into. Also presented and researched i-totally open functions in topological spaces.

Keywords: i-open set, Clopen set, Totally continuity, i-totally continuity, i-totally open map

1. Introduction

i-open sets [1,2,6] are crucial for generalizing continuous functions in the study of general topological spaces. Numerous researchers have presented and investigated various kinds of generalizations of continuity using these sets. N. Levine [5] first described the category of semi-continuous functions in 1963. In 1980, Jain [4] introduced completely continuous functions. As a generalization of "totally continuous functions", T. M. Nour [7] developed the idea of "totally semi-continuous functions", and numerous features of these functions were discovered. This study introduces and studies i-totally continuity, a new generalization of strong continuity that is stronger than totally continuity. Further research is done into the fundamental characteristics of these functions, preservation theorems for i-totally continuous functions, and connections between i-totally continuous functions and other varieties of continuous functions. Also presented and researched i-totally open functions in topological spaces.

Throughout the entire paper, topological spaces are referred to as *X* and *Y*. Let *A* be a subset of *X*. *Cl(A)* and *Int(A)* respectively represent for the closure and interior of *A*. The term "i-open" refers to a subset A of X [1,2,6] (in short (*IOS*)) if $A \subset Cl(A \cap G), G \in \tau, G \neq \emptyset, X$. i-closed set, or *ICS* for short, is the complement of i-open set. *IOS(X)* stands for the family of i-open sets of X. The i-interior of *A* [1,2,6] is the union of all the i-open sets included in *A*, and it is represented by *IInt(A)*. The i-closure of *A* [1,2,6] is the intersection of all i-closed sets that contain *A*, and it is denoted by *ICl(A)*. Clopen set (abbreviated *COS*) is a set that is both open and closed. *COS*(X) stands for the family of all clopen sets.

2. Preliminaries

Definition 2.1. A function $f : X \to Y$ is called:

(i) i-continuous (in short (*ICONTF*) [1,2,6] if $f^{-1}(O^*) \in IOS(X), \forall O^* \in OS(Y)$.
(ii) Totally continuous (in short (*TCONTF*)) [4] if $f^{-1}(O^*) \in COS(X), \forall O^* \in OS(Y)$.
(iii) Strongly continuous (in short (*STCONTF*)) [8] if $f^{-1}(O^*) \in COS(X), \forall O^* \subseteq Y$.
(iv) Totally i-continuous (in short (*TICONTF*)) if $f^{-1}(O^*) \in ICOS(X), \forall O^* \in OS(Y)$.
(v) Strongly i-continuous (in short (*STICONTF*) if $f^{-1}(O^*) \in ICOS(X), \forall O^* \subseteq Y$.
(vi) i-irresolute (in short (*IIRREF*) [1,2,6] if $f^{-1}(O^*) \in IOS(X), \forall O^* \in IOS(Y)$.

3. Main results

The concept of i-totally continuous functions is presented in this section. It is possible to

Received 21 January 2024; accepted 11 April 2024.
Available online 20 August 2025

* Corresponding author.
E-mail address: sabihqaqos@uomosul.edu.iq (S.W. Askandar).

https://doi.org/10.29350/2411-3514.1247

2411-3514/© 2025 College of Science University of Al-Qadisiyah. This is an open access article under the CC-BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

characterize and establish certain connections between i-totally continuous functions and other related functions. Additionally, several fundamental i-totally continuous function characteristics are examined.

Definition 3.1. A function $f: X \to Y$ is called i-totally continuous (in short(*ITCONTF*) if $f^{-1}(G^*) \in COS(X), \forall G^* \in IOS(Y)$

Example 3.2. Let $X = Y = \{9,7,5,3\}, \tau = \{X, \varnothing, \{9,5\}, \{7,3\}\}$ and $\sigma = \{Y, \varnothing, \{9,5\}\}$. Then $IOS(Y) = \{Y, \varnothing, \{9\}, \{5\}, \{9,7\}, \{9,5\}, \{9,3\}, \{7,5\}, \{5,3\}, \{9,7,5\}, \{9,7,3\}, \{7,5,3\}, \{9,5,3\}\}$.
Define $f = (X,\tau) \to (Y,\delta)$ by $f(9) = f(5) = 5$ *and* $f(7) = f(3) = 9$. f is *ITCONTF*.

Theorem 3.3. $f: X \to Y$ is *ITCONTF iff* the inverse image of each i-closed subset of Y is *COS* in X.

Proof. Let C be any *ICS* in Y. Then $Y - C$ is *IOS* in Y. $f^{-1}(Y - C)$ *is COS in* X . *That is* $X - f^{-1}(C)$ is *COS* in X . Henceforth, $f^{-1}(C)$ *is COS in* X .

Conversely, if O is *IOS* in Y, then $Y - O$ is *ICS* in Y By hypothesis ,$f^{-1}(Y - O) = X - f^{-1}(O)$ is *COS* in X, we get, $f^{-1}(O)$ is *COS* in X . Accordingly, f is *ITCONTF*.

Proposition 3.4. [1,2,6] Each open set is i-open.

Theorem 3.5. Every *ITCONTF* is a *TCONTF*.

Proof. Suppose $f: X \to Y$ is *ITCONTF* and O is any open subset of Y, by ("Proposition 3.4") we get, O is *IOS* in Y and $f: X \to Y$ is *ITCONTF*, it follows $f^{-1}(O)$ is *COS* in X. Accordingly, f is *TCONTF*.

Example 3.6. Let $X = Y = \{7,5,3,1\}, \tau = \{X, \varnothing, \{7,3\}, \{5,1\}\}$ and $\sigma = \{Y, \varnothing, \{7,3\}\}$. Then $IO(Y) = \{Y, \varnothing, \{7\}, \{3\}, \{7,5\}, \{7,3\}, \{7,1\}, \{5,3\}, \{3,1\}, \{7,5,3\}, \{7,5,1\}, \{5,3,1\}, \{7,3,1\}\}$. Define $f = (X,\tau) \to (Y,\delta)$ by $f(7) = 3, f(5) = 1, f(3) = 7, f(1) = 5$. f is *TCONTF*. But *it* is not *ITCONTF*, because for the *IOS* $\{7,5\}$ *in* Y, $f^{-1}(\{7,5\}) = \{3,1\}$ isn't *COS* in X .

Theorem 3.7. Every *STCONTF* is *ITCONTF*.

Proof. Suppose $f: X \to Y$ is *STCONTF* and M be any open set in Y. We get, $f^{-1}(M)$ is *COS* in X and by ("Proposition 3.4") we get, M is *IOS* in . Accordingly, f is *ITCONTF*.

Theorem 3.8. Every *ITCONTF* is *TICONTF*.

Proof. Suppose $f: X \to Y$ is *ITCONTF* and M is any open set in Y. by (Proposition 3.4) we and $f: X \to Y$ is *ITCONTF*, it follows that $f^{-1}(M)$ is *COS* and hence *ICOS* in X. Accordingly, f is *TICONTF*.

Example 3.9. Let $X = Y = \{5,3,1\}, \tau = \{X, \phi, \{3\}, \{1\}, \{3,1\}\}$ and $\sigma = \{Y, \phi, \{3\}\}$. Then $iO(X) = \{X, \phi, \{3\}, \{1\}, \{5,3\}, \{5,1\}, \{3,1\}\}$ and $IOS(Y) = \{Y, \phi, \{3\}, \{5,3\}, \{3,1\}\}$. Define $f: (X,\tau) \to (Y,\tau)$ as $f(3) = 3$ and $f(5) = f(1) = 1$. f is *TICONTF*. But *it* is not *ITCONTF*, because for the *IOS*, $\{3\}$ in Y , $f^{-1}\{3\} =$ is not *COS* in X .

Theorem 3.10. Every *ITCONTF* is *ICONTF*.

Proof. Suppose $f: X \to Y$ is a *ITCONTF* and M is any "open set" in Y. Since f is *ITCONTF*, $f^{-1}(M)$ is *COS* in X, henceforth it is *ICOS* in X . We get, $f^{-1}(M)$ is *IOS* in X . Accordingly, f is *ICONTF* .

Example 3.11. *Let* $X = \{6,4,2\}, \tau = \{X, \phi, \{4\}, \{6,4\}\}$ *and* $\sigma = \{X, \phi, \{4\}\}$. Then $IOS(X) = \{X, \phi, \{6\}, \{4\}, \{6,4\}, \{6,2\}, \{4,2\}\}$.
$IOS(Y) = \{Y, \phi, \{4\}, \{6,4\}, \{4,2\}\}$ *Define* $f: (X,\tau) \to (Y,\sigma)$ *as* $f(6) = 6, f(4) = 4$ *and* $f(2) = 2$. Clearly f is *ICONTF* . But it is not *ITCONTF*, because for the set $\{b\}$ in Y , $f^{-1}\{4\} =$ is not *COS* in X .

As a result, there is the following relationship.

Strongly continuity $\longrightarrow$ ***i -totally continuity*** $\underset{\not\longleftarrow}{\longrightarrow}$ ***totally continuity*** $\underset{\not\longleftarrow}{\longrightarrow}$ ***totally i-continuity*** $\underset{\not\longleftarrow}{\longrightarrow}$ ***i-continuity.***

Theorem 3.12. Let $f : M \to N$, then the following statements are equivalent:

(i) f is *ITCONTF.*
(ii) for each $z \in M$ and each *IOS*, E in N with $f(z) \in E$, there is a *COS*, O in M s.t. $z \in O$ and $f(O) \in E$.

Proof. (i) $\Rightarrow$ (ii): Suppose $f : M \to N$ is *ITCONTF* and E be any *IOS* in N containing $f(z)$ so that $z \in f^{-1}(E)$. Since f *ITCONTF*, $f^{-1}(E)$ is *COS* in M. Let $O = f^{-1}(E)$, then O is *COS* in M and $z \in O$. Also $f(O) = f(f^{-1}(E)) \subset E$. This implies $f(O)) \subset E$.

(ii) $\Rightarrow$ (i): Let E be *IOS* in N Let $z \in f^{-1}(E)$ be any arbitrary point This implies $f(z) \in E$. Accordingly, by (ii) there exist a *COS* $f(O_z) \subset M$ containing z s.t. $f(O_z) \subset E$, which implies $O_z \subset f^{-1}(E)$, we have $z \in O_z \subset f^{-1}(E)$ This implies $f^{-1}(E)$ is "clopen neighborhood" of z Given that z is random, it implies $f^{-1}(E)$ is "clopen neighborhood" of each of its points . Hence it is *COS* in *M.* Accordingly, f is *ITCONTF.*

Theorem 3.13. The composition of two *ITCONTF* is *ITCONTF.*

Proof. Let $f : M \to N$ and $g : N \to K$ be any two *ITCONTF*. Let E be *IOS* in K. Since g is *ITCONTF*, $g^{-1}(E)$ is *COS* and hence open in . by ("Proposition 3.4") we get, $g^{-1}(E)$ is *IOS* in N. Further, since f is *ITCONTF*, $f^{-1}(g^{-1}(E)) = (g \, o \, f)^{-1}(E)$ is *COS* in . Hence $g \, o \, f : M \to K$ is *ITCONTF.*

Theorem 3.14. *If* $f : M \to N$ is *ITCONTF* and $g : N \to K$ is *IIRREF*, then $gof : M \to K$ is *ITCONTF.*

Proof. Let $f : M \to N$ be *ITCONTF* and $g : N \to K$ be *IIRREF*. Let E be *IOS* in K . Since g is *IIRREF*, $g^{-1}(E)$ is *IOS* in N. Now since f is *ITCONTF*, $f^{-1}(g^{-1}(E)) = (gof)^{-1}(E)$ is *COS* in M. Hence $gof : M \to K$ is *ITCONTF.*

Theorem 3.15. *If* $f : M \to N$ is *ITCONTF* and $g : N \to K$ is *ICONTF*, then $gof : M \to K$ is *TCONTF.*

Proof. Let E be open in K . Since g is *ICONTF*, $g^{-1}(E)$ is *IOS* in N. Now since f is *ITCONTF*, $f^{-1}(g^{-1}(E)) = (g \, of)^{-1}(E)$ is *COS* in M, Hence $gof : M \to K$ is *TCONTF.*

Theorem 3.16. Let $f : M \to N$ be *ITCONTF* and $g : N \to K$ be any function. Then $g^{o}f : M \to K$ is *ITCONTF iff* g is *IIRREF.*

proof. Let $g : M \to K$ be *IIRREF*. The proof then comes from "Theorem 3.14."
"Conversely", let $gof : M \to K$ is *ITCONTF*, let E be *IOS* in *K*. Since $gof : M \to K$ is *ITCONTF*, $(gof)^{-1}(E) = f^{-1}(g^{-1}(E))$ is *COS* in *M*. Since f is *ITCONTF* , $g^{-1}(E)$ is *IOS* in. Hence g is *IIRREF.*

Definition 3.17. $f : M \to N$ is called i-totally open (*ITOF*) if the image of each *IOS* in *M* is *COS* in *N.*

Theorem 3.18. If a bijective function [3] $f : M \to N$ is *ITOF*, then the image of each *ICS* in *M* is *COS* in *N.*

Proof. Let C be *ICS* in X. Then *$M - C$ is IOS in M. Since f is ITOF* , $f(M - C) = N - f(C)$ *is COS in N. This implies* $f(C)$ *is COS in N.*

Theorem 3.19. $f : M \to N$ A surjective function [3] and *ITOF iff* for every subset B of N and for every *ICS* O containing $f^{-1}(B)$ there exist a *COS* set E of N s.*t.* $B \subset E$ and $f^{-1}(E) \subset O$.

Proof. Suppose $f : M \to N$ is a surjective *ITOF* and $B \subset N$. Let O be *ICS* of X s.*t.* $f^{-1}(B) \subset O$. Then $E = N - f(M - O)$ is *COS* subset of N containing B s.*t.* $f^{-1}(E) \subset O$. On the other hand, suppose C is *ICS* of M . Then $f^{-1}(N - f(C)) \subset M - C$ and $M - C$ is *IOS* . By hypothesis, there exists *COS* E of N s.t. $N - f(C) \subset E$, which implies $f^{-1}(E) \subset M - C$ Accordingly, $C \subset M - f^{-1}(E)$. Hence $N - E \subset f(C) \subset f(M - f^{-1}(E)) \subset N - E$. This implies X 1, $f(C) = N - E$, which is *COS in* N. Accordingly, f is *ITOF.*

Theorem 3.20. The following statements are identical, For any bijective function $f : X \to Y$ (i) f^{-1} is *ITCONTF.* (ii) f is *ITOF* .

Proof. (i)$\Rightarrow$(ii): Assume that O be *IOS* of X . By assumption $(f^{-1})^{-1}(O) = f(O)$ is *COS* in Y. Hence f is *ITOF.*

(ii) $\Rightarrow$(i): Suppose that C be *IOS* in X . *We get,* $f(C)$ is *COS* in *Y*. Then, $(f^{-1})^{-1}(C)$ is *COS* in *Y*. Accordingly, f^{-1} *is ITCONTF.*

Theorem 3.21. When two *ITOF* are combined, they form another *ITOF*.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are *ITOF*. Then their composition is $g\,o\,f : X \to Z$. Let V be *IOS* in X . Consider $(g\,o\,f)(V) = g(f(V))$. Since f is *ITOF*, $f(V)$ is *COS* in Y. So it's open in Y. But by (Proposition 3.4) we get, $f(V)$ is *IOS* in . Since g is *ITOF*, $g(f(V))$ is *COS* in Z. Accordingly, $g\,o\,f : X \to Z$ is *ITOF*.

Theorem 3.22. Let $f : M \to N$ and $g : N \to K$, s.t. $g\,o\,f : M \to K$ is *ITCONTF*. Then (*i*) g is *ITOF* if f is *IIRREF* and surjective. (*ii*) f is *ITOF* if g is *TCONTF* and injective.

Proof. (i) Let E be *IOS in* N. Then $f^{-1}(E)$ is *IOS* in M, because f is *IIRREF*. Since $(g\,o\,f)$ is *ITCONTF*, $(g\,o\,f)$ $(f^{-1}(E)) = g(E)$ is *COS* in K. Then g is *ITOF*. (ii) Since g is injective, we have, $f(O) = g^{-1}\,(g\,o\,f)(O)$ is true for each subset O of M . Let J be any *IOS* in M . Accordingly, $(g\,o\,f)(J)$ is *COS*. Henceforth it is open in K . Since g is *TCONTF*, $g^{-1}\,(g\,o\,f)(J) = f(J)$ is *COS* in N. This shows that f is *ITOF*.

4. Conclusions

From above we concluded that Every *ITCONTF* is a *TCONTF*, Every *STCONTF* is *ITCONTF*, Every *ITCONTF* is *TICONTF*, Every *ITCONTF* is *ICONTF*, The composition of two *ITCONTF* is *ITCONTF* and If $f : M \to N$ is *ITCONTF* and $g : N \to K$ is *IIRREF*, then $g^{\circ}f : M \to K$ is *ITCONTF*.

Acknowledgment

The Authors are very grateful to the University of Mosul/ College of Education for Pure Sciences for their provided facilities, which helped to improve the quality of this work.

References

[1] Askandar SW. On i-Separation Axioms. Int J Sci Eng Res 2016; 7(5):367−73.
[2] Askandar SW. The property of extended and non-extended topologically for semi-open, α-open and i-open sets with the application. 2012. M.Sc., Thesis, Mathematics Department, College of Education. Mosul University; Iraq: 2012.
[3] Biswas N. Some mappings in topological spaces. Bull Calcutta Math Soc 1969;61:127−35.
[4] Jain RC. The role of regularly open sets in general topology, *Ph.D. thesis*. Meerut-India: Meerut University, Institute of advanced studies; 1980.
[5] Levins N. Semi-open sets and semi-continuity in topological spaces. Am Math Mon 1963;70:36−41.
[6] Mohammed AA, Askandar SW. I-open sets in bitopological spaces. Raf J Comp Math 2018;12(1):13−23.
[7] Nour TM. Totally semi-continuous functions, Indian. J Pure Appl Math 1995;26(7):675−8.
[8] Stone M. Applications of the theory of boolean rings to general topology. Truns Amer Math Soc 1937;vol. 41:374.