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ARTICLE

Preserving Some Orthogonal Operators on Banach Spaces

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Abstract

In this article, the Birkhoff−James orthogonality on a real Banach space X from linear operators and the category of conjugate linear mapping $\varnothing : B(H) \to B(H)$ that keep the strong orthogonality in both spaces is explored. We also investigate left orthogonal Birkhoff-James symmetry of linear operators defined on X. We also prove, for every strictly convex and smooth real Banach space X, $T \in L(X)$ is left symmetric if and only if T operators are zero. We also examine operator spaces $\ell_p^2 (p \geq 2, p \neq \infty)$ and determine its left symmetric operators.

Keywords: Birkhoff-James orthogonality, Linear operators, Symmetry, Banach space

1. Introduction

Birkhoff-James orthogonality is one of the most important concepts of orthogonality in soft space and is a powerful tool for identifying geometric features of soft space. Due to the importance of this concept, we will first describe its history. In 1935, in [1], Birkhoff realized another concept of orthogonality in soft space, taking the idea of perpendicularity of two lines in Euclidean space. A decade later, James in [2] extended this concept of orthogonality in soft space and expressed more characteristics of it according to Birkhoff's ideas. Between 1945 and 1947, in his three famous papers [2−4], Robert James studied Birkhoff James orthogonality and obtained equations for geometric properties such as flatness, strictly convexity, and uniformly convexity of soft spaces. He derived equations for the inner multiplicative space based on this orthogonality. James is most famous for his efforts in the geometry of Banach spaces, and in particular, the exact equivalence that he provided reflecting Banach spaces.

Recently in [5], Sain and Paul proved that Birkhoff-James orthogonality for linear operators on finite-dimensional Banach spaces yields orthogonality on the space for certain points. They have equated the Birkhoff-James orthogonality of the resulting soft linear operators on the finite-dimensional Hilbert space with respect to the orthogonality of the space itself. Gosh et al. [6] have investigated orthogonal symmetry concept and showed that the necessary and sufficient condition for the left symmetry of a linear operator on a finite-dimensional Hilbert space is the zeroing of the linear operator. Also, they proved that the necessary and sufficient condition for the right symmetry of a linear operator on a Hilbert space with finite dimension is the equality of the sphere of the space and the set of soft yielding points of the linear operator. Four corollaries for explaining the importance of characterization of smooth spaces with operator norm attainment and Birkhoff−James orthogonality is obtained in [7]. Comparing exposition of the big category of orthogonality concepts with fundamental properties is considered in [8]. Article [9] has expressed the connection of Birkhoff−James orthogonality and norm. In [10−14] the study on nonlinear equivalence of Banach spaces with respect to structure of Birkhoff-James orthogonality was investigated.

The aim of this paper is to equate the orthogonal Birkhoff-James space of linear operators on a real

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Banach space with finite dimension. The different parts of the article are as follows.

In section 2, we examine the basic properties of metric space, norm space, Hilbert space, and strictly convex and smooth space. In section 3, we will review the basic properties of Birkhoff-James orthogonality and the equations obtained based on this orthogonality. We also state the relation of Birkhoff-James orthogonality with linear and bounded subscripts and superspaces. Finally, we examine the relationship between orthogonality and bounded linear operators. In section 4, we equate Birkhoff-James orthogonality of linear operators on a real Banach space with finite dimension. Also, we examine the concept of Birkhoff-James orthogonal left and right symmetry of linear operators on real Banach spaces X with finite dimension and express the relationship between left and right symmetry of linear operators on the space with the left and right symmetry of the space X. Finally, according to the obtained results, we prove that the linear operator T on the space $\ell_p^2(p \geq 2, p \neq \infty)$ is left symmetric with respect to Birkhoff-James orthogonality if and only if the operator T is zero.

2. Preliminaries

The introductory definitions in this section are used from source [5].

Definition 1. Let X be an infinite set. A meter by X is a function of $d: X \times X \to [0, \infty)$, which has the following characteristics:

a) For every $x, y \in X$, we have $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
b) For every $x, y \in X$, we have $d(x, y) = d(y, x)$,
c) For every $x, y, z \in X$, the relation $d(x, z) \leq d(x, y) + d(y, z)$ is established.

Any set equipped with a meter is called a metric space.

Often, we denote the metric space of X, whose meter is d, by (X, d).

Definition 3. We call any norm space X which is complete with respect to the induced meter by norm, a Banach space.

Definition 4. If the space of inner multiplication X with the norm induced by inner multiplication is complete, then we call it a Hilbert space.

Definition 5. We call the norm space $(X, \|\cdot\|)$ strictly convex, whenever we can conclude $\| \frac{x+y}{2} \| < 1$ from $\| x\| = \|y\| = 1$ for every $x, y \in X$ which is $x \neq y$. In other words, the midpoint of each line segment with endpoints located on the unit sphere should be inside the unit sphere.

3. Properties of Birkhoff-James orthogonality

For any two elements $x, y \in \mathrm{X}, x$ is orthogonal to y in the base of Birkhoff-James, written as $x \perp_B v$, if for all $\lambda \in \mathbb{R}$, we have:

$$\| x\| \leq \|x + \lambda y \|$$

Likewise, for any two elements $T, A \in \mathrm{L}(\mathrm{X}), T$ is said to be orthogonal to A in the sense of BirkhoffJames, written as $T \perp_B A$, if

$$\| T\| \leq \|T + \lambda A\| \text{ for all } \lambda \in \mathbb{R}$$

Theorem 1. Suppose X is an inner multiplicative space and x, y∈X. In this case, $x \perp y$ if and only if $x \perp_B y$.

Proof. Suppose it is $\langle x, y\rangle = 0$. In this case, for every $\lambda \in \mathbb{R}$, we have:

$$\| x + \lambda y\|^2 = \langle x + \lambda y, x + \lambda y\rangle = \|x\|^2 + \lambda^2 \| y\|^2 \geq \|x\|^2$$

So, $\| x + \lambda y \| \geq \| x \|$ and therefore: $x \perp_B y$.
On the contrary, suppose $x \perp_B y$ but $\langle x, y\rangle \neq 0$. In this case, for $\lambda = -\frac{\langle x,y\rangle}{\langle y,y\rangle}$, we have:

$$\| x + \lambda y\|^2 = \left\langle x - \frac{\langle x,y\rangle}{\langle y,y\rangle} y, x - \frac{\langle x,y\rangle}{\langle y,y\rangle} y \right\rangle$$

$$= \langle x,x\rangle - 2\frac{\langle x,y\rangle^2}{\langle y,y\rangle} + \frac{\langle x,y\rangle^2}{\langle y,y\rangle^2}\langle y,y\rangle$$

$$= \langle x,x\rangle - \frac{\langle x,y\rangle^2}{\langle y,y\rangle} < \langle x,x\rangle = \| x\|^2$$

Therefore $\| x + \lambda y \| < \| x \|$. So x by y is not Birkhoff-James orthogonal, which contradicts the assumption.
For a linear operator T defined on a Banach space X, let M_T write the set of all unit vectors in X at which T attains norm as,

$$M_T = \{x \in S_X : \|T'x\| = \|T'\|\}$$

In a finite exact dimensional Hilbert space $\mathbb{H}$, for any two members $T, A \in \mathbb{L}(\mathrm{H}), T \perp_B A$ if and only if there exists $x \in M_T$ such that $Tx \perp_B Ax$. Sain and Paul [5] generalized the result for linear operators defined on finite dimensional real Banach spaces. For any two members x, y in a real line normed space X , let us for all $\lambda \geq 0$, denote that $y\in x^{+}$ if $\| x+\lambda y\| \geq \|x\|$. Accordingly, we denote that for all $\lambda \geq 0$, $y\in x^{-}$ if $\| x+ \lambda y\| \geq \|x\|$. By using this concept, we completely determine the orthogonality of linear operators defined on real Banach.

Proposition 2. Suppose X is a vector space and H is a subspace of X. In this case, H is a superspace if and only if there exists a nonzero linear subscript like $f : X \longrightarrow \mathbb{R}$ such that $H = \ker f = \{x \in X : f(x) = 0\}$

proof. For proof, refer to Theorem 3-4 of source [9]. In the following theorem, James has stated the connection between Birkhoff-James orthogonality with linear and bounded subscripts and superspaces.

Theorem 3. Suppose X is a smooth space. If $f : X \longrightarrow \mathbb{R}$ is a nonzero bounded linear function and $x\in$ X, then:
a) $x \perp_B$ kerf if and only if $f(x) = \| f\| \|x\|$.
b) Suppose H is a subspace of X. then is $X \perp_B H$, if and only if there exists a nonzero bounded linear function $f : X \longrightarrow \mathbb{R}$ such that $f(x) = \| f\| \|x\|$ and for every $y\in H$ Let's have $f(y) = 0$.

proof. For the proof, see [5], Corollary 2-2.

Corollary 4. For every non-zero vector X in the soft space X , there is a superspace H in X such that $x \perp_B H$
Next we consider the left symmetry of Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Banach space X.

Definition 6. For an element $x\in$X, let us say that x is left symmetric (with respect to Birkhoff-James orthogonality) if $x \perp_B y$ implies $y \perp_B x$ for any $y\in$ X.

Theorem 5. Suppose H is a real Hilbert space with finite dimension and $T\in L(H)$. In this case, T is left symmetric if and only if T operator is zero.

proof. For the proof, refer to reference [6] of Theorem 10-2.
Also, we know that in a real Banach space of finite dimension X, which is not a Hilbert space, there exist linear nonzero operator $T\in \mathbb{L}(\mathbf{X})$ such that T is a symmetric left point in $\mathbb{L}(\mathbf{X})$.

4. Birkhoff-James orthogonality on a Banach real space of linear operators

In this section, taking ideas from the geometry of Banach spaces, we equate the Birkhoff-James orthogonality of linear operators on a Banach space with finite dimension X. For this purpose, for every $x, y\in X$, two concepts $y\in x^{+}$ and $y\in x^{-}$ help us.

Theorem 6. Let X be a fintte dimenstonal real Banech space. Let $T, A\in \mathbb{L}(\mathbf{X})$. Then $T \perp_B A$ if and only if there exists $x, y\in M_T$ such that $Ax\in Tx^{+}$ and $Ay\in Ty^{-}$.

Proof. : Suppose there are $T, A\in L(X)$ such that $T \perp_B A$. According to inverse process proof, let's assume that for all $x, y\in M_T$ we have, $Ax \notin Tx^{+}$ and $Ay \notin Ty^{-}$. Therefore, one of the following statements is true:
i. For every $x\in M_T$ and $Ax\in Tx^{+}$ then $Ax \notin Tx^{-}$.
ii. For every $x\in M_T$ and $Ax\in Tx^{-}$ then $Ax \notin Tx^{+}$.
Suppose that (i) holds. We prove the theorem for case (ii) similarly. We define the function $g : S_x \times [-1, 1] \longrightarrow \mathbb{R}$ for every $x\in S_x$ and $\lambda\in[-1, 1]$ as follows:

$$g(x, \lambda) = \|Tx + \lambda Ax\|$$

Since Banach space X is finite-dimensional, then operators T and A are continuous. On the other hand, the norm function is also continuous, so function g is continuous.
Note that for every $x\in M_T, Ax \notin Tx^{-}$, then there are $\| Tx\| = \| T \|$ and $\lambda_x < 0$ so that $\| Tx + \lambda_x Ax \| < \|Tx\|$. So for any $x\in M_T$, we can write:

$$g(x, \lambda_x) < \| T \|$$

Set $\{B(x, r_x) \cap S_x : x \in M_T\} \cup \{B(z, r_z) \cap S_x : z \in S_x \setminus M_T\}$ is an open cover for sphere S_x. On the other hand, since X is a finite-dimensional Banach space, it is S_y compact according to the
On the other hand, since X is a finite-dimensional Banach space, it is S_X compact according to the Heine borel's theorem. Therefore, every open cover has a finite subcover. As a result, there are natural numbers n_1 and n_2 so that for $x_i\in M_T$ and $z_k\in S_X \setminus M_T$ we have:

$$S_x \subset \bigcup_{i=1}^{n_1} B(x_i, r_{x_i}) \bigcup_{k=1}^{n_2} B(z_k, r_{z_i}) \cap S_x$$

We consider the arbitrary term $\lambda_0 \in (\bigcap_{i=1}^{n_1} (\lambda_{x_1}, 0)) \cap (\bigcap_{k=1}^{n_2} (-\delta_{z_2}, \delta_{z_2}))$.
So, $\| T + \lambda_0 A \| = \| (T + \lambda_0 A) w_0 \| < \| T \|$ and therefore, it is in contradiction with assumption $T \perp_B A$. On the contrary, suppose there are $x, y\in M_T$, which

are $Ax \in Tx^{+}$ and $Ay \in Ty^{-}$. Since $Ax \in Tx^{+}$ then for every $\lambda \geq 0$, $\| Tx \| \leq \| Tx + \lambda Ax \|$. On the other hand, $x \in M_T$ therefore $\| Tx \| = \| T \|$. We have:

$$\| T \| = \| Tx \| \leq \| Tx + \lambda Ax \| \leq \| T + \lambda A \| \| x \| = \| T + \lambda A \|$$

As a result, $\| T \| \leq \| T + \lambda A \|$. In a similar way, for all $\lambda \leq 0$, the result is $\| T \| \leq \| T + \lambda A \|$. So for every $\lambda \in \mathbb{R}$, we have $\| T \| \leq \| T + \lambda A \|$, that is, $T \perp_B A$. Therefore, the ruling is confirmed.

Theorem 2.4. Let X be a finite dimension strictly convex real Banach space. If $T \in \mathbb{L}(X)$ is a left symmetric point then for each $x \in M_T, Tx$ is a lefi symmetric point.

Proof. First we see that the theorem is trivially true if Tis the zero operator. So, we can suppose T be nonzero. Since X is finite dimensional, M_T should be nonempty. If possible suppose that there exists $x_1 \in M_T$ such that Tx_1 is not a symmetric left point. Since T is nonzero so, we have $Tx_1 \neq 0$. Then there exists $y_1 \in S_x$ such that $Tx_1 \perp_B y_1$. Since X is strictly convex, x_1 should be an exposed point of the unit ball B_X. Let H be the hyperplane of codimension in X such that $x_1 \perp_B H$. Clearly, any member x of X can be uniquely written in the form $x = \alpha_1 x_1 + h$, where $\alpha_1 \in \mathbb{R}$ and $h \in H$. Define a linear operator $A \in \mathbb{L}(\mathbf{X})$ as follows:

$$Ax_1 = y_1, Ah = 0 \text{ for all } h \in H.$$

Since $x_1 \in M_T$ and $Tx_1 \perp_B Ax_1$, it follows that $T \perp_B A$. Since T is left symuetric, it follows that $A \perp_B T$.
It ls easy to check that $M_A = \{\pm x_1\}$, since **X** is strictly convex. Applying Theorem 2.1 of [5] to A, it results from $A \perp_B T$ that $Ax_1 \perp_B Tx_1$. It contrary to our initial assumption. So, by this contradiction the proof of the theorem is complete.

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