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A Note on Multiplication Modules and Pure Submodules

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REVIEW

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Abstract

Let R be a commutative ring with non zero identity and M be R-module. An R - module M is said to be a multiplication module if for any submodule N of M, there exists an ideal I such that N = I M. Ideal {r∈R : rM ⊆N} is denoted as (N : M) and ideal (0 : M) be annihilator of M. We will study the relation between multiplication R-module M and pure R-submodule in M. In this note, will be seen what characteristics can be brought to the multiplication modules and its relation with pure submodules.

Keywords: Modules, Multiplication modules, Pure submodules

1. Introduction

In this note all rings are commutative rings with identity and all modules are unital. Multiplication modules have been investigated in Ref. [2]. The aim of this paper is to study multiplication modules and pure submodules. Now we define the concepts that we will use. If R is a ring and N is a submodule of an R-module, the ideal {r∈R : rM ⊆N} will be denoted by (N : M). Then (0 : M) is the annihilator of M. An R-module M is called a multiplication module if for each submodule N of M then N = I M for some ideal I of R. In this case we can take I = (N : M) and we say ideal I is presentation ideal or presentation of N. In this paper, we approach some concepts of pure submodule in different way, for example, we do not use tensor product for defining pure submodule. Let N be an R-submodule of M, N is said pure if $v \in M \setminus N$ then $rv \notin N$ for all $r \neq 0 \in R$.

2. Research method

The study in this article is a literature review where all references used are included in the bibliography.

3. Result and discussion

Before we jump into pure submodules, consider this example:

Suppose M is R-Modul with N < M (N is submodule of M), is there any submodule with property $v \in M \setminus N$ then $rv \notin N$ for all $r \neq 0$ in scalar of M?

1. Suppose Z − Module Z with 3Z as submodule, choose $3 \in Z \setminus 2Z$, for $r = 2 \in Z$ we have $6 = 3.2 = 2.3 \in 2Z$, that is even thought $3 \notin 2Z$ ($3 \in Z \setminus 2Z$) but there is scalar r such that $r.3 \in 2Z$
2. Choose R^2 as R − module, it is clear that $R^2 = <\begin{pmatrix}1\\0\end{pmatrix}> \oplus <\begin{pmatrix}0\\1\end{pmatrix}>$, take $v \in R^2 \setminus <\begin{pmatrix}1\\0\end{pmatrix}>$, for arbitrary r in R it is clear that $rv \in R^2 \setminus <\begin{pmatrix}1\\0\end{pmatrix}>$ and $rv \notin <\begin{pmatrix}1\\0\end{pmatrix}>$.

From the above points we can see that not all modules will satisfy the condition that if $v \in M \setminus N$ then $rv \notin N$ for all $r \neq 0 \in R$. So we formalize the condition with definition below:

Definition 1. [1].
Let M be an R-module, a submodule N of M is said to be pure in M, if $v \in M \setminus N$ then $rv \notin N$ for all $r \neq 0 \in R$.

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Theorem 1. [1]. From the definition we have these properties:

(1) Submodule N is pure if and only if $v \in N$ and $v = rw$ for all $r \in R$ then $w \in N$.
(2) Submodule N is pure if and only if M/N is torsion-free.
(3) Let L and N are pure submodules in M, then so is $L \cap N$ in M.
(4) Let N pure submodule in M, then so $L \cap N$ pure submodule in L for any submodules L of M.

Proof.
1. Proof for part one is contrapositive of the definition of pure submodule. For the other parts are straightforward.

Before we characterize the pure submodule, we have a counterexample about this statement:
"For right R-modules $N \subseteq M$, if $N \cap Mr = Nr$ for every $r \in R$ then $N \cap MI = NI$ for every left ideal $I \subseteq R$"

This notion comes from Ref. [3]. Let R = k[x,y], where k is a field. Let $M = R^2$, and N = (x,y). $R \subseteq M$. Then $N \cap Mr = Nr$ for all $r \in R$. For, if (f,g) r = (x,y) s where $s \in R$, then, assuming $r \neq 0$, we can show by unique factorization that $f = xf_0$, $g = yg_0$ for suitable f_0, $g_0 \in R$. Now $f_0 r = s = g_0 r$ implies that $f_0 = g_0$, so we have

$(f,g) = (x,y)(f_0 r) \in Nr$. On the other hand, for the ideal $I = Rx + Ry$, we have $MI = (R\oplus R)I = I \oplus I \supset N$, so $N \cap MI \neq NI$, as desired.
As the example above, we come to the fact that the condition will be true if the submodule N is pure.

Theorem 2. [2]. An R-submodule N is said to be pure if and only if $I N = N \cap I M$ for all ideal I of R.

Proof.
($\Rightarrow$) Clearly $IN \neq \emptyset$. Take $v \in I N$ arbitrarily, write $v = rw$, for $r \in I \subseteq R$, and $w \in N \subseteq M$. Since N is pure then clearly $w \in N$. By remembering N is a submodule, then we have $rw \in N$. So, $rw \in N \cap I M$, in other words we have $IN \subseteq N \cap I M$. On the other side, let $rw \in N \cap I M$, which means $rw \in N$ and $rw \in N \cap I M$ with $r \in I \subseteq R$ and $w \in M$. Since N is pure, so $w \in N$. Then we have $rw \in I N$ and $N \cap I M \subseteq I N$. Finally, we obtain $IN = N \cap IM$.
($\Leftarrow$) Straight forward.

Theorem 3. [2]. Let M be a multiplication R -module. If N is a submodule of M such that $N \cap IM = I N$, for all ideal I of R, then N is multiplication modules.

Proof.
Take any submodules S of N. Since M is multiplication then there exists an ideal U such that S = UM. From hypothesis, we have $N \cap UM = UN$. We also have $S = UM = S \cap UM \subseteq N \cap UM = UN \subseteq UM = S$, i.e S = UN. Which means N is also a multiplication module.
Theorem 3 states that for any pure submodules of multiplication modules they also multiplication modules.

Theorem 4. [2]. Let M be a divisible module over principal ideal domain. Then M is multiplication if and only if M is simple (so cyclic).

Proof.
($\Rightarrow$) Let M be a divisible module and multiplication. Take any submodule N of M.
Consider these cases:

1. If N = <0>, choose I = <0> such that N = I M = <0> M.
2. If N ≠ <0>. Since M is divisible module, choose $m \notin 0$ such that N = <m>. Since M is multiplication, choose ideal I which is generated by $r \neq 0 \in R$ such that N = I M. So we have N = <m> = I M = M.

So we arrive that N only has submodules <0> and itself, i.e M is simple and cyclic.
($\Leftarrow$) Suppose M is simple (so cyclic). Take any submodule $N \neq 0$ of M. Since M is divisible, simple, and cyclic, write N = rM = M, choose ideal I which is finitely generated by $r \notin 0$ i.e I = <r>, such that N = <r> M = M. So we have N = I M, i.e M is multiplication.
We also have the property that a pure submodule is equivalent to a strongly pure submodule on a free module.

Theorem 6. [2]. Suppose R commutative ring, M a free R -module, and N an R-submodule of N Then N is strongly pure submodule if and only if N is pure submodule.

Proof.
($\Rightarrow$) Since N is a strongly pure submodule, then there is a map $f{:}M \to N$ such that $f(x_i) = x_i$, for finite tuples $x_i \in N$. Let $rw \in N$, which clearly $f(rw) = rw$, with $0 \notin r \in R$. Choose r = 1, so we have $w = 1w = f(1w) \in N$. So we can conclude that N is pure submodule.

($\Leftarrow$) Straight forward.

Theorem 7. [2]. Let R be a commutative ring, M be an R -module and $N \neq 0$ is a strongly pure submodule in M. Then ideal (N : M) is idempotent.

Proof.
Since M is multiplication, write $(N:M)^2 = (N:M)(N:M) = (N:M) \cap (N:M) = (N:M)$. So, $(N:M)^2 = (N:M)$, i.e (N : M) is idempotent.

Theorem 8. [2]. Let R be a commutative ring, and let N be a representable multiplication module. Then M is finitely generated.

Proof.
See [3, theorem 2.2].

Theorem 9. [2]. Let R be a commutative ring and M be a prime multiplication module. Then M be an R-module which is finitely generated.

Proof.
Take element $0 \neq a \in M$. Then $R a = \theta(M) R a$, so there is $r \in \theta(M)$ with $ra = a$. We obtain $(1 - r) a = 0$ and $(1 - r)^m M = 0$, for some $m \in N$. Since M is prime, we have $(1 - r)^m \in Ann(M) \subseteq \theta(M)$, with $(1 - r)^m = 1 - s$, for some $s \in \theta(M)$. So, clearly $1 = (1 - s) + s \in \theta(M)$, i.e $\theta(M) = R$. So, M is finitely generated.

Lemma 1. Let R be a commutative ring and M be a free multiplication module which is a weakly prime submodule. Then M is finitely generated.
Based on the discussion above, we have the following properties with the condition that the ring is principal ideal domain (PID).

Theorem 10. Let R be principal ideal domain, let M be a free R-module, and S be a submodule of M. Consider the following statements:

(1) S is complemented.
(2) M/S is free.
(3) If $x \in S$ and $x = ay$ for some $y \in M$, $a \notin 0 \in R$ then $y \in S$.
(4) S cyclic and pure.
(5) S multiplication and pure.

then (1) $\Rightarrow$(2) $\Rightarrow$(3) $\Rightarrow$(4) $\Rightarrow$(5) $\Rightarrow$(3). Furthermore, if M is a finitely generated module then (3) $\Rightarrow$(1).

Proof.
(1)$\Rightarrow$(2). Since S is complemented, then there exists a submodule $T \subset M$ such that.
$S \oplus T = M$, so $M/S \cong T$. Since T is a submodule of a free module over principal ideal domain, so T is free. So we can conclude M/S is also free.
(2)$\Rightarrow$(3). Let M/S be free. Suppose $x \in S$ with $x = ay$, for some $y \in M$, $a \notin 0 \in R$. Then we have $a(y + S) = S \in M/S$. Since free module is torsion-free, we have $y + S = S$, i.e $y \in S$.
(3)$\Rightarrow$(4). Clearly from definition S is pure. Since R is principal ideal domain, we can construct S as Ry i.e S = <y>. So S is cyclic.
(4)$\Rightarrow$(5). Clearly the statement for pure is clear. From Theorem 4, S is multiplication.
(5)$\Rightarrow$(3). Clear.
(3)$\Rightarrow$(1). Let M be a module that is finitely generated over principal ideal domain, and let.
$S \subset M$ satisfies condition (3). From the proof of (2)$\Rightarrow$(3) it can be concluded that M/S is torsion-free, and also we have M/S is free. Furthermore, we have the following split exact sequence,

$$0 \dashrightarrow S \dashrightarrow M \dashrightarrow M/S \dashrightarrow 0,$$

So, $M \cong S \oplus M/S$, i.e S complemented.
From the statement we can also conclude that S is also a projective R-module of M.

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