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Shaimia Qais Latif

Salwa Salman Abed

Haider Ahmed Shihab

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ORIGINAL STUDY

Fixed Points of Multi-Valued Graph Maps in Strong b-Metric

Shaimia Qais Latif [a,*], Salwa Salman Abed [b], Haider Ahmed Shihab [a]

[a] Department of Computer, College of Education, Al-Iraqia University, Baghdad, Iraq
[b] Department of Mathematics, College of Education for Pure Science Ibn Al-Haitham, University of Baghdad, Baghdad, Iraq

ABSTRACT

This paper involves adopting a well-known generalization method in the branches that dealing with fixed points via weakening the suppositions. As appearing in previous sources, letting (Ω, S_b, K) be a strong b-MS, and $F,\ H$ be two multi-valued maps on Ω equipped with a graph σ s.t the set of vertices of σ, $\Lambda(\sigma) = \Omega$ and the set of edges of σ, $\Xi(\sigma) \subseteq \Omega \times \Omega$. The acceptable assupmtions have been adopted to finding a common fixed point for in Ω. Our result is a generalization to other statuses.

Keywords: Fixed point, Strong b-MS, Hausdorff distance, Graph contractive
2020MSC: 47H10, 47H04, 54C60, 54H25

1. Introduction and Preliminaries

The existence and uniqueness of FP for various collection of maps is a central topic in nonlinear functional analysis and FP theory. Contractive type conditions that utilize metric structure ensure maps have FP through Banach's contraction principle and its numerous generalizations [1]. Multi-valued maps, where images are sets rather than points, have been essential in applications to differential and integral equations [2]. Graph maps combine aspects of single and multi-valued maps using graphs to relate domains and images [3]. Recently, there has been growing interest in studying FP theory in more generalized MS. b-MS endow a real constant $K \geq 1$ that relaxes the triangle inequality [4]. Matthews introduced the notion of a partial MS as well, where self-distances need not be zero [5]. Various contractive-type maps have now been investigated in the settings of b-MS and partial MS [6]. Integral equations have also been studied in these spaces [7, 8]. Also, Latif and Abed [9, 10] dealt with the existence of FP and coupled FP of multi-valued cases for comparable variables or for some vertices a graph in the same path in *G*-MS. An and Dung [10] provided an answer to the two questions (1.7–1.8) of Kirk, which discussed the existence of FP in strong *b*-MS and the density of a complete strong *b*-MS by providing suitable examples. The work on this aspect is to merge the ideas of multi-valued maps, graph structure, and strong b-MS. New FP results are obtained for classes of multi-valued graph maps defined on strong b-MS and prove uniqueness as well as existence results.

In this article, we take inspiration from the results by Chifu and Petruşel [11] which had been extended by Jiddah et al. [12] for a class of contractive maps in strong b-MS. Additionally, the results in [13–16] will be combined to prove a new version of CFP theorem for multi-valued single-valued and maps. To begin, recalling some strong b-MS. concepts. Some arguments and considerations were taken from [17–21].

Definition 1.1 ([10, 14]): Let $\Omega \neq \emptyset$ and $k \geq 1$ $S_b : \Omega\ x\ \Omega \to [0, \infty)$, a map S_b is called a strong b -metric on Ω if

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* Corresponding author.
E-mail addresses: shaymaa.q.lateef@aliraqia.edu.iq (S. Qais Latif), salwaalbundi@yahoo.com, salwa.s.a@ihcoedu.uobaghdad.edu.iq (S. Salman Abed), haider.a.shihab@aliraqia.edu.iq (H. Ahmed Shihab).

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1) $S_b(\omega, \upsilon) = 0$ if and only if $\omega = \upsilon$
2) $S_b(\omega, \upsilon) = S_b(\nu, \omega)\ \forall\omega, \upsilon \in \Omega$
3) $S_b(\omega, \nu) \preccurlyeq S_b(\omega, \eta) + KS_b(\eta, \upsilon) \forall\omega, \upsilon, \eta \in \Omega$

then (Ω, S_b, K) is called a strong *b*–MS.

For more details about strong *b*-MS see [14].

Example 1.2: *Let* $\Omega = \{0, 2, 4\}$ *and* $S_b : \Omega\ x\ \Omega \to [0, \infty)$ *be defind by* $S_b(0, 0) = S_b(2, 2) = S_b(4, 4) = 0,\ S_b(0, 2) = \frac{1}{2}S_b(2, 0),\ S_b(0, 4) = 8 = S_b(4, 0), S_b(2, 4) = 5 = S_b(4, 2)$.
Therefore,

$$S_b(0, 2) \preccurlyeq S_b(0, 4) + KS_b(4, 2), \forall K \geq 1,$$

$$S_b(2, 0) \preccurlyeq S_b(2, 4) + KS_b(4, 0), \forall K \geq 1,$$

$$S_b(0, 4) \preccurlyeq S_b(0, 2) + KS_b(2, 4), \forall K \geq \frac{3}{2}$$

$$S_b(4, 0) \preccurlyeq S_b(4, 2) + KS_b(2, 0), \forall K \geq 6,$$

$$S_b(2, 4) \preccurlyeq S_b(2, 0) + KS_b(0, 4), \forall K \geq 1,$$

$$S_b(4, 2) \preccurlyeq S_b(4, 0) + KS_b(0, 2), \forall K \geq 6,$$

So $(\Omega, S_b, K = 6)$ *is a strong b-MS, but it is not MS, because* $S_b(0, 4) > S_b(0, 2) + S_b(2, 4)$.

Definition 1.3 ([13])**:** Let (Ω, S_b, K) be a *strong b*-MS. Let $\{\omega_n\}$ be a sequence in Ω and $\omega \in \Omega$ then

(a) A sequence $\{\omega_n\}$ is called convergent to ω if $S_b(\omega_n, \omega) = 0$, denoted by $lim_{n\to\infty}\ \omega_n = \omega$ or $\omega_n \to \omega,\ n \to \infty$.
(b) A sequence $\{\omega_n\}$ is called Cauchy sequence in Ω if $lim_{n\to\infty}\ S_b(\omega_n, \omega_m) = 0$.
(c) The S_b–MS (Ω, S_b, k) is called complete if every Cauchy sequence in Ω is convergent in Ω.

Proposition 1.4 ([16])**:** Let (Ω, S_b, K) be a strong *b*-MS and $\{\omega_n\}$ be a sequence in Ω, Then

(a) If $\{\omega_n\}$ converges to $\omega \in \Omega$ and $\{\omega_n\}$ converges to $y \in \Omega$, then $\omega = \upsilon$
(b) If $lim_{n\to\infty}\ \omega_n = \omega \in \Omega$ and $lim_{n\to\infty}\ \upsilon_n = \upsilon \in \Omega$ then $lim_{n\to\infty}\ S_b(\omega_n, \upsilon_n) = S_b(\omega, \upsilon)$.

Proposition 1.5 ([16])**:** Let $\{\omega_n\}$ be a sequence in a strong *b*-MS (Ω, S_b, K) where $\sum_{n=1}^{\infty} S_b(\omega_n, \omega_{n+1}) < +\infty$, then $\{\omega_n\}$ is a Cauchy sequence.

The beginning presents the definition of the Hausdorff distance by S_b-metric, and then some of its required properties are proven. Let (Ω, S_b, K) be a strong *b*-MS,

Let $CB(\Omega) := \{A \subset \Omega : \emptyset \neq A$ is closed and bounded$\}$, the Hausdorff distance H_{S_b} on $CB(\Omega)$ derived from S_b is $H(A, B) := Max\{sup_{\omega\in B}\ d(\omega, A); sup_{\omega\in A} d(\omega, B)\}$, where A, B, $CB(\Omega)$ and $d(\omega, A) := inf_{\upsilon\in A}\ S_b(\omega; \upsilon)$. Now, we state some of the required lemmas in the sense of the references.

Lemma 1.6: *If* (Ω, S_b, K) *is a strong b-MS and* $A, B \in CB(C\Omega)$. *If* $H(A, B) > 0$ *then* $\forall h > 1$ *and* $a \in A,\ \exists b \in B$ *such that,* $S_b(a, b) < h\,H(A, B)$.

Proof: Using the characterized of infimum, with $\in = (h - 1)\ H(A, B) > 0, \exists b \in B$ such that $S_b(a, b) < d(a, B) + \in$. And by the definition of $H(A, B)$, getting $d(a, B) \leq H(A, B)$, So, $S_b(a, b) < h.H(A, B)$.

Lemma 1.7: *If* $A, B \in CB(\Omega)$ *with* $S_b(A, B) < \varepsilon$ *then for each* $a \in A\ \exists b \in B$ *such that* $d(a, b) < \varepsilon$.

Proof: By Haudorff distance, $H_{S_b}(A, B) < \varepsilon$ means that $sup_{a\in A} inf_{b\in B} S_b(a, b) < \in$ and $sup_{b\in b} inf_{a\in A} S_b(b, a) < \in$. This ordinarily gives $\forall a \in A$ $\exists b \in B$ such that $d(a, b) < \varepsilon$.

Lemma 1.8: *Let* $\{A_n\}$ *be a sequence in* $CB(\Omega)$ *and* $S_b(A_n, A) = 0$, *for* $A \in CB(\Omega)$. *If* $\omega_n \in A_n$ *and* $S_b(\omega_n, \omega) = 0$, *then* $\omega \in A$.

Proof: Let $\{A_n\} \subset CB(\Omega)$ and $S_b(\omega_n, \omega) = 0$ for $A \in CB(\Omega)$. Assume that $\omega_n \in A_n$ and $S_b(\omega_n, \omega) = 0$ for some $\omega \in \Omega$. To show that $\omega \in A$. Hausdorff metric S_b implies

$$S_b(A_n, A) = max\left\{sup_{\omega\in A_n} S_b(\omega, A), sup_{\upsilon\in A} S_b(\upsilon, A_n)\right\}$$

Since $S_b(A_n, A) = 0$, so,

$$sup\ sup\ \{\omega \in A_n,\ S_b(\omega, A)\} = 0\ and$$

$$sup\ sup\ \{\upsilon \in A,\ S_b(\upsilon, A_n)\} = 0.$$

By hypothesis, $\omega_n \in A_n$ and $S_b(\omega_n, \omega) = 0 \Rightarrow$ for any $\varepsilon > 0$ $M \in N$ such that $\forall n \geq M$, $S_b(\omega_n, \omega) < \varepsilon$. Assume that $\omega \notin A$. Then, $\exists\delta > 0$ such that $S_b(\omega, A) \geq \delta$.

Since $sup\ sup\ \{\upsilon \in A\}\ S_b(\upsilon, A_n) = 0$ there exists an $M' \in N$ such that for all $n \geq M'$, $sup\{\upsilon \in A\}\ S_b(\upsilon, A_n) < \delta/2$.

Choose $n_0 = max\{M, M'\}$. Then, for $n \geq n_0$, getting

$$S_b(\omega_n, A) \preccurlyeq S_b(\omega_n, \omega) + S_b(\omega, A) < \varepsilon + \delta$$

Now, choose $\varepsilon = \delta/2$. Then for $n \geq n_0$, $S_b\ (\omega_n, A) < \delta/2 + \ \delta = 3\delta/2$. However, since $\omega_n \in A_n$ and $n \geq n_0$, then

$$S_b\ (\omega_n, A) \preccurlyeq\ sup\{\upsilon \in\ A)\ S_b\ (\upsilon, A_n) \geq S_b\ (\omega, A_n)$$
$$\geq\ S_b\ (\omega, A) \geq \delta.$$

This is a contradiction, as $S_b\ (\omega_n, A) < 3\delta/2$ and $S_b\ (\omega_n, A) \geq \delta$ it cannot be true at the same time. Thus, the assumption $\omega \notin A$ must be false. Hence, $\omega \in A$.

To endow strong b-MS, Ω with a directed graph σ where σ is ordered pair $(\Lambda,\ \Xi)$ such that

$\Lambda\ (\sigma) = \Omega$ is the set of vertices,

$\Xi\ (\sigma) = \{(\omega, \upsilon) : (\omega, \upsilon) \in \Omega \times \Omega, \omega \neq \upsilon\}$

is the set edges of σ,

so, the graph $\sigma := (\Lambda\ (\sigma)\ ,\ \Xi\ (\sigma))$.

σ^{-1} := The conversion of the graph σ, (obtained by reversing the direction of $\Xi(\sigma)$).
$\sigma^{\sim}$:= By disregarding the orientation of σ, edges, the undirected graph was produced.

Here, $\sigma^{\sim}$ considered as a directed graph with a symmetric set of edges, hence, $\Xi\ (\sigma^{\sim}) = \Xi(\sigma) U \Xi\ (\sigma^{-1})$ [15].

Definition 1.9 ([23]): A subgraph of σ is a graph H such that $\Lambda\ (Z) \subseteq \Lambda\ (\sigma)$ and $\Xi\ (Z) \subseteq \Xi(\sigma)$ and, for any edge $(\omega, \nu) \in \Xi(Z)\ \omega,\ \upsilon\ \in \Lambda\ (Z)$.

Definition 1.10 ([15]): Let ω and υ be vertices σ, a Path in G from Ω to y of length n ($n\ \in\ N \cup \{0\}$) is a sequence $(\omega_i)_{i=0}^{n}$ of $n+1$ vertices such that, $\omega_0 = \omega$, $\omega_n = \upsilon$ and $(\omega_{i-1}, \omega_i)\ \in \Xi(\sigma)$ for $(i)_{i=1}^{n}$.

Definition 1.11 ([15]): The path length is the number of edges in G that form the path.

Definition 1.12 ([15]): If a path connects any two vertices in a graph σ, then the graph is linked. Otherwise, σ is called disconnected. Also, σ is weakly connected if $\sigma^{\sim}$ is connected.

Suppose that $\Xi(\sigma)$ is symmetrical and ω is a vertex in σ. If subgraph σ_ω consisting of all edges and vertices, that are belonged to some path in σ starting from ω, then σ_ω is called the component of σ including ω.

Here, using the rule $R(\omega R y$ if there is a path from ω to $y)$ to define the equivalence class $[\omega]_\sigma$ on $\Lambda(\sigma)$ such that $\Lambda(\sigma_\omega) = [\omega]_\sigma$.

Property A ([16]): For any sequence $(\omega_n)_{n\in N}$ in Ω, *if* $\omega_n \to \omega$ *and* $(\omega_n, \omega_{n+1}) \in\ \Xi(\sigma)$ *for* $n\ \in N$, *then* $(\omega_n, \omega) \in \Xi(\sigma)$.

As in [9, 16, 20], the following definition is reformed.

Definition 1.13: Let (Ω, S_b) be a *strong* b-MS and $F, H : \Omega \to CB(\Omega)$. The maps $F,\ H$ are called graph contractive in the event that $K \in (0, 1)$

s.t $(\omega \neq \upsilon)$, $(\omega, \upsilon)\ \in \Xi(\sigma)$ then $S_b\ (f_\omega, T\upsilon) < kd(\omega, \upsilon)$ and if $u\ \in f_\omega$ and $c \in\ H_\upsilon$ are such that then $(u, c) \in \Xi\ (\sigma)$, $d\ (u, c) <\ d\ (\omega, \upsilon)$.

Definition 1.14 ([15]): A binary relation $\preccurlyeq$ on set Ω is PO if for ω, υ and η in Ω, A set Ω with a PO $\preccurlyeq$ is called a POS if for all ω, υ and η in Ω.

1. $\preccurlyeq$ is reflexive ($\omega \preccurlyeq \omega$);
2. $\preccurlyeq$ is antisymmetry (if $\omega\ \preccurlyeq \upsilon$ and $\upsilon \preccurlyeq \omega$, then $\omega = \upsilon$);
3. $\preccurlyeq$ is transitivity (if $\omega\ \preccurlyeq \upsilon$ and $\upsilon \preccurlyeq \eta$, then $\omega \preccurlyeq \eta$);

Note that two elements ω and υ are called comparable if either $\omega \preccurlyeq \upsilon$ *or* $\upsilon \preccurlyeq \omega$.

2. Main results

In the following, the existence of a CFP in strong b-MS endowed with a graph for the multi-valued maps is proved. Firstly, suppose that $(\Omega,\ S_b,\ \sigma)$ is a complete strong b-MS and σ is a directed graph s.t $\Xi(\sigma)$ is symmetric.

Theorem 2.1: *Let $F, H :\ \Omega \to CB(\Omega)$ be graph contractive maps and let the triple (Ω, S_b, σ) have the Property A. Set $\Omega_F = \{\omega \in \Omega :\ (\omega, u)\ \in \Xi(\sigma)$ for some then the following statements hold.*

1. *For any $\omega \in\ \Omega_F$, $F,\ H|_{[\omega]_\sigma}$ have a CFP.*
2. *If $\Omega_F \neq \emptyset$ and σ has a poor connection, then $F,\ H$ have a CFP in Ω.*
3. *If $\Omega' :=\ U\{[\omega]_\sigma : \omega \in\ \Omega_F\}$, then $F, H|\omega'$ have a CFP*
4. *If $F \subseteq\ \Xi(\sigma)$ then F, H have a CFP.*

Proof: Let $\omega_0 \in \Omega$ then $\exists \omega_1\ \in F\omega_0\ \ni (\omega_0,\ \omega_1),\ \in \Xi(\sigma)$. If $H(F\omega_0,\ T\omega_1) = 0$ then $F\omega_0 =\ T\omega_1$, thus ω_1 is ordinarily CFP of T and F. If $H(F\omega_0,\ T\omega_1)\ > 0$, by Lemma 1.7
then for each $h_1\ > 1$ there exists $\omega_2 \in T\omega_1 \ni (\omega_1,\ \omega_2),\ \in \Xi(\sigma)$ and $S_b(\omega_1,\ \omega_2) < h_1 H(F\omega_0,\ T\omega_1)$.

Now if $H(F\omega_1,\ T\omega_2) = 0$. Then $F\omega_1 =\ T\omega_2,\ \omega$, thus ω_2 is CFP. If $H(F\omega_1,\ T\omega_2) > 0$ by Lemma 1.7 then for each $h_2 > 1$, $\exists \omega_3\ \in F\omega_2$ such that $(\omega_2,\ \omega_3),\ \in \Xi(\sigma)$ *and* $S_b(\omega_2,\ \omega_3) < h_2 H(F\omega_1,\ T\omega_2)$.

Continuing in this manner, if $H(F\omega_{2n-1}, T\omega_{2n-2}) = 0$ then $F\omega_{2n-1} = T\omega_{2n-2}, \omega_1$, thus ω_{2n} is CFP of F and T. If $H(F\omega_{2n-1}, T\omega_{2n}) > 0$ by Lemma 1.7 then for each $h_{2n} > 1, \exists\omega_{2n-1} \in T\omega_{2n-2}$, such that $(\omega_{2n-1}, \omega_{2n}), \in \Xi(\sigma)$

and $S_b(\omega_{2n-1}, \omega_{2n}) < h_n H(F\omega_{2n-1}, T\omega_{2n-2})$.

If at step k, in a bove process, satrsfy, $H(F\omega_{k-1}, T\omega_k) = 0$ then ω_k is CFP of F and T.

If not then are get the sequence $\langle\omega_n\rangle$ *in* Ω, and $\langle h_n\rangle$ $h_n > 1, \forall n$ such that $\omega_{2n+1} \in F\omega_{2n}$ and $\omega_{2n+2} \in T\omega_{2n+1}$.

Also $(\omega_{2n}, \omega_{2n+1}) \in \Xi(\sigma)$ and $S_b(\omega_{2n}, \omega_{2n+1})$

$$< h_{2n} H(F\omega_{2n}, T\omega_{2n+1}) \tag{1}$$

Since $\frac{1}{s+1} d(\omega_{2n}, F\omega_{2n}) < \frac{1}{s+1} S_b(\omega_{2n}, \omega_{2n+1}) \preccurlyeq (S_b(\omega_{2n}, \omega_{2n+1}))$
and by hypothesis, implies that

$$H(F\omega_{2n}, T\omega_{2n+1}) < K\, S_b(\omega_{2n}, \omega_{2n+1}) \tag{2}$$

By (1) and (2)

$$S_b(\omega_{2n}, \omega_{2n+1}) < h_2 K\, S_b(\omega_{2n}, \omega_{2n+1})$$

It is possible to choose $h_{2n} = \frac{r}{k}, r \in (0, 1), \ 0 < k < 1$
then $S_b(\omega_{2n}, \omega_{2n+1}) < \frac{k}{1-k} S_b(\omega_{2n-1}, \omega_{2n})$
thus $S_b(\omega_{2n}, \omega_{2n+1}) < (\frac{k}{1-k})^n S_b(\omega_0, \omega_1), \forall\, n \geq 1$

So, $\sum_{n=1}^{\infty} S_b(\omega_{2n}, \omega_{2n+1}) \preccurlyeq S_b(\omega_0, \omega) \sum_{n=1}^{\infty} (\frac{k}{1-k})^n < \infty$

By Proposition 1.5, getting $\langle\omega_n\rangle$ Cauchy sequence in Ω, $\exists\, \underline{\omega} \in \Omega \ni = \underline{\omega.}$

To prove $\underline{\Omega} \in f_\omega(F)\, f_\omega(T)$.

For n even by Property (A), obtaining $(\omega_n, \omega) \in \Xi(\sigma)$ therefore we must prove that

$$\frac{1}{s+1} d(\omega_n, T\omega_n) \preccurlyeq S_b(\omega_n, \underline{\omega}). \tag{3}$$

If not

$$\frac{1}{s+1} d(\omega_n, T\omega_n) \preccurlyeq S_b(\omega_n, \underline{\omega}) + SS_b(\omega_{n+1}, \omega)$$

$$\times S_b(\omega_n, \underline{\omega}) <$$

$$< \frac{1}{s+1} d(\omega_n, T\omega_n) + \frac{s}{s+1} d(\omega_{n+1}, T\omega_{n+1})$$

$$\preccurlyeq \frac{1}{s+1} d(\omega_n, \omega_{n+1}) + \frac{s}{s+1} d(\omega_{n+1}, \omega_{n+2}).$$

This contradicts
By (3) and hypothesis, getting

$$S_b(F\omega_n, T\omega) < kd(\omega_n, \omega)$$

Since $\omega_{n-1} \in F\omega_n$ and $\omega_n \to \underline{\omega}$ therefore by Lemma 1.8 $\underline{\omega} \in T\omega$ then $\underline{\omega} \in Fix(T)$. For n odd by similar way, getting $\underline{\omega} \in Fix(F)$.

Then $\underline{\omega} \in Fix(T) \cap Fix(F)$.

2. Since $\Omega_f \neq \emptyset$, so there exists $\omega_0 \in \Omega_f$, and since σ is weakly connected, therefore $[\omega_0]_\sigma = \Omega$ and by I, maps F and H have a CFP in Ω,
3. It follows easily from 1 and 2.
4. F $\subseteq \Xi(\sigma)$ implies that all $\omega \in \Omega$ are s.t $\exists\, u \in F_\omega$ with $(\Omega, u) \in \Xi(\sigma)$ so $\Omega_f = \Omega$ and by 2 and 3 H, F, have a FP.

Remark 2.2: Replace Ω_F by $\Omega_H := \{\omega \in \Omega : (\omega, u) \in \Xi(\sigma)$ For a few u $\in H\omega\}$ given the circumstances 1–3 of Theorem 2.1, then the conclusion is still accurate. Specifically, if $\Omega_F U\ \Omega_H \neq \emptyset$, then we have $FixF \cap FixH \neq \emptyset$, it readily flows from 1–3. Likewise, in condition 4, it is possible to replace

$$F \subseteq \Xi(\sigma) \text{ by } H \subseteq \Xi(\sigma).$$

The following consequence is directly related to the Theorem 2.1(1).

Corollary 2.3: *Let* (Ω, S_b, K) *be a complete strong b-MS and allow the triple* (Ω, S_b, σ) *possess the characteristic A. In the event that G has weak connections, graph contractive mappings*

$F, H : \Omega\ \ CB(\Omega)$ such that $(\omega_0, \omega_1) \in \Xi(\sigma)$

for some $\omega_1 \in F\omega_0 \in$ have a CFP.

Definition 2.4: A *strong* $b - MS$ (Ω, S_b) is called a $\varepsilon-$chainable strong b-MS for some $\varepsilon > 0$ if given $\omega, v \in \Omega$ there is $n \in N$ and a sequence $(\omega_i)_{i=0}^n$ s.t, $\omega_0 = \omega$, $\omega_n = v$ and $d(\omega_{i-1}, \omega_i) < \varepsilon$ for $i = 1, 2, \ldots, n$.

Example 2.5:

Consider $\Omega = R$ with $S_b(\omega, v)$

$$= \begin{cases} |\omega - v| \ if \ |\omega - v| \leq 1 \\ 2|\omega - v| \ if \ |\omega - v| > 1 \end{cases}$$

for $\omega, v \in \Omega$. The pair (Ω, S_b) easily satiies Definition 1.1. But strong triangle inequality deserves illustration as follows and we show the general case where $|\omega - z| > 1, \ |\omega - v| > 1$ and $|v - z| < 1$, so

$|\omega - z| \leq |\omega - \nu| + |\nu - z|$ then $S_b(\omega, z) \leq 2|\omega - z| \leq 2(|\omega - \nu| + |\nu - z|)$.

And $S_b(\omega, \nu) \geq |\omega - \nu|$, $S_b(\nu, z) \geq |\nu - z|$.

Thus $S_b(\omega, z) \leq 2(|\omega - \nu| + |\nu - z|) \leq S_b(\omega, \nu) + 2S_b(\nu, z)$. This demonstrates the strong triangle inequality at $K = 2$.

Here, Ω is ε-chainable sine, for $\varepsilon > 0$ and for any $\omega, \nu \in \Omega$ $S_b(\omega, z) < \varepsilon$ there are $\omega = \omega_1, \omega_2, \ldots .\omega_n = \nu$ such that $S_b(\omega_{j-1}, \omega_j) < \frac{\varepsilon}{2}$ by dividing the interval $[\omega, \nu]$ into equal parts of length $< \varepsilon$.

Corollary 2.6: *Let (Ω, S_b, K) be a ε−chainable complete strong b-MS for some $\varepsilon > 0$. Let $F, H : \Omega \to CB(\Omega)$ be such that there exists $K \in (0, 1)$ with*

$$0 < S_b(\omega, \nu) < \varepsilon \Rightarrow S_b(F\omega, H\nu) < KS_b(\omega, \nu)$$

Then F and H have a CFP.

Proof: Consider the graph σ as $\Lambda(\sigma) := \Omega$ and $\Xi(\sigma) := \{(\omega, \upsilon) \in \Omega \times \Omega : 0 < S_b(\omega, \nu) < \varepsilon\}$.

The ε-chainability of (Ω, S_b) means σ is connected. If $(\omega, \nu) \in \Xi(\sigma)$, then

$$S_b(F\omega, H\nu) < K\, S_b(\omega, \nu) < K\varepsilon < \varepsilon k.$$

And by using Lemma 1.7, for each $u \in F\omega$ we have the existence of $\nu \in H\nu$ s.t $S_b(\omega, \nu) < \varepsilon$ which suggests $(\omega, \nu) \in \Xi(\sigma)$. Hence F and H are contractive mappings of graphs.

Also, (Ω, S_b, σ) possesses property A. Indeed, if and $\omega_n \to \omega$ and $d(\omega_n, \omega_{n+1}) < \varepsilon$ for $n \in N$ then $S_b(\omega_n, \omega_{n+1}) < \varepsilon$ for sufficiently large n, therefore $(\omega_n, \omega) \in \Xi(\sigma)$. Accordingly, by Theorem 2.1 (2), F and H have a CFP.

Theorem 2.7: *Let $F : \Omega \to CB(\Omega)$ be graph contractive maps and let the triple (Ω, S_b, σ) have the property A. Set $\Omega_F = \{\omega \in \Omega : (\omega, u) \in \Xi(\sigma)$ for some $u \in F\omega$. Consequently, the following claims are true.*

1. *For any $\omega \in \Omega_F$, F, $H|_{[\omega]_\sigma}$ have a CFP.*
2. *If $\Omega_F \neq \emptyset$ and σ is not well connected, then F, has a FP in Ω.*
3. *If $\Omega' := U\{[\omega]_\sigma : \omega \in \Omega_F\}$, then $F, H|\omega'$ has a FP.*
4. *If $F \subseteq \Xi(\sigma)$ then F, H has a CFP.*
5. *If $\Omega_F \neq \emptyset$, then Fix $F \neq \emptyset$.*

Proof: It is possible to prove statements 1–4 by taking $F = H$ in Theorem 2.1 and 5 obtained from Remark 2.2.

Note that our Theorem 2.7 does not require that $\Xi(\sigma)$ be symmetric.

Corollary 2.8:

1. *Let $F : \Omega \to CB(\Omega)$ be contractive graph maps, and let the triple (Ω, S_b, σ) possess property A. Set $\Omega_F = \{\omega \in \Omega : (\omega, u) \in \Xi(\sigma)$ for some $u \in F\omega$ Consequently, the following claims are true.*
   a. *If $\Omega_F \neq \emptyset$ and σ is weakly connected, then F, has a FP in Ω.*
   b. *If $\Omega_F \neq \emptyset$, then Fix $F \neq \emptyset$.*
2. *Let $(\Omega, \preccurlyeq)$ be a POS and suppose that there exists a strong b−MS in Ω such that (Ω, S_b) is a complete strong b−MS Assume that Ω satisfies if a nondecreasing sequence $\{\omega_n\} \to \omega$ in Ω, then $\omega_n \preccurlyeq \omega, \forall\, n$. Let $f : \Omega \to \Omega$ be a monotone nondecreasing mapping such that there exists $k \in [0, 1)$ with $S_b(f(\omega), f(\upsilon)) \preccurlyeq kS_b(\omega, \upsilon), \forall \omega \geq \upsilon$. If there exists $\omega_0 \in \Omega$ with $\omega_0 \preccurlyeq f(\omega_0)$, then f has a FP.*
3. *Let Ω be a POS such that every pair $\omega, \upsilon \in \Omega$ has a lower bound and an upper bound. Furthermore, let S_b metric on Ω such that (Ω, S_b) is a complete strong b−MS. If μ is a continuous, monotone (i.e., either order-preserving or order-reversing) map from μ into μ such that*
   (1) $\exists 0 < c < 1 : S_b(\mu(\omega), \mu(\upsilon)) \preccurlyeq cS_b(\omega, \upsilon), \forall \omega \geq \upsilon$,
   (2) $\exists\, \omega_0 \in \Omega : \omega_0 \preccurlyeq \mu(\omega_0)$ or $\omega_0 \geq \mu(\omega_0)$, then μ has a unique FP $\underline{\omega}$. Moreover, for every $\omega \in \Omega$, $\mu^n(\omega) = \underline{\omega}$.
4. *Let (Ω, S_b) be complete, and let the triple (Ω, S_b, σ) have the following property: (for any (ω_n) $n \in N$ in Ω, if $\omega_n \to \omega$, and $(\omega_n, \omega_{n+1}) \in \Xi(\sigma)$ for $n \in N$, then there is a subsequence (ωk_n) $n \in N$ with $(\omega k_n, \omega) \in \Xi(\sigma)$ for $n \in N$.*
5. *Let Ω be a complete strong b−MS ε-chainable space, f a mapping of Ω into itself which is (ε, δ)−uniformly locally contractive, then a singular point is present. $\omega \in \Omega$ such that $f(\omega) = \omega$.*

As a special case of the above work in the usual metric, one can consider the following remark.

Remark 2.9:

1. If we assume Ω is such that $\Xi(\sigma) := \Omega \times \Omega$. Then, it is evident that Ω and our Theorem 2.7, are related. See [22] enhances the Banach contraction theorem if F is single-valued, and it also enhances Nadler's theorem.
2. If F is a single-valued mapping, then Theorem 2.7(2, 5) with the graph G_1 improves [25, Theorem 2.2].

3. If F is a single-valued mapping, then Theorem 2.7(2, 5) with the graph G_2 improves [24, Theorem 2.1].
4. If $F = H$ is a single-valued mapping, then Theorems 2.1 and 2.7 partially generalize [25, Theorem 3.2].
5. If $F = H$ we take as single-valued mappings in Corollary 2.6, then we have [26, Theorem 5.2].

3. Conclusions

Fixed point theorems are a tool for various branches of mathematics, topology, equations, approximation theory, potential theory, functional analysis and statistics. Even in computer science, fixed points help in modify algorithms, especially in iterative schemes. This work prompted us and other researchers to present results with respect to graph concepts.

In future work, as fractal graph, we aspire to generate the results in [27] to produce some non-classical variants of Julia set, utilizing the Jungck-Ishikawa fixed point iteration system with respect to graph tools.

Abbreviation

Common fixed point (CFP); Fixed point (FP); strong b-metric space (strong b-MS); b-metric space (b-MS); Metric space (MS); Partial ordered set (POS).

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Conflict of interest

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