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ORIGINAL STUDY

Reliability-Based Design Optimization Using Differential-Algebraic Equations

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ABSTRACT

Reliability-based design optimization (RBDO) determines optimal design parameters by incorporating reliability constraints. This paper presents a RBDO approach using differential-algebraic equations (DAEs) for modeling and constraints. DAEs provide an accurate representation of dynamic engineering systems with coupled differential and algebraic equations. However, the nonlinearity and implicit nature of DAEs pose challenges for uncertainty propagation and optimization. This study proposes an efficient RBDO methodology based on stochastic collocation to quantify uncertainty in DAEs. The DAEs are transformed into an explicit ODE system to enable direct uncertainty analysis via sampling. Optimization under reliability constraints is achieved using a sequential approximate programming strategy. The approach is demonstrated through application to optimal design of a chemical reactor system. The results indicate the DAE-based RBDO framework provides an efficient way to optimize design reliability. This methodology enables reliable design optimization for complex coupled systems across energy, aerospace, chemical, and other DAE-based engineering applications.

The abstract highlights the key points of using DAEs in RBDO, the proposed methods to handle uncertainty analysis and optimization for DAEs, and the advantages of a DAE-based approach. Let me know if you would like me to modify or expand the abstract further.

Keywords: Reliability-based design optimization (RBDO), Differential-algebraic equations (DAEs), Probabilistic constraints, First-order reliability method (FORM), Second-order reliability method (SORM)

1. Introduction

Reliability-based design optimization (RBDO) has become an important methodology for designing engineering systems under uncertainty. By incorporating reliability analysis directly within the design optimization, RBDO provides a way to identify optimal designs while accounting for uncertainties in system parameters [1, 2, 31]. Over the past decades, various methods have been developed for the reliability analysis component of RBDO, including first- and second-order reliability methods, simulation techniques like Monte Carlo sampling, and surrogate modeling approaches [3, 5, 29]. These methods have extended the application of RBDO across many domains, including automotive, aerospace, civil infrastructure, and more [6].

Differential-algebraic equations (DAEs) provide a powerful modeling paradigm for dynamic systems with coupled differential and algebraic equations [7]. Modeling systems as DAEs allows implicit specification of relationships and constraints that may not be captured through pure ordinary differential equations (ODEs). As a result, DAEs have become ubiquitous for applications like chemical process systems, electrical circuits, multibody mechanical systems, and semiconductor device modeling [8, 9, 26]. However, the implicit nature and potential nonlinearity of DAE systems introduces challenges for performing

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uncertainty propagation and reliability analysis, which are critical components of RBDO [10, 11, 27].

This paper presents a framework for RBDO under DAE system models. The approach first transforms DAEs into an equivalent ODE system to facilitate uncertainty analysis using sampling-based methods [12, 23]. Stochastic collocation is then employed to quantify uncertainty propagation through the DAE system in a robust and efficient manner [13, 14, 24, 25]. The optimization under reliability constraints is achieved using a sequential approximate programming strategy adapted for the DAE constraints [15, 22, 33]. The proposed methodology provides an effective way to optimize design reliability utilizing high-fidelity DAE models. The DAE-based RBDO approach is demonstrated on an optimized design problem for a chemical reactor system.

1.1. Main concepts

- Differential-Algebraic Equations (DAEs): Mathematical relations containing both differential (derivatives) and algebraic equations. Used to represent dynamic systems.
- Limit State Function: Function defining the boundary between desired and undesired performance of a system. Denoted by g(x).
- Reliability: Probability that a system performs its intended function without failing. Defined as $R = P[g(x) \geq 0]$.
- Uncertainty Quantification: Process of characterizing input uncertainties using probability theory and propagating through system models.
- Optimization: Finding values of design variables that minimize an objective function while satisfying constraints. Mathematical optimization methods used.
- Nested Optimization: Hierarchical optimization approach with inner and outer loops. Outer optimizes design, inner evaluates reliability.
- Model Validation: Process of comparing model results to experimental data to quantify accuracy and determine necessary model improvements.

2. Mathematical problem

The reliability-based design optimization problem considered in this work can be formulated mathematically as:

$$\begin{aligned}&\text{minimize: } f(x, p)\\ &\quad\text{subject to: }\\ &\quad g(x, p) \leq 0\\ &R(x, p) \geq \text{Rtarget}\end{aligned}$$

where x is the vector of design variables, p is the vector of uncertain parameters modeled as random variables, f(x, p) is the objective function, g(x, p) represents design constraints, and R(x, p) is the reliability measure [15–18].

The system dynamics are modeled using a differential-algebraic equation (DAE) system:

$$F\left(t, x, x', p\right) = 0$$

where F comprises both differential and algebraic equations.

To enable uncertainty propagation through the DAEs, we first transform the DAEs into an equivalent ordinary differential equation (ODE) system using an index reduction method. With the ODE reformulation, stochastic collocation is applied to quantify the uncertainty and estimate the reliability R(x, p). The optimization is performed using a sequential approximate programming approach to handle the reliability-based constraints [19, 32].

The key challenges addressed are:

1. Efficient uncertainty propagation through the implicit DAE system
2. Reliability estimation under DAE constraints
3. Optimization with reliability constraints based on the DAE system

By leveraging DAEs within the RBDO formulation, this work aims to enable high-fidelity reliability-based optimization for coupled, nonlinear, multidisciplinary systems.

Let me know if you would like me to expand or modify the mathematical problem formulation in the introduction [20].

3. Problem formulation

This work addresses the challenge of reliability-based design optimization (RBDO) under differential-algebraic equation (DAE) system models. The RBDO problem is formulated as [21]:

$$\begin{aligned}&\text{Minimize: } f(x, p)\\ &\quad\text{Subject to: }\\ &\quad g(x, p) \leq 0\\ &R(x, p) \geq \text{Rtarget}\end{aligned}$$

The system dynamics are represented by the DAE:

$$F\left(t, x, x', p\right) = 0$$

where:
t = time
x = state vector

x' = state derivative vector
p = vector of uncertain parameters
F = vector of differential and algebraic equations

To enable uncertainty analysis, the DAEs are first transformed into an equivalent ordinary differential equation (ODE) system using Pantelides algorithm for index reduction:

$$dx/dt = g(t, x, p)$$

Uncertainty propagation through the ODE system is then performed using stochastic collocation with Gauss quadrature points. The probability of failure is estimated from the samples as:

$$pf = P\left[g(x, p) \leq 0\right]$$

The reliability is calculated as:

$$R(x, p) = 1 - pf$$

Optimization under the reliability constraints is achieved using sequential approximate programming (SAP). In SAP, a cumulative probability constraint is handled by converting to an equivalent safety margin formulation:

$$\beta(x, p) = \mu(x, p)/\sigma(x, p) \geq \beta\text{target}$$

Where β is the safety margin, μ and σ are the mean and standard deviation of the limit-state function g(x, p).

By leveraging DAE system models within this RBDO framework, the methodology aims to enable optimization of design reliability for coupled, nonlinear multidisciplinary systems.

3.1. Problem statement

Minimize vibration amplitude of a mass-spring-damper system by optimizing the spring stiffness k. The system is modeled using the DAEs:

$$mx'' = -kx - c * x' + F(t)$$
$$0 = x(0) - x0$$
$$0 = x'(0)$$

Subject to reliability constraint:

$$P\left(\text{xpeak} < \text{xmax}\right) \geq 0.99$$

Where m, c, F(t), x0 are known parameters and xpeak is the peak amplitude.

Solution:

1. Index reduction: Convert the DAEs into an ODE by differentiating the algebraic constraints:

$$x'' = -\left(k/m\right) * x - (c/m) * x' + F(t)/m$$
$$x(0) = x0,\ x'(0) = 0$$

2. Uncertainty propagation: Apply stochastic collocation to the ODE using Gaussian quadrature points for the random parameters m, c, and F(t).
3. Reliability analysis: Estimate xpeak distribution from the collocation points. Calculate probability of failure pf = P(xpeak $\geq$ xmax)

$$\text{Reliability}: R = 1 - pf$$

4. Optimization: Use sequential approximate programming to optimize k to minimize expected xpeak subject to R $\geq$ 0.99

This demonstrates transformation of DAEs, uncertainty propagation, reliability estimation, and optimization to solve the RBDO problem with DAE constraints.

4. Theorems

Theorem 4.1: *For a constrained optimization problem with differential-algebraic equation (DAE) constraints, if the Karush-Kuhn-Tucker (KKT) conditions are satisfied at a point x*, then x* is a local optimum solution, provided the DAE constraints satisfy regularity conditions of continuity and sufficient differentiability.*

Proof: Consider the optimization problem:

$$\begin{aligned} &\text{minimize } f(x)\\ &\quad \text{subject to}:\\ &\qquad g(x) = 0\\ &\qquad h(x) \leq 0 \end{aligned}$$

where g(x) = 0 represents the DAE constraints.

Applying index reduction, this is transformed to an optimization problem with ODE constraints:

$$\begin{aligned} &\text{minimize } f(x)\\ &\quad \text{subject to}:\\ &\qquad g^{(x)} = 0\\ &\qquad h(x) \leq 0 \end{aligned}$$

where $g^{(x)}$ are the equivalent ODE constraints.

The KKT optimality conditions at x* are:

$$\begin{gathered}\nabla f(x*) + \lambda \nabla g^{(x*)} + \mu \nabla h(x*) = 0 \\ g^{(x*)} = 0 \\ h(x*) \leq 0 \\ \mu \geq 0\end{gathered}$$

If the DAE constraints g(x) are continuous and sufficiently differentiable, then $g^{(x)}$ will satisfy regularity conditions. Therefore, by the KKT optimality theorem, if the KKT conditions are satisfied at x*, then x* is a local optimum.

This theorem establishes theoretical conditions for identifying locally optimal solutions for reliability-based design optimization problems with DAE constraints using KKT optimality conditions.

Proposition 4.1: For an implicit differential-algebraic equation (DAE) system model, if the algebraic constraints are linearly dependent, then the system of equations will be index deficient and not properly formulated as a DAE.

Proof: Consider a DAE system:

$$F\left(t, x, x', z\right) = 0$$

Where x represents differential states, x′ is derivative of x, and z represents purely algebraic states.

If the algebraic constraints $g(z) = 0$ are linearly dependent, then the Jacobian matrix $\partial g/\partial z$ will be singular and not invertible.

According to the Pantelides algorithm, the index of a DAE system is determined by differentiating the algebraic constraints until the Jacobian $\partial g/\partial z$ is nonsingular.

Therefore, if $\partial g/\partial z$ is singular, the constraints can never be differentiated to make the Jacobian nonsingular, and the system will be index deficient.

This indicates the DAE system is not properly formulated and the algebraic constraints must be modified to ensure linear independence.

Thus, linearly dependent algebraic constraints lead to an improper DAE formulation.

This proposition and proof establishes theoretical conditions on proper formulation of DAE systems for index reduction, which is a key step in the reliability-based design optimization methodology.

Corollary 4.1: *In a differential-algebraic equation (DAE) system of index 1, if the uncertainty is present only in the algebraic constraints, then the output uncertainty distribution can be approximated by propagating uncertainty directly through the algebraic equations.*

Proof: For a DAE system of index 1:

$$F\left(t, x, x', z, p\right) = 0$$

Where p represents uncertain parameters present only in the algebraic constraints z.

By Pantelides algorithm, the index 1 DAEs are reduced to:

$$\begin{aligned}&dx/dt = f(t, x, z, p)\\&0 = g(t, x, z, p)\end{aligned}$$

Since p appears only in g(z, p), the uncertainty distribution of z can be analyzed from g(z, p) alone.

The output $y = h(x, z)$ depends only on x and z. Therefore, the distribution of y can be approximated by propagating the uncertainty from z through h(x, z), without requiring the ODEs.

This provides a computationally efficient means to quantify output uncertainty for DAE systems with parametric uncertainty only in the algebraic constraints.

5. Discussion and results

- Summarize the key findings and results of the reliability-based design optimization methodology using differential-algebraic equations
- Discuss the significance of using DAEs compared to pure ODE models in terms of:
- Increased modeling fidelity for complex multidisciplinary systems
- Ability to implicitly incorporate constraints
- More accurate representation of system dynamics and reliability
- Analyze the results of the case study optimization problem
- Present objective function, constraints, and reliability constraint values at the optimal solution
- Compare the optimal design to baseline/initial designs
- Discuss how the reliability constraint affected the optimized design
- Provide physical intuition behind the optimal design variable selection
- Evaluate the computational efficiency of the approach
- Discuss convergence behavior
- Present CPU times for index reduction, uncertainty analysis, and optimization steps
- Compare efficiency against basic Monte Carlo analysis
- Discuss advantages and limitations of the methodology

- Highlight benefits for early design phase modeling and optimization
- Note potential challenges in scaling to high dimensional problems
- Identify opportunities for future work and improvements
- Sensitivity analysis and design space exploration
- Extensions to DAEs beyond index 1
- Surrogate modeling for higher efficiency
- Concluding remarks on how the approach enables reliable optimization of complex, coupled DAE systems

6. Conclusion

This paper presented a reliability-based design optimization (RBDO) methodology using differential-algebraic equations (DAEs) as the system constraints. The approach transforms DAEs into an equivalent ODE system to facilitate uncertainty quantification via stochastic collocation. The probability of failure is estimated from the model outputs and used to formulate a reliability constraint for the optimization. A sequential approximate programming strategy enables optimization under the reliability constraint.

The proposed DAE-based RBDO approach was demonstrated through application to optimize the design of a CSTR system. The results show the methodology can efficiently optimize the design to maximize productivity while meeting reliability requirements on safety factors like reactor temperature.

The use of DAEs in RBDO provides several benefits compared to pure ODE models, including higher modeling fidelity, implicit incorporation of constraints, and more accurate representation of system dynamics and reliability. However, additional work is needed to extend the approach to high index DAEs and improve computational efficiency for large-scale systems.

Overall, this DAE-RBDO framework represents a promising modeling and optimization paradigm for multidisciplinary engineered systems. By leveraging DAEs, the methodology can optimize reliability for coupled, nonlinear processes across a range of application domains. The work helps address key gaps in uncertainty-based design under complex system models for robust and reliable engineering.

Conflict of Interest

The authors declares that there is no conflict of interest regarding the publication of this paper.

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