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RESEARCH ARTICLE

# Analysis of Minimal and Maximal Closed Submsets in M-topology

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## ABSTRACT

In the realm of mathematical analysis, the concept of a Multiset (in short mset), which allows for the inclusion of repeated elements within a collection, has garnered attention. This is particularly relevant in real-life scenarios where the presence of duplicates holds significance. The emergence of mset topology, a specialized branch of topology designed to accommodate the unique characteristics of msets, has provided a valuable framework for understanding the topological properties of these diverse collections. This paper delves into the nuanced exploration of mset topology, specifically focusing on various properties associated with minimal closed submsets and maximal closed submsets. These submsets are then scrutinized in terms of their interior, closure, and counts, providing a comprehensive understanding of their structural intricacies within the mset topology context. Expanding the analysis, this research also investigates submsets with diverse combinations of designations, including minimal open, maximal open, minimal closed, and maximal closed. This study contributes to the establishment of a detailed taxonomy of submsets within the mset topology framework, elucidating the interplay between openness and closedness in different contexts. By uncovering and explicating the properties of minimal and maximal closed submsets, as well as their varied combinations of designations, this paper makes a substantial contribution to the broader mathematical discourse on msets and their intricate topological characteristics.

**Keywords:** Closed submset, Maximal closed sets, Minimal closed sets, Multiset, Mset, M-topology, Submset

## Introduction

Deviations from the conventional notions of logic and analysis always yielded fruitful results and enriched mathematics. The rationale behind such alterations is quite often for dealing with problems of imprecision, ambiguity, and repetition. Such resultant structures gained prominence following the introduction of fuzzy sets and rough sets, which offer distinct perspectives compared to traditional or orthodox approaches.[1,2] Also, other such resultant structures including soft sets and msets are rapidly emerging as significant areas of study within mathematics.[3–5] Identical entities are natural in data processing and information retrieval, and this justifies the need for msets, a structure that supports duplicates. Msets or bags are collection in which duplicates or repetitions are allowed. Fundamental results and theory developed can be seen in.[6,7]

The topological study of objects includes the exploration of qualitative properties of objects which do not change under certain types of transformations called continuous mappings.[8,9] Compared with classical sets, msets can represent situations more effectively if the objects under study are non-distinct. Similarities between various universes can be measured more effectively in the mset topology structure.

Several research publications by Rajish et al.[10] and Girish et al.[11] presented the study of topological structures on msets. In recent times, numerous

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investigations have been conducted on M-topological concepts, yielding a plethora of findings.[12–15] In 2001, Nakaoka and N. Oda[16,17] presented the notions of minimal open sets and maximal open sets within the realm of general topology. Their investigation delved into the relationships between these concepts and their connections with similar ideas defined in the context of closed sets. Moreover, the presence of minimal open sets and maximal closed sets is notable in spaces characterized by local finiteness, as exemplified in the digital line.[16,18] The concept of maximal and minimal open submsets in M-topology was explored by the same authors in the year 2024.

The paper's findings can be summarized as follows: Section 2 gathers all the fundamental definitions and concepts necessary for subsequent discussions. Section 3 introduces the notion of minimal closed subsets and maximal closed subsets within M-topology. It provides definitions for the whole core and whole complement of a subset and examines their significance in M-topology concerning maximal and minimal closed subsets. The section also investigates the impact of maximal and minimal closed subsets on the disconnectedness of an M-topological space. Considering clopen subsets and one of the introduced concepts, the restriction on the count of elements in such sets is explored, leading to the establishment of several results.

## Materials and methods

The pre-requisites, including definitions, concepts and properties discussed in,[8,11,12] that are necessary for this work are discussed in this section.

**Definition 1:** [11] For any ordinary set $Y$, an mset $P$ drawn from the set $Y$ is a function Count $P$ denoted by $C_P$ and defined as $C_P : Y \to \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Let us denote the number of occurrences of an element $e$ in the mset $P$ by $C_P(e)$. It also represents as the multiplicity of the element $e$ in $P$.

If $Y = \{y_1, \ldots, y_r\}$ and the multiplicity or count of $y_i$ in $P$ is $m_i$, then the mset $P$ is represented as $P = \{m_1/y_1, m_2/y_2, \ldots, m_r/y_r\}$.

Any element in $P$ that occurs zero times is not included in this representation.

**Example 1:** *Let $A = \{a, b, c\}$, then $P = \{9/a, 8/b\}$ is an mset drawn from $A$.*

**Note 1:**[11] Consider two msets $C$ and $D$ drawn from an ordinary set $Y$. Then the mset operations are as follows:

(i) $C = D \Leftrightarrow C_C(y) = C_D(y), \forall y \in Y$.
(ii) $C \subseteq D \Leftrightarrow C_C(y) \le C_D(y), \forall y \in Y$.
(iii) $E = C \cup D \Leftrightarrow C_E(y) = \max\{C_C(y), C_D(y)\}$, $\forall y \in Y$.
(iv) $E = C \cap D \Leftrightarrow C_E(y) = \min\{C_C(y), C_D(y)\}$, $\forall y \in Y$.
(v) $E = C \oplus D \Leftrightarrow C_E(y) = C_C(y) + C_D(y), \forall y \in Y$.
(vi) $E = C \ominus D \Leftrightarrow C_E(y) = \max\{C_C(y) - C_D(y), 0\}$, $\forall y \in Y$, where $\oplus$ is the mset addition and $\ominus$ is the mset subtraction.

**Definition 2:** [11] The support set or root set of an mset $P$ drawn from an ordinary set $Y$ is the ordinary subset of $Y$ defined by $P^* = \{y \in Y : C_P(y) > 0\}$.

**Definition 3:** [11] The underlying set $Y$, from which the mset is constructed, is termed the domain. The family of all msets drawn from $Y$ such that the multiplicity of each element is not more than $w$, is denoted by $[Y]^w$. The family of all msets drawn from $Y$ such that there is no limit on the number of occurrences of an element is denoted by $[Y]^\infty$.

**Definition 4:** [11] If count of every element of the domain $Y$ is zero in an mset, then it is called a null mset or an empty mset, that is, an mset $P$ is empty if and only if $C_P(y) = 0, \forall y \in Y$.

**Definition 5:** [10] Let $P$ be an mset and $Q$ be a partial whole submset of $P$, then $p \in P$ is called a whole element of $P$ if $C_P(p) = C_Q(p)$ and a part element of $P$ if $C_P(p) < C_Q(p)$.

In mset theory, the concept of a submset is defined using the count function, giving rise to diverse categories of submsets as follows:

**Definition 6:** [11] If the count of every element of a submset $Q$ of $P$ is equal to that of $P$, that is, each element has full multiplicity as in $P$, then $Q$ is called a whole submset of $P$.

**Definition 7:** [11] If there exists an element in submset $Q$ of $P$ such that its count in $Q$ is equal to its count in $P$, then $Q$ is categorized as a partial whole submset of $P$.

**Definition 8:** [11] If $Q \subseteq P$ with support sets of $Q$ and $P$ are equal, then $Q$ is called a full submset of $P$.

**Definition 9:** [10] Let $N$ be an mset and $Q$ be a partial whole submset of $N$, then $q \in Q$ is called a whole element of $Q$ if $C_Q(q) = C_N(q)$ and called a part element of $Q$ if $C_Q(q) < C_N(q)$.

**Definition 10:** [11] Let $Q$ be an mset. The power mset of $Q$ which is denoted by $\mathcal{P}(Q)$ is the collection of all submsets of $Q$. That is, $D \in \mathcal{P}(Q)$ if and only if $D \subseteq Q$.

**Definition 11:** [11] The power set of $Q$ is the support set of $\mathcal{P}(Q)$ and it is denoted by $\mathcal{P}^*(Q)$.

Moving forward, we will explore the definition of $M$-topology and discuss select concepts that are crucial for our study.

**Definition 12:** [11] Let $Q$ be an mset and $\tau \subset \mathcal{P}^*(Q)$. Then $\tau$ is called an M-topology on $Q$ if it satisfies the following conditions:

i. $\emptyset, Q \in \tau$.
ii. $\tau$ is closed under mset union.
iii. $\tau$ is closed under finite mset intersection.

Let $N$ be an M-topological space with M-topology $\tau$, then a submset $V$ of $N$ is said to be open, if $V$ belongs to the collection $\tau$.

**Definition 13:** [11] Let $N$ be an M-topological space with M-topology $\tau$ and $M \subseteq N$. The collection $\tau_M = \{M \cap U; U \in \tau\}$ is an M-topology on $M$ and is called the open subspace $M$-topology or subspace $M$-topology on $M$. In this case, $M$ is an M-topological subspace of $N$ and the open submsets of $M$ with respect to subspace M-topology on $M$, are obtained by intersecting open submsets of $N$ with $M$.

**Definition 14:** [11] A submset $U$ of an M-space $N$ is said to be closed if its mset complement is open, i.e., $N \ominus U \in \tau$, where $\ominus$ is the mset subtraction.

**Definition 15:** [11] Let $U$ be a submset of an M-topological space $M$ in $[Y]^w$. The interior of $U$ is defined as the mset union of all submsets which are open in $M$ and contained in $U$, and denotes the interior of $U$ by int$(U)$. i.e., $C_{\text{int }(U)}(y) = C_{\cup H}(y)$, where the mset union is over all $H$ which are open in $M$ and $H \subset U$. The closure of $U$ is defined as the mset intersection of all submsets which are closed in $M$ and contain $U$ and is denoted by $cl(U)$. i.e., $C_{cl(U)}(y) = C_{\cap K}(y)$ where the mset intersection is over all $K$ which are closed in M and $U \subset K$.

**Definition 16:** [10] Let $V$ be a submset of an M-topological space $N$ with M-topology $\tau$. The closed subspace M-topology on $V$ is defined by $\tau_c = \{V \ominus (V \cap U^c) : U \text{ is open in } N\}$, where $\ominus$ is the mset subtraction.

**Definition 17:** [10] [1] Let $N$ be an mset and $V$ be a submset of $N$, then $v \in V$ is called a whole element of $V$ if $C_V(v) = C_N(v)$ and called a part element of $V$ if $C_V(v) < C_N(v)$.

## Results and discussion

In the context of M-topology, this section introduces and analyzes the properties of maximal and minimal closed submsets, drawing connections to the properties of their open counterparts.

**Definition 18:** A proper nonempty closed submset $P$ of an M-topological space $N$ is called a maximal closed submset if there is no proper closed submset of $N$ properly containing $P$.

**Definition 19:** A proper nonempty closed submset $P$ of an M-topological space $N$ is called a minimal closed submset if there is no nonempty closed submset properly contained in $P$.

**Definition 20:** Let $N$ be an mset equipped with the M-topology $\tau$. A proper nonempty open submset $P$ of $N$ is called a maximal open submset if there is no proper open submsets properly containing $P$.

**Definition 21:** A nonempty open submset $P$ of an $M$-topological space $N$ is called a minimal open submset if there is no nonempty open submset properly contained in $P$.

**Definition 22:** An $M$-topological space $N$ is said to be disconnected if $N = A \cup$ B, where $A$ and $B$ are nonempty disjoint whole open submsets of $N$.

**Example 2:** *Let $N = \{8/c, 8/d\}$ and $\tau = \{N,\ \emptyset,\ \{4/c, 8/d\},\ \{8/c, 4/d\},\ \{4/c, 4/d\}\}$. Then, $\tau$ is clearly an M-topology on N. The maximal open submsets are $\{4/c, 8/d\}$, $\{8/c, 4/d\}$ and $\{4/c, 4/d\}$ is a minimal open submset. The closed submsets are $N$, $\emptyset$, $\{4/c\}$, $\{4/d\}$,* and *$\{4/c, 4/d\}$. Here, $\{4/c, 4/d\}$ is a maximal closed submset of N and $\{4/c\}$, $\{4/d\}$ are minimal closed submsets of N.*

Now, consider $\tau_1 = \{N, \emptyset, \{4/c, 8/d\}, \{8/c, 4/d\}, \{4/c, 4/d\}\}$ as a topology on $N$ and take $A = \{8/c\}$ and $B = \{8/d\}$. Then, $A$ and $B$ are nonempty disjoint whole open submsets of $N$ and $N$ is disconnected with respect to the topology $\tau_1$. But we cannot find two such sets in the topology $\tau$, and hence $N$ is not disconnected with respect to the topology $\tau$.

**Theorem 1:** *If $P$ is a maximal closed submset of an M-topological space $N$ and $Q$ is any closed submset of $N$, then either $P \cup Q = N$ or $Q \subseteq P$. If $Q \nsubseteq P$, then $P^w \subset Q$.*

**Theorem 2:** *If a submset $P$ of an M-topological space $N$ is minimal closed and $Q$ is any closed submset, then either $P \subseteq Q^c$ or $P \subseteq Q$.*

**Theorem 3:** *If $P$ and $Q$ are closed submsets of an M-topological space $N$ such that $P$ is maximal closed and $Q$ is minimal closed, then either $Q \subseteq P$ or $P$ and $Q$ are whole submsets and complement to each other. (i.e., $\cup Q = N, P \cap Q = \emptyset$).*

**Proof:** Since $P$ is maximal closed submset, $P \cup Q = P$ or $N$. So, either (a) $Q \subseteq P$ or (b) $P \cup Q = N$. Since $Q$ is minimal $\Rightarrow P \cap Q = \emptyset$ or $Q$. Consequently either (c) $Q \subseteq P$ or (d) $P \cap Q = \emptyset$. Now, (a) and (c) $\Rightarrow P \subset Q$, (a) and (d) $\Rightarrow Q = \emptyset$, which is not possible. And (b) and (c) $\Rightarrow P = N$, which is also not possible. Considering (b) and (d), it follows that $P \cup Q = N$ and $P \cap Q = \emptyset$. Therefore $Q$ and $P$ are whole submsets of $N$ and complements to each other.

**Theorem 4:** *If $P$ is a maximal closed submset of an M-topological space N, then either of the following is true:*

*(a) $P$ is a full submset of $N$ with $C_P(y) \geq \frac{C_N(y)}{2}, \forall y \in N$.*
*(b) $\text{int}(P) = \text{int}(\tilde{P})$.*

**Proof:** If $H = \text{int}(P)$, then $H^c$ is a closed submset of $N$ and hence $P \subseteq P \cup H^c \subseteq N$. Since $P$ is maximal closed, either $P \cup H^c = P$ or $P \cup H^c = N$.

Suppose that $P \cup H^c = P$. So $H^c \subseteq P \Rightarrow P^c \subseteq H = \text{int}(P) \subseteq P$. That is, $P^c \subseteq P$. Hence $P$ is a full submset of $N$ with $C_P(y) \geq \frac{C_N(y)}{2}, \forall y \in N$.

On the other hand, consider $P \cup H^c = N$. Then $P^w \subseteq H^c \Rightarrow H \subseteq (P^w)^c = \tilde{P} \Rightarrow \text{int}(P) \subseteq \tilde{P} \Rightarrow \text{int}(\text{int}(P)) \subseteq \text{Int}(\tilde{P}) \Rightarrow \text{int}(P) \subseteq \text{int}(\tilde{P})$. Since $\tilde{P} \subseteq P$, $\text{int}(\tilde{P}) \subseteq \text{int}(P)$. Hence $\text{int}(P) = \text{int}(\tilde{P})$.

**Theorem 5:** *If $P$ is a maximal closed submset of an M-topological space $N$ with $\text{int}(\tilde{P}) = \emptyset$, then $P$ is the one and only maximal closed submset of $N$.*

**Proof:** Suppose $Q$ is a closed submset of $N$ different from $P$. Then $P \subseteq P \cup Q \subseteq N$. Since $P$ is maximal, either $P = P \cup Q$ or $P \cup Q = N$. If $P = P \cup Q$, then $Q \subseteq P$.

If $P \cup Q = N$, then $\max\{C_P(y), C_Q(y)\} = C_N(y)$. Hence $y \notin \tilde{P} \Rightarrow C_P(y) < C_N(y)$. But $\max\{C_P(y), C_Q(y)\} = C_N(y)$, so $C_Q(y) = C_N(y)$, implies that $y \notin Q^c$. That is, $y \notin \tilde{P} \Rightarrow y \notin Q^c$ and it implies that $Q^c \subseteq \tilde{P}$. Also, $Q^c$ is open and a submset of $\tilde{P}$. It is also given that $\text{int}(\tilde{P}) = \emptyset$. So $Q^c = \emptyset$ and hence $Q = N$ and $Q$ is not a proper closed submset of $N$. Thus, for every closed submset $Q$, either $Q \subseteq P$ or $Q$ is not a proper submset and it follows that every proper closed submset of $N$ is a submset of $P$. Hence $P$ is the only maximal closed submset of $N$.

The proof of the following corollary is straightforward.

**Corollary 1:** *If $P$ is a maximal closed submset of an M-topological space $N$ without whole elements, then $P$ is the only maximal closed submset of $N$.*

**Theorem 6:** *Let $P$ be a maximal closed submset of an M-topological space $N$ and $y \in P^w$. If every nonempty open submset contains $y$, then $P$ is the only maximal closed submset of $N$.*

**Proof:** Suppose $P$ is a maximal closed submset and $y \in P^w$. Hence, $y \notin \tilde{P}$ and $C_P(y) < C_N(y)$. If $Q$ is a proper closed submset of $N$, then $N \ominus Q$ is an open submset containing $y$ and $C_Q(y) < C_N(y)$. Therefore, $C_{P\cup Q}(y) = \max\{C_P(y), C_Q(y)\} \neq C_N(y) \Rightarrow P \cup Q \neq N$. Since $P$ is maximal and $P \cup Q \neq N \Rightarrow P \cup Q = P$. So $Q \subseteq P$. That is, every proper closed submset $Q$ is contained in $P$. Hence $P$ is the only maximal closed submset of $N$.

**Corollary 2:** *If $N$ is an M-topological space with the property that the intersection of all nonempty open submsets is not empty and $P$ is maximal closed submset, then there is only one maximal closed submset.*

**Theorem 7:** *If a submset $P$ of an M-topological space $N$ is both open and closed with maximal in one among them, then $C_P(y) \geq \frac{C_N(y)}{2}, \forall y \in P$.*

**Proof:** Suppose that $P$ is open and maximal closed. Then $P$ is a clopen submset and $P^c$ is also clopen. Hence $P^c$ is closed and $P$ is maximal closed $\Rightarrow$ either $P^c \subseteq P$ or $P \cup P^c = N$.

If $P^c \subset P$, then $C_P(y) \geq \frac{C_N(y)}{2}, \forall y \in P$.

If $P \cup P^c = N$, then $P$ is a whole submset and hence $C_P(y) = C_N(y) \geq \frac{C_N(y)}{2}, \forall y \in P$.

In either case, $C_P(y) \geq \frac{C_N(y)}{2}, \forall y \in P$.

In a similar way, the other case can also be proved.

**Theorem 8:** *If a submset $P$ of an M-topological space $N$ is both open and closed with minimal in one among them, then either $C_P(y) \leq \frac{C_N(y)}{2}, \forall y \in P$ or $P$ is a whole submset of $N$.*

**Proof:** First assume that $P$ is open and minimal closed. Then $P$ is a clopen submset and $P^c$ is also clopen. Hence $P^c$ is closed and $P$ is minimal closed $\Rightarrow$ either $P \subseteq P^c$ or $P \cap P^c = \emptyset$.

If $P \subseteq P^c$, then $C_P(y) \leq \frac{C_N(y)}{2}, \forall y \in P$. when $P \cap P^c = \emptyset$, $P$ is a whole submset of $N$.

The proof for the other case proceeds in a similar fashion.

**Theorem 9:** *Let $N$ be an mset with count of every element is even and $\tau$ be an M-topology on $N$. If a submset $P$ of $N$ is both maximal open and minimal closed, then either of the following is true:*

*(i) $C_P(y) = \frac{C_N(y)}{2}, \forall y \in N$, in this situation if $P$ is denoted by $\frac{N}{2}$, every open submset $H$ of $N$ is contained in $\frac{N}{2}$ and every closed set $K$ contains $\frac{N}{2}$, i.e., $H \subseteq \frac{N}{2} \subseteq K$, and $\frac{N}{2}$ is the only proper nontrivial clopen submset.*

*(ii) $P$ is a proper whole submset of $N$ and the M-topology reduces to $\tau = \{\emptyset, P, P^c, N\}$. Consequently $P$ and $P^c$ are the only proper nonempty open submsets and they are closed also.*

If $N$ has an element with odd count, then (ii) is the only valid case.

**Proof:** Suppose $P$ is both maximal open and minimal closed. Then $P$ is a clopen submset and consequently $P^c$ is clopen. If $P^c$ as open and $P$ as maximal open, then $P \subseteq P \cup P^c \subseteq N$. It follows that, either $P = P \cup P^c$ or $P \cup P^c = N$. Consequently either (a) $P^c \subseteq P$ or (b) $P \cup P^c = N$.

On the other way, if $P^c$ is closed and $P$ is minimal closed, then $\emptyset \subseteq P \cap P^c \subseteq P$ and hence either $P = P \cap P^c$ or $P \cap P^c = \emptyset$. Consequently either (c) $P \subseteq P^c$ or (d) $P \cap P^c = \emptyset$.

Now, (a) and (c) $\Rightarrow P = P^c$. Hence $C_P(y) = C_N(y) - C_P(y) \Rightarrow 2C_P(y) = C_N(y) \Rightarrow C_P(y) = \frac{C_N(y)}{2}, \forall y \in N$.

Also,

(a) and (d) $\Rightarrow P^c = \emptyset$ and hence $P = N$, which is not possible.

(b) and (c) $\Rightarrow P^c = N$ and hence $P = \emptyset$, which is not possible.

(b) and (d) $\Rightarrow P$ is a whole clopen submset of $N$.

Suppose $P$ is whole submset and $L$ is an open submset of $N$. Then $P$ is maximal open $\Rightarrow$ either (e) $L \subseteq P$ or (f) $L \cup P = N$. Since $P$ is minimal closed also, either (g) $P \subseteq L^c$ or (h) $P \cap L^c = \emptyset$.

Now, (e) and (g) $\Rightarrow L \subseteq P$ and $P \subseteq L^c \Rightarrow L \subseteq P$ and $L \subseteq P^c \Rightarrow L \subseteq P \cap P^c = \emptyset$. Hence $L = \emptyset$.

The conditions (e) and (h) $\Rightarrow L \subseteq P$ and $P \cap L^c = \emptyset \Rightarrow L \subseteq P$ and $P^c \cup L = N \Rightarrow L \subseteq P$ and $P \subseteq L$. Therefore $L = P$.

By (f) and (g) $\Rightarrow L \cup P = N$ and $P \subseteq L^c \Rightarrow P^c \subseteq L$ and $P \subseteq L^c \Rightarrow L^c \subseteq P$ and $P \subseteq L^c \Rightarrow L = P^c$

Now, by (f) and (h) $\Rightarrow L \cup P = N$ and $P \cap L^c = \emptyset \Rightarrow P^c \subseteq L$ and $P \cap L^c = \emptyset \Rightarrow L^c \subseteq P$ and $P \cap L^c = \emptyset$. If $L^c \subseteq P$ then $P \cap L^c = L^c = \emptyset$. Hence $L = N$. If $L$ is an open submset of $N$, then the different possibilities for $L$ are $\emptyset, P, P^c$ or $N$. So $\tau = \{\emptyset, P, P^c, N\}$.

**Corollary 3:** *If $P$ is both minimal closed and maximal open submset of an M-topological space $N$, then $P$ and $N \ominus P$ are the only proper nonempty clopen submsets in the space.*

**Remark 1:** If $N$ has an element of odd count and there is a submset $P$ which is both maximal open and minimal closed, then the M-topology on $N$ is fixed and is given by $\tau = \{\emptyset, P, P^c, N\}$. But when count of every element of $N$ is even and $P = \frac{N}{2}$, then there may be more than one M-topologies for which $P$ is both maximal open and minimal closed.

**Example 3:** *Suppose $N = \{8/c, 8/d\}$ and $P = \{4/c, 4/d\}$. Then, $P$ is both maximal open and minimal closed in the topologies $\tau_1 = \{N, \emptyset, P\}$ and $\tau_2 = \{N, \emptyset, P, \{2/c, 3/d\}\}$.*

In general, if $\tau_{\frac{N}{2}}$ is any M-topology on $\frac{N}{2}$, then $\tau = \tau_{\frac{N}{2}} \cup \{N\}$ is an M-topology on $N$ and in this topology, $\frac{N}{2}$ is both maximal open and minimal closed.

**Corollary 4:** *If $P$ is both maximal open and minimal closed in an M-topology $\tau$ on $N$, then either of the following is true:*

*(i) $P$ is a whole submset and $\tau = \{\emptyset, P, P^c, N\}$.*

*(ii) $P = \frac{N}{2}$ and $\tau = \tau_{\frac{N}{2}} \cup \{N\}$ for any M-topology $\tau_{\frac{N}{2}}$ on $\frac{N}{2}$.*

**Theorem 10:** *Let $N$ be an mset with count of every element is even and $\tau$ be an M-topology on $N$. If a submset $P$ of $N$ is both maximal closed and minimal open, then either of the following is true:*

*(i) $C_P(y) = \frac{C_N(y)}{2}, \forall y \in N$, in this situation if $P$ is denoted by $\frac{N}{2}$, every closed submset $K$ of $N$ is contained in $\frac{N}{2}$ and every open set $H$ contains $\frac{N}{2}$, i.e., $K \subseteq \frac{N}{2} \subseteq H$, and $\frac{N}{2}$ is the only proper nontrivial clopen submset.*

*(ii) $P$ is a proper whole submset of $N$ and the M-topology reduces to $\tau = \{\emptyset, P, P^c, N\}$. Consequently $P$ and $P^c$ are the only proper nontrivial open submsets and they are closed also.*

If $N$ has an element with odd count, then (ii) is the only valid case.

**Corollary 5:** *If $P$ is a maximal closed submset of an M-topological space $N$ which is also minimal open, then $P$ and $N \ominus P$ are the only proper nonempty clopen submsets in the M-topological space.*

**Corollary 6:** *If $P$ is both maximal closed and minimal open in an M-topology $\tau$ on $N$, then either of the following is true.*

*(i) $P$ is a whole submset and $\tau = \{\emptyset, P, P^c, N\}$.*
*(ii) $P = \frac{N}{2}$ and $\tau = \{U \oplus \frac{N}{2} : U \in \tau_{\frac{N}{2}}\} \cup \{\emptyset\}$ for any M-topology $\tau_{\frac{N}{2}}$ on $\frac{N}{2}$.*

**Theorem 11:** *If a submset $P$ of an M-topological space $N$ is both minimal and maximal closed, then one among the following is true:*

*(a) $P$ is the one and only one proper closed submset of the space.*
*(b) If $Q$ is a closed submset of $N$, then $Q$ and $P$ are whole submsets and complements to each other (i.e., $Q \cup P = N$ and $Q \cap P = \emptyset$).*

**Proof:** Let $Q$ be a closed submset of $N$. Since $P$ is maximal closed, either $Q \subseteq P$ or $P \cup Q = N$. Also, since $P$ is minimal closed, either $P \subseteq Q$ or $P \cap Q = \emptyset$. Considering all these combinations, it follows that $Q \subseteq P$ and $P \subseteq Q \Rightarrow Q = P$.

Now, $Q \subseteq P$ and $P \cap Q = \emptyset \Rightarrow Q = \emptyset$.

Also, $P \cup Q = N$ and $P \subseteq Q \Rightarrow Q = N$ and $P \cup Q = N$ and $P \cap Q = \emptyset \Rightarrow P$ and $Q$ are whole and complement to each other.

Therefore, for every nonempty proper closed set $Q$ of $N$, either $Q = P$ or $P$ and $Q$ are whole and complement to each other.

**Corollary 7:** *If a nonempty submset $P$ of an M-topological space $N$ is both maximal and minimal closed and $H$ is an open submset of $N$, then one among the following is true:*

*(i) $P = N \ominus H$*
*(ii) $P$ is a clopen and a whole submset of $N$ with $P = H$ and the space $N$ is disconnected.*

**Theorem 12:** *If $P$ is a minimal closed submset of $N$, then either of the following is true:*

*(i) $C_{\text{int }(P)}(y) \leq \frac{C_N(y)}{2}, \forall y \in N$.*
*(ii) $P$ is clopen.*

**Proof:** Consider $Q = (\text{int}(P))^c$. Since $Q$ is a closed submset and $P$ is minimal closed, either $P \subseteq Q$ or $P \cap Q = \emptyset$. Now, $P \subseteq Q \Rightarrow P \subseteq (\text{int}(P))^c$ and by taking the complement on both sides, we get $int(P) \subseteq P^c$. Hence $int(P) \subseteq P \cap P^c$, since $int(P) \subseteq P$. Then, by Note 1(ii), $C_{\text{int }(P)}(y) \leq \frac{C_N(y)}{2}, \forall y \in N$.

If $P \cap Q = \emptyset$, then $P \cap (int(P))^c = \emptyset \Rightarrow (int(P))^c \subseteq P^c \Rightarrow P \subseteq int(P)$. As $\text{int}(P) \subseteq P$, it follows that $P = int(P)$ and $P$ is open. Hence $P$ is clopen submset of $N$.

**Theorem 13:** *If $K$ is a submset of an M-topological space $N$ and there is a minimal closed submset $P$ of $N$ such that $K \subset P$, then*

*(i) The closed subspace M-topology on $K$ is indiscrete and it has no maximal open submsets.*
*(ii) Every proper open submset in open subspace M-topology of $K$ contains part elements of $K$ with count less than or equal to half of the full multiplicity in $N$.*

**Proof:** By assuming the condition that there is a minimal closed submset $P$ such that $K \subseteq P$, we get $K \ominus (K \cap P) = \emptyset$. Let $H$ be an open submset of $N$. Now, since $P$ is minimal closed $\Rightarrow$ either $P \subseteq H^c$ or $P \cap H^c = \emptyset$. If $P \subseteq H^c$, then $K \subseteq P \subseteq H^c \Rightarrow K \subseteq K \cap P \subseteq K \cap H^c \Rightarrow (K \ominus (K \cap H^c)) \subseteq (K \ominus (K \cap P)) = \emptyset \Rightarrow K \ominus (K \cap H^c) = \emptyset$. Hence the open submset corresponding to $H$ in the closed subspace M-topology on $K$ is empty.

If $P \cap H^c = \emptyset$, then $K \subseteq P$ and $P \cap H^c = \emptyset \Rightarrow K \cap H^c = \emptyset \Rightarrow K \ominus (K \cap H^c) = K$. Hence the open submset corresponding to $H$ in closed subspace M-topology on $K$ is $K$. That is, the open submsets corresponding to any open submset $H$ of $N$ in closed subspace M-topology on $K$ is either $\emptyset$ or $K$ and any closed subspace M-topology on $K$ is indiscrete. Hence it has no nonempty proper open submsets and no maximal open submsets.

Now consider any open subspace M-topology on $K$. If $H$ is an open submset of $N, K \cap H$ is the corresponding open submset in $K$. Since $K$ is a submset of the minimal closed submset $P$, either $H^c \cap P = \emptyset$ or $H^c \cap P = P$.

Suppose $H^c \cap P = \emptyset$. Then $H^c \cap K = \emptyset$ and hence $H \cup K^c = N$. It follows that $H^w \subseteq K^c \Rightarrow K \subseteq \tilde{H} \subseteq H$. Hence $K \cap H = K$, not a proper submset.

If $H^c \cap P = P$, then $P \subseteq H^c$ and hence $K \subseteq H^c$. So $H \subseteq K^c$ and consequently $K \cap H \subseteq K \cap K^c$. Since $K \cap K^c$ contains part elements $y$ of $K$ with count $\leq \frac{\bar{C}_N(y)}{2}$, it follows that $K \cap H$ contains part elements of $K$ with count less than or equal to half of the full multiplicity in $N$.

## Conclusion

In this paper, we introduced the innovative concepts of minimal closed subsets and maximal closed subsets within the framework of $M$-topology. Our analysis delved into their intricate relationships with key $M$-topological concepts such as the whole core, whole complement, closure, interior, and some conditions for disconnectedness in $M$-topological spaces. This paper also explored their behavior in both subspace $M$-topologies of a given subset, discussing various properties arising from different combinations of the introduced designations. The exploration suggests the need for further investigation into these concepts in conjunction with other $M$-topological properties. Researchers are encouraged to seize this opportunity for a more in-depth analysis and exploration of the implications and applications of these findings.

## Authors' declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Manipal Academy of Higher Education, Manipal, India.

## Authors' contribution statement

R. K. studied and wrote the manuscript and S. J. J., B. T. revised the manuscript and approved the final version.

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# تحليل الحد الأدنى والحد الأقصى للمجموعات المغلقة في طوبولوجيا M

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## المستخلص

في التحليل الرياضي، اكتسب مفهوم المجموعة المتعددة (باختصار mset)، الذي يسمح بإدراج عناصر متكررة داخل مجموعة، اهتمامًا كبيرًا. وهذا مهم بشكل خاص في السيناريوهات الواقعية حيث تكون العناصر المكررة ذات أهمية. وقد أدى ظهور طوبولوجيا mset، وهو فرع متخصص من الطوبولوجيا مصمم لاستيعاب الخصائص الفريدة لـ msets، إلى توفير إطار قيم لفهم الخصائص الطوبولوجية لهذه المجموعات المتنوعة. يتعمق هذا البحث في الاستكشاف الدقيق لطوبولوجيا mset، مع التركيز بشكل خاص على الخصائص المرتبطة بالمجموعات الفرعية المغلقة الدنيا والمجموعات الفرعية المغلقة القصوى. ثم يتم فحص هذه المجموعات الفرعية من حيث داخلها وإغلاقها وعددها، مما يوفر فهمًا شاملاً لتعقيداتها البنيوية في سياق طوبولوجيا mset. وبتوسيع نطاق التحليل، يدرس هذا البحث أيضًا مجموعات فرعية ذات مجموعات متنوعة من التسميات، بما في ذلك المجموعات المفتوحة الدنيا، والمجموعات المفتوحة القصوى، والمجموعات المغلقة الدنيا، والمجموعات المغلقة القصوى. تساهم هذه الدراسة في إنشاء تصنيف مفصل للمجموعات الفرعية ضمن إطار طوبولوجيا المجموعات الفرعية، وتوضيح التفاعل بين الانفتاح والانغلاق في سياقات مختلفة. ومن خلال الكشف عن خصائص المجموعات الفرعية المغلقة الدنيا والقصوى وتفسيرها، فضلاً عن مجموعاتها المتنوعة من التسميات، تقدم هذه الورقة مساهمة كبيرة في الخطاب الرياضي الأوسع حول المجموعات الفرعية وخصائصها الطوبولوجية المعقدة.

**الكلمات المفتاحية:** مجموعة فرعية مغلقة، مجموعات مغلقة قصوى، مجموعات مغلقة دنيا، مجموعة متعددة، مجموعة فرعية، طوبولوجيا M، مجموعة فرعية.