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RESEARCH ARTICLE

The Influence of Hunting Cooperation and Fear on the Dynamics of the Eco-Epidemiological Model with Disease in Both Populations

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ABSTRACT

Predicting and controlling environmental dynamics requires an understanding of the intricate interaction between hunting cooperation, fear, and other biological elements in eco-epidemiological models. Studying these models is crucial for environmental sustainability and conservation, as seen by the rise in infectious diseases caused by population growth and interactions among organisms. The goal of this study is to create a novel mathematical model that takes into account contagious diseases that impact both predators and prey, as well as how cooperative hunting behavior on the part of predators causes anxiety in the prey population. Important features of the model are examined, including the existence, boundedness, uniqueness, and positivity of solutions, as well as the determination of equilibrium locations and the local stability criteria that support them. Around the equilibrium points, bifurcation analyses are performed, exposing a variety of dynamic behaviors, including multi-stability events. The theoretical conclusions are confirmed and control settings are determined by numerical simulations using MATLAB R2021a's 4th-order Runge-Kutta method.

Keywords: Eco-epidemiological model, Fear, Hunting cooperation, Stability, Bifurcation

Introduction

The predation process is essential in advancing life evolution and maintaining ecological balance and biodiversity. Moreover, cooperation between the species' individuals is a fundamental feature of animal social life and is important in biological systems. Group hunting has many advantages, such as the rate of hunting success increasing with the number of adults, chasing distance decreasing and the probability of capturing large prey increasing.[1] Population dynamics are regulated by several factors: availability of resources, predation, diseases, etc., see for instance articles[2–4] which describe the role of additional resources and disease, while articles[5,6] which study the role of fear and reaction-diffusion. Among these factors, the interaction between prey and predators is probably the most studied in ecology due to its importance, dating back to the works of Lotka and Volterra in the early 20th century. Since then, several prey-predator models have been proposed and studied, see the excellent review.[7] Thus, the study of prey-predator models plays a crucial role in understanding the predation relationships between species in ecosystems.[8–10] Predators feed on prey to ensure their survival, so they typically attempt to enhance their ability to capture and kill prey, which is more conducive to their long-term survival. To enhance their ability to capture and kill prey, some animals commonly employ the

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strategy of hunting cooperation.[11] In a mathematical modeling approach, many authors[12,13] investigated the impacts of hunting cooperation and the fear effect, which is a natural behavior that results in feedback of the intensity of predation, in prey-predator systems, for example.[14–16]

Epidemiology on the other hand, is one of the hot topics in the study of mathematical biology. The literature on epidemiology models is rich; see for example[17–19] and the references therein. The study of the eco-epidemiological model, which connects ecology with epidemiology, with the addition of the fear effect is one of the most fascinating new developments in mathematical modeling research. A Lotka-Volterra eco-epidemiological model that included fear variables and disease in the prey population was studied by Sha et al.[20] Later on, many researchers developed eco-epidemiological systems including different biological factors, for example the articles,[21–23] while articles[24,25] are proposed and studied eco-epidemiological systems with the impact of fear, refuge, and harvesting on the system dynamics.

It's also critical to remember that predators find hunting more difficult if their prey is afraid of them. Moreover, overcrowding makes ecological animals more susceptible to infectious diseases, which has an impact on the evolution of particular species, such as those engaged in prey-predator relationships. This disease may make some predators less strong and efficient hunters, which raises the possibility of their extinction. In the context of infectious diseases, many studies have looked at the prey-predator relationship, including hunting cooperation or fear.[26–28] In actuality, infectious diseases arise when tainted foreign objects enter the body. Numerous physical symptoms, including discomfort and raised body temperature, can accompany these infections in addition to other symptoms that differ based on the type, location, and intensity of the infection. It is possible to have an illness that shows little symptoms; in this scenario, medical intervention is not necessary. But there are also serious situations that require medical attention since they may be lethal.

In contrast to previous studies, the goal of the current research is to create a prey-predator model that takes hunting cooperation and the fear that predators impose on their prey into account when there are infectious diseases in both populations. Additionally, to investigate the combined effects that hunting cooperation and fear can have on population dynamics in an eco-epidemiological system, we develop and analyze an eco-epidemiological model in this work that incorporates both of these phenomena.

Materials and methods

In the following, the adopted assumptions to build the mathematical model that describes the eco-epidemiological system are stated.

1. The prey biomass density at time T is denoted by $N(T) = X(T) + Y(T)$, where $X(T)$ is the susceptible part while $Y(T)$ is the infected part of the prey population. The predator biomass density at time T is denoted by $P(T) = Z(T) + W(T)$, where $Z(T)$ is the susceptible part and $W(T)$, is the infected part of the predator population.
2. From a logistical standpoint, prey populations flourish in the absence of predators. Conversely, when food sources dwindle, predators experience rapid decline.
3. Assume that the illness in the prey community is limited to the prey population and that it is not genetically inherited. This means that only the vulnerable prey can procreate, with the sick prey fighting for the resource alone. However, the sickness that affects predators is believed to be of the SIS variety, and rather than being genetically transmitted, it can only spread between individual predators through contact between an infected and a healthy predator. The illness is also cured by the medication given to the diseased predator.
4. Based on the Lotka-Volterra functional response, suppose that the predator works together to hunt the prey and consumes both populations of the prey.
5. A population of prey experiences fear of predators because of the predator's cooperative hunting behavior, which causes it to attack its prey in groups.
6. Many animals have an innate fear of becoming prey because it helps them survive and stay out of harm's way. Animals that are vigilant and fearful of predators in the wild have a higher chance of avoiding capture and death. They take precautions because of this concern, like decreasing playing in open spaces, staying in smaller groups, finding cover, or decreasing mating there. It is an evolutionary survival strategy meant to secure the species' survival. As a result, predation anxiety thereby alters the prey population's

hunting habits, lowering the chance of disease transmission among individuals of the same species due to the reduction of crowding.

7. The capacity of hunting collaboration results in an increase in the attack rate of the predator population, say $\alpha_1 > 0$, by the cooperation term to become $(\alpha_1 + \alpha_2 Y)$, where $\alpha_2 \geq 0$ denotes the level of predator cooperation during hunting.[27]

Accordingly, the stated eco-epidemiological system's dynamic can be represented by the following set of nonlinear first-order differential equations.

$$\begin{aligned}
\frac{dX}{dT} &= \frac{r}{1+\gamma_1(Z+W)}X\left[1-\frac{X+Y}{k}\right] - \frac{\beta_1 XY}{1+\gamma_2(Z+W)} - [\alpha_1+\alpha_2(Z+W)]X(Z+W)\\
\frac{dY}{dT} &= \frac{\beta_1 XY}{1+\gamma_2(Z+W)} - [\alpha_1+\alpha_2(Z+W)]Y(Z+W) - d_1 Y\\
\frac{dZ}{dT} &= [\alpha_1+\alpha_2(Z+W)](Z+W)(c_1X+c_2Y) - \beta_2 ZW + \frac{\mu W}{\sigma+W} - d_2 Z\\
\frac{dW}{dT} &= \beta_2 ZW - \frac{\mu W}{\sigma+W} - d_3 W.
\end{aligned} \tag{1}$$

To non-dimensionalize the system 1, the following transformation is used.

$$rT = t,\ \frac{X}{k} = x_1,\ \frac{Y}{k} = x_2,\ \frac{\alpha_2}{\alpha_1}Z = x_3,\ \frac{\alpha_2}{\alpha_1}W = x_4.$$

Then, system 1 reduces to the following form

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{x_1(1-x_1-x_2)}{1+w_1(x_3+x_4)} - \frac{w_2x_1x_2}{1+w_3(x_3+x_4)} - (1+x_3+x_4)w_4x_1(x_3+x_4)\\
\frac{dx_2}{dt} &= \frac{w_2x_1x_2}{1+w_3(x_3+x_4)} - (1+x_3+x_4)w_4x_2(x_3+x_4) - w_5x_2\\
\frac{dx_3}{dt} &= w_6(1+x_3+x_4)(x_3+x_4)(c_1x_1+c_2x_2) - w_7x_3x_4 + \frac{w_8x_4}{w_9+x_4} - w_{10}x_3\\
\frac{dx_4}{dt} &= w_7x_3x_4 - \frac{w_8x_4}{w_9+x_4} - w_{11}x_4,
\end{aligned} \tag{2}$$

where:

$$w_1 = \gamma_1\frac{\alpha_1}{\alpha_2},\ w_2 = \frac{\beta_1 k}{r},\ w_3 = \gamma_2\frac{\alpha_1}{\alpha_2},\ w_4 = \frac{\alpha_1^2}{r\alpha_2},\ w_5 = \frac{d_1}{r},$$
$$w_6 = \frac{\alpha_1 k}{r},\ w_7 = \frac{\beta_2\alpha_1}{r\alpha_2},\ w_8 = \frac{\mu\alpha_2}{r\alpha_1},\ w_9 = \frac{\alpha_2\sigma}{\alpha_1},\ w_{10} = \frac{d_2}{r},\ w_{11} = \frac{d_3}{r}.$$

It is clear from system 2 that, the interaction functions $x_i f_i(x_1, x_2, x_3, x_4); i = 1, 2, 3, 4$ in the right-hand side of the system 2, are continuous and have continuous partial derivatives on the domain $\mathbb{R}^4_+ = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0, x_4 \geq 0\}$. Hence, they are locally Lipschitz functions in $\mathbb{R}^4_+$. Consequently, due to the fundamental existence and uniqueness theorem, it is obtained that system 2 with any non-negative initial condition $x_1(0) \geq 0$, $x_2(0) \geq 0$, $x_3(0) \geq 0$, and $x_4(0) \geq 0$ there exists $T > 0$ so that system 2 has a unique solution defined in $\mathbb{R}^4_+$.

Properties of the solution

This section shows properties of the solution of system 1, such as positivity and bounded as presented in the next theorems.

Theorem 1: *All system 2's solutions with the initial conditions belong to int.$\mathbb{R}^4_+$ are positively invariant.*

Proof: From the first equation of system 2, it is obtained:

$$\frac{dx_1}{x_1} = \left[\frac{(1 - x_1 - x_2)}{1 + w_1 (x_3 + x_4)} - \frac{w_2 x_2}{1 + w_3 (x_3 + x_4)} - (1 + x_3 + x_4) w_4 (x_3 + x_4)\right] dt = f_1 (x_1, x_2, x_3, x_4) dt.$$

Then integrating the above equation within the limit [0, t], gives that:

$$x_1 (t) = x_1 (0) e^{\int_0^t f_1(x_1(s), x_2(s), x_3(s), x_4(s)) ds} > 0;\ \forall t > 0.$$

Similarly, the other equations, gives

$$x_2 (t) = x_2 (0) e^{\int_0^t f_2(x_1(s), x_2(s), x_3(s), x_4(s)) ds} > 0;\ \forall t > 0,$$

$$x_3 (t) = x_3 (0) e^{\int_0^t f_3(x_1(s), x_2(s), x_3(s), x_4(s)) ds} > 0;\ \forall t > 0,$$

$$x_4 (t) = x_4 (0) e^{\int_0^t f_4(x_1(s), x_2(s), x_3(s), x_4(s)) ds} > 0;\ \forall t > 0,$$

where

$$f_2 (x_1,\ x_2,\ x_3,\ x_4) = \frac{w_2 x_1}{1 + w_3 (x_3 + x_4)} - (1 + x_3 + x_4) w_4 (x_3 + x_4) - w_5.$$

$$f_3 (x_1,\ x_2,\ x_3,\ x_4) = \frac{w_6 (1 + x_3 + x_4)(x_3 + x_4)(c_1 x_1 + c_2 x_2)}{x_3} - w_7 x_4 + \frac{w_8 x_4}{x_3 (w_9 + x_4)} - w_{10}.$$

$$f_4 (x_1,\ x_2,\ x_3,\ x_4) = w_7 x_3 - \frac{w_8}{w_9 + x_4} - w_{11}.$$

This completes the proof.

Theorem 2: *All system 2's solutions with initial conditions belonging to $\mathbb{R}_+^4$ are uniformly bounded*

Proof: From system 2, it is easy to verify that

$$\frac{dx_1}{dt} \leq x_1 (1 - x_1)$$

Then according to the lemma 2.2,[29] it is obtained that

$$x_1 (t) \leq \left[1 + \left(\frac{1}{x_1 (0)} - 1\right) e^{-t}\right]^{-1}$$

Hence for $t \to \infty$, it is obtained that $x_1(t) \leq 1$.
Let $= x_1 + x_2 + x_3 + x_4$, then using the fact that $c_i \in (0, 1]; i = 1, 2$ system 2 gives that:

$$\frac{dQ}{dt} \leq 2x_1 - MQ,$$

where $M = min\{1, w_5, w_{10}, w_{11}\}$. Hence, simple manipulation yields

$$\frac{dQ}{dt} + MQ \leq 2.$$

Then according to the lemma 2.1,[29] it is obtained that

$$Q (t) \leq \frac{2}{M} \left[1 + (Q (0) - 1) e^{-Mt}\right]$$

Therefore, for $t \to \infty$, it is obtained that:

$$Q(t) \le \frac{2}{M}.$$

That completes the proof.

According to the above theorems, system 2 is a well-post biological system as the population in the environment is always nonnegative and bounded by the habitat carrying capacity.

Equilibria and stability analysis

This section determines the stability analysis of each probable equilibrium point. The following equilibrium points (EPs) exist in System 2:

1) The vanishing equilibrium point (VEP), $E_0 = (0, 0, 0, 0)$ always exists.
2) The axial equilibrium point (AEP), $E_1 = (1, 0, 0, 0)$ always exists.
3) The predator-free equilibrium point (PFEP), $E_2 = (\frac{w_5}{w_2}, \frac{w_2 - w_5}{w_2(1+w_2)}, 0, 0)$ exists provided the following condition holds

$$w_5 < w_2. \tag{3}$$

4) The disease-free equilibrium point (DFEP), $E_3 = (\hat{x}_1, 0, \hat{x}_3, 0) = (\frac{w_{10}}{c_1 w_6 (1+\hat{x}_3)}, 0, \hat{x}_3, 0)$, where $\hat{x}_3$ is a positive root for

$$\begin{aligned} c_1 w_1 w_4 w_6 x_3^4 + (c_1 w_4 w_6 + 2c_1 w_1 w_4 w_6) x_3^3 + (2c_1 w_4 w_6 + c_1 w_1 w_4 w_6) x_3^2 \\ + (-c_1 w_6 + c_1 w_4 w_6) x_3 - c_1 w_6 + w_{10} = 0. \end{aligned} \tag{4}$$

Note that, Eq. (4) has a unique positive root $\hat{x}_3$ if the following condition is met:

$$w_{10} < c_1 w_6. \tag{5}$$

5) The predator-disease-free equilibrium point (PDFEP), $E_4 = (\check{x}_1, \check{x}_2, \check{x}_3, 0)$, is determined as:

$$\check{x}_1 = \frac{(1 + w_3 \check{x}_3)\left(w_5 + w_4 \check{x}_3 + w_4 \check{x}_3^2\right)}{w_2}, \quad \check{x}_2 = \frac{w_2 w_{10} - c_1 w_6 (1 + \check{x}_3)(1 + w_3 \check{x}_3)\left(w_5 + w_4 \check{x}_3 + w_4 \check{x}_3^2\right)}{c_2 w_2 w_6 (1 + \check{x}_3)},$$

which are positive if the following condition holds:

$$c_1 w_6 (1 + \check{x}_3)(1 + w_3 \check{x}_3)\left(w_5 + w_4 \check{x}_3 + w_4 \check{x}_3^2\right) < w_2 w_{10}. \tag{6}$$

While $\check{x}_3$ is a positive root of the following equation:

$$\rho_1 x_3^5 + \rho_2 x_3^4 + \rho_3 x_3^3 + \rho_4 x_3^2 + \rho_5 x_3 + \rho_6 = 0, \tag{7}$$

where

$$\rho_1 = (c_2 - c_1) w_3 w_4 w_6 (w_1 w_2 + w_3),$$

$$\rho_2 = (c_2 - c_1) w_4 w_6 \left[w_1 w_2 + 2w_3 + w_2 w_3 + 2w_1 w_2 w_3 + 2w_3^2\right],$$

$$\begin{aligned} \rho_3 = {} & (c_2 - c_1) w_4 w_6 \lceil 1 + w_2 + 2w_1 w_2 + 4w_3 + 2w_2 w_3 + w_1 w_2 w_3 + w_3^2 \rceil \\ & - c_1 w_1 w_2 w_3 w_5 w_6 + (c_2 - c_1) w_3^2 w_5 w_6, \end{aligned}$$

$$\begin{aligned} \rho_4 = {} & -c_2 w_2 w_3 w_6 + (c_2 - c_1) w_4 w_6 [2 + 2w_2 + w_1 w_2 + 2w_3 + w_2 w_3] \\ & - c_1 w_1 w_2 w_5 w_6 + (c_2 - c_1) w_3 w_5 w_6 [2 + w_3] - c_1 (1 + w_1) w_2 w_3 w_5 w_6, \end{aligned}$$

$$\rho_5 = -c_2 w_2 w_6 (1 + w_3) + (c_2 - c_1) w_6 [w_4 (1 + w_2) + w_5 (1 + 2w_3)] \\ - c_1 (1 + w_1) w_2 w_5 w_6 - (2 + w_2) w_3 w_5 w_6 + (w_1 w_2 + w_3) w_2 w_{10},$$

$$\rho_6 = -(c_2 w_2 + c_1 w_5) w_6 + (c_2 - c_1 w_2) w_5 w_6 + (w_2 + 1) w_2 w_{10}.$$

Following the Descartes rule of sign, Eq. (6) may have at least one positive root for different cases as displayed in Table 1.

Table 1. Number of positive roots of Eq. (6) regarding sign changes.

Cases	ρ^1	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	No. of sign changes	No. of Positive roots
1	+	+	+	+	+	–	1	1
2	+	+	+	+	–	–	1	1
3	+	+	+	–	+	–	3	1, 3
4	+	+	+	–	–	–	1	1
5	+	+	–	+	+	–	2	0, 2
6	+	+	–	+	–	–	3	1, 3
7	+	+	–	–	+	–	2	0, 2
8	+	+	–	–	–	–	1	1
9	–	–	+	+	+	+	1	1
10	–	–	+	+	–	+	3	1, 3
11	–	–	+	–	+	+	3	1, 3
12	–	–	+	–	–	+	3	1, 3
13	–	–	–	+	+	+	1	1
14	–	–	–	+	–	+	3	1, 3
15	–	–	–	–	+	+	1	1
16	–	–	–	–	–	+	1	1

Clearly, according to the cases 1, 2, 4, 8, 9, 13, 15, and 16 given in Table 1 with condition 6, the PDFEP can exist uniquely.

6) The healthy prey equilibrium point (HPEP), $E_5 = (\tilde{x}_1, 0, \tilde{x}_3, \tilde{x}_4)$ is computed by:

$$\left.\begin{aligned} \tilde{x}_3 &= \frac{w_8 + w_9 w_{11} + w_{11}\tilde{x}_4}{w_7 (w_9 + \tilde{x}_4)} \\ \tilde{x}_1 &= \frac{w_7 (w_9 + \tilde{x}_4)[w_{10}(w_8 + w_9 w_{11} + w_{11}\tilde{x}_4) + w_7 \tilde{x}_4 (w_9 w_{11} + w_{11}\tilde{x}_4)]}{c_1 w_6 [\tilde{x}_4 w_7 (w_9 + \tilde{x}_4) + w_8 + w_9 w_{11} + w_{11}\tilde{x}_4][w_7 (1 + x_4)(w_9 + \tilde{x}_4) + w_8 + w_9 w_{11} + w_{11}\tilde{x}_4]} \end{aligned}\right\}. \tag{8}$$

While $\tilde{x}_4$ is a positive root of the following equation.

$$\sigma_1 x_4^{10} + \sigma_2 x_4^9 + \sigma_3 x_4^8 + \sigma_4 x_4^7 + \sigma_5 x_4^6 + \sigma_6 x_4^5 + \sigma_7 x_4^4 + \sigma_8 x_4^3 + \sigma_9 x_4^2 + \sigma_{10} x_4 + \sigma_{11} = 0. \tag{9}$$

Where the coefficients $\sigma_i; i = 1, 2, \ldots, 11$ are computed using the Mathematica. However, their large and intricate forms are not included here. It is well known that, the existence and uniqueness of the HPEP depend on the number of positive roots of Eq. (9), which may have at least one positive root when the coefficients $\sigma_1 = c_1 w_1 w_4 w_6 w_7^5 > 0$ and σ_{11} of opposite signs.

7) Calculating the coexistence equilibrium point (CEP), $E_6 = (x_1^*, x_2^*, x_3^*, x_4^*)$, is done by

$$\left.\begin{aligned} x_3^* &= \frac{w_8 + w_9 w_{11} + w_{11} x_4^*}{w_7 (w_9 + x_4^*)} = \frac{\mathrm{M}}{\mathrm{N}} \\ x_1^* &= \frac{[\mathrm{N} + w_3 (x_4^* \mathrm{N} + \mathrm{M})][w_5 \mathrm{N}^2 + w_4 (x_4 \mathrm{N} + \mathrm{M})((1 + x_4^*)\mathrm{N} + \mathrm{M})]}{w_2 \mathrm{N}^3} \\ x_2^* &= \frac{(w_7 x_4^* + w_{10}) x_3^* (w_9 + x_4^*) - w_8 x_4^* - c_1 w_6 (1 + x_3^* + x_4^*)(x_3^* + x_4^*) x_1^* (w_9 + x_4^*)}{c_2 w_6 (1 + x_3^* + x_4^*)(x_3^* + x_4^*)(w_9 + x_4^*)} \end{aligned}\right\}. \tag{10}$$

While x_4^* is a positive root of the higher-order equation obtained from the first equation of system 2 after substituting the values of x_1^*, x_2^*, and x_3^*, which are given in Eq. (10). Therefore, the number of CEP depends on the number of positive roots x_4^* and the following condition.

$$w_8 x_4^* + c_1 w_6 (1 + x_3^* + x_4^*)(x_3^* + x_4^*) x_1^* (w_9 + x_4^*) < (w_7 x_4^* + w_{10}) x_3^* (w_9 + x_4^*). \tag{11}$$

The local stability analysis of the previously described EPs can be studied using the computed Jacobian matrix (JM) that follows.

$$J = [s_{ij}]_{4\times 4}, \tag{12}$$

Where

$$s_{11} = -w_4(x_3+x_4)(1+x_3+x_4) - \frac{-1+2x_1+x_2}{1+w_1(x_3+x_4)} - \frac{w_2x_2}{1+w_3(x_3+x_4)}.$$

$$s_{12} = -x_1\left(\frac{1}{1+w_1(x_3+x_4)} + \frac{w_2}{1+w_3(x_3+x_4)}\right),$$

$$s_{13} = x_1\left(-w_4(1+2x_3+2x_4) + \frac{w_1(-1+x_1+x_2)}{(1+w_1(x_3+x_4))^2} + \frac{w_2w_3x_2}{(1+w_3(x_3+x_4))^2}\right),$$

$$s_{14} = x_1\left(-w_4(1+2x_3+2x_4) + \frac{w_1(-1+x_1+x_2)}{(1+w_1(x_3+x_4))^2} + \frac{w_2w_3x_2}{(1+w_3(x_3+x_4))^2}\right),$$

$$s_{21} = \frac{w_2x_2}{1+w_3(x_3+x_4)},$$

$$s_{22} = -w_5 - w_4(x_3+x_4)(1+x_3+x_4) + \frac{w_2x_1}{1+w_3(x_3+x_4)},$$

$$s_{23} = x_2\left(-w_4(1+2x_3+2x_4) - \frac{w_2w_3x_1}{(1+w_3(x_3+x_4))^2}\right),$$

$$s_{24} = x_2\left(-w_4(1+2x_3+2x_4) - \frac{w_2w_3x_1}{(1+w_3(x_3+x_4))^2}\right),$$

$$s_{31} = c_1w_6(x_3+x_4)(1+x_3+x_4),$$

$$s_{32} = c_2w_6(x_3+x_4)(1+x_3+x_4),$$

$$s_{33} = -w_{10} - w_7x_4 + w_6(c_1x_1+c_2x_2)(1+2x_3+2x_4),$$

$$s_{34} = -w_7x_3 + \frac{w_8w_9}{(w_9+x_4)^2} + w_6(c_1x_1+c_2x_2)(1+2x_3+2x_4),$$

$$s_{41} = 0,\ s_{42} = 0,\ s_{43} = w_7x_4,\ s_{44} = -w_{11} + w_7x_3 - \frac{w_8w_9}{(w_9+x_4)^2}.$$

Accordingly, the JM given by 12 at E_0 becomes

$$J_{E_0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -w_5 & 0 & 0 \\ 0 & 0 & -w_{10} & \frac{w_8}{w_9} \\ 0 & 0 & 0 & -\frac{w_8}{w_9} - w_{11} \end{bmatrix}. \tag{13}$$

Therefore, the eigenvalues of J_{E_0} are given by

$$\lambda_{01} = 1 > 0,\ \lambda_{02} = -w_5 < 0, \lambda_{03} = -w_{10} < 0,\ \lambda_{04} = -\left(\frac{w_8}{w_9} + w_{11}\right) < 0. \tag{14}$$

As one of the eigenvalues is positive and the others are negative, hence, E_0 is a saddle point.
The JM that is determined by Eq. (12) at E_1 becomes

$$J_{E_1} = \begin{bmatrix} -1 & -1-w_2 & -w_4 & -w_4 \\ 0 & w_2-w_5 & 0 & 0 \\ 0 & 0 & c_1w_6 - w_{10} & c_1w_6 + \frac{w_8}{w_9} \\ 0 & 0 & 0 & -\frac{w_8}{w_9} - w_{11} \end{bmatrix}. \tag{15}$$

Therefore, the eigenvalues of J_{E_1} are given by

$$\lambda_{11} = -1 < 0,\ \lambda_{12} = w_2 - w_5, \lambda_{13} = c_1 w_6 - w_{10},\ \lambda_{14} = -\left(\frac{w_8}{w_9} + w_{11}\right) < 0. \tag{16}$$

Hence, the AEP is locally asymptotically stable (LAS), provided that the following conditions are met.

$$w_2 < w_5. \tag{17}$$

$$c_1 w_6 < w_{10}. \tag{18}$$

Otherwise, it will be a non-hyperbolic point if at least one of the above two inequality becomes equality. Moreover, it is a saddle point if at least one of the above inequality is reflected.

Now, the JM that is given by Eq. (12) at E_2 becomes

$$J_{E_2} = \begin{bmatrix} -\frac{w_5}{w_2} & -\frac{(1+w_2)w_5}{w_2} & -\frac{w_5[w_4(1+w_2)+(w_2-w_5)(w_1-w_3)]}{w_2(1+w_2)} & -\frac{w_5[w_4(1+w_2)+(w_2-w_5)(w_1-w_3)]}{w_2(1+w_2)} \\ \frac{w_2-w_5}{1+w_2} & 0 & -\frac{(w_2-w_5)(w_4+w_3w_5)}{w_2(1+w_2)} & -\frac{(w_2-w_5)(w_4+w_3w_5)}{w_2(1+w_2)} \\ 0 & 0 & \frac{w_6[c_2(w_2-w_5)+c_1(1+w_2)w_5]}{w_2(1+w_2)} - w_{10} & \frac{w_6[c_2(w_2-w_5)+c_1(1+w_2)w_5]}{w_2(1+w_2)} + \frac{w_8}{w_9} \\ 0 & 0 & 0 & -\frac{w_8}{w_9} - w_{11} \end{bmatrix}. \tag{19}$$

Hence, the characteristic equation of $J_{E_2} = [a_{ij}]_{4\times4}$ can be written as:

$$\left[\lambda^2 - a_{11}\lambda - a_{12}a_{21}\right][a_{33} - \lambda][a_{44} - \lambda] = 0. \tag{20}$$

Direct computation gives the following roots

$$\left.\begin{array}{l} \lambda_{21} = \frac{a_{11}}{2} + \frac{1}{2}\sqrt{(a_{11})^2 - 4(a_{12}a_{21})} \\ \lambda_{22} = \frac{a_{11}}{2} - \frac{1}{2}\sqrt{(a_{11})^2 - 4(a_{12}a_{21})} \\ \lambda_{23} = \frac{(c_2(w_2-w_5)+c_1(1+w_2)w_5)w_6}{w_2(1+w_2)} - w_{10} \\ \lambda_{24} = -\frac{w_8}{w_9} - w_{11} < 0 \end{array}\right\}. \tag{21}$$

Direct computation shows that the eigenvalues λ_{21} and λ_{22} have always negative real parts, while λ_{23} will be negative if the following condition holds.

$$\frac{(c_2(w_2 - w_5) + c_1(1 + w_2)w_5)w_6}{w_2(1 + w_2)} < w_{10}. \tag{22}$$

Therefore, the PFEP is a LAS if condition 22 is satisfied. Otherwise, it will be a non-hyperbolic point if inequality in condition 22 becomes equality, and it is a saddle point if the inequality in condition 22 is reflected.

The JM that is given by Eq. (12) at E_3 becomes

$$J_{E_3} = \left[\hat{s}_{ij}\right]_{4\times4}, \tag{23}$$

Where

$$\hat{s}_{11} = -\frac{w_{10}}{c_1 w_6(1 + \hat{x}_3)(1 + w_1\hat{x}_3)},$$

$$\hat{s}_{12} = -\frac{w_{10}[1 + w_3\hat{x}_3 + w_2(1 + w_1\hat{x}_3)]}{c_1 w_6(1 + \hat{x}_3)(1 + w_3\hat{x}_3)(1 + w_1\hat{x}_3)},$$

$$\hat{s}_{13} = -\frac{w_{10}\left(w_4\hat{x}_3 + w_4(1 + \hat{x}_3) + \frac{w_1(c_1w_6(1+\hat{x}_3)-w_{10})}{c_1w_6(1+\hat{x}_3)(1+w_1\hat{x}_3)^2}\right)}{c_1 w_6(1 + \hat{x}_3)},$$

$$\hat{s}_{14} = -\frac{w_{10}\left(w_4\hat{x}_3 + w_4\left(1+\hat{x}_3\right) + \frac{w_1\left(c_1w_6\left(1+\hat{x}_3\right)-w_{10}\right)}{c_1w_6\left(1+\hat{x}_3\right)\left(1+w_1\hat{x}_3\right)^2}\right)}{c_1w_6\left(1+\hat{x}_3\right)},$$

$$\hat{s}_{21} = 0,\ \hat{s}_{22} = -w_5 - w_4\hat{x}_3\left(1+\hat{x}_3\right) + \frac{w_2w_{10}}{c_1w_6\left(1+\hat{x}_3\right)\left(1+w_3\hat{x}_3\right)},\ \hat{s}_{23} = 0,\ \hat{s}_{24} = 0,$$

$$\hat{s}_{31} = c_1w_6\hat{x}_3\left(1+\hat{x}_3\right),\ \hat{s}_{32} = c_2w_6\hat{x}_3\left(1+\hat{x}_3\right),$$

$$\hat{s}_{33} = \frac{w_{10}\hat{x}_3}{1+\hat{x}_3},\ \hat{s}_{34} = \frac{w_8}{w_9} - w_7\hat{x}_3 + w_{10}\left(2 - \frac{1}{1+\hat{x}_3}\right),$$

$$\hat{s}_{41} = 0,\ \hat{s}_{42} = 0,\ \hat{s}_{43} = 0,\ \hat{s}_{44} = -\frac{w_8}{w_9} - w_{11} + w_7\hat{x}_3.$$

Hence, the characteristic equation of J_{E_3} can be written as:

$$\left[\lambda^2 - \left(\hat{s}_{11} + \hat{s}_{33}\right)\lambda + \left(\hat{s}_{11}\hat{s}_{33} - \hat{s}_{13}\hat{s}_{31}\right)\right]\left[\hat{s}_{22} - \lambda\right]\left[\hat{s}_{44} - \lambda\right] = 0. \tag{24}$$

Direct computation gives the following roots

$$\left.\begin{array}{c} \lambda_{31} = \frac{\left(\hat{s}_{11}+\hat{s}_{33}\right)}{2} + \frac{1}{2}\sqrt{\left(\hat{s}_{11}+\hat{s}_{33}\right)^2 - 4\left(\hat{s}_{11}\hat{s}_{33} - \hat{s}_{13}\hat{s}_{31}\right)} \\ \lambda_{33} = \frac{\left(\hat{s}_{11}+\hat{s}_{33}\right)}{2} - \frac{1}{2}\sqrt{\left(\hat{s}_{11}+\hat{s}_{33}\right)^2 - 4\left(\hat{s}_{11}\hat{s}_{33} - \hat{s}_{13}\hat{s}_{31}\right)} \\ \lambda_{32} = -w_5 - w_4\hat{x}_3\left(1+\hat{x}_3\right) + \frac{w_2w_{10}}{c_1w_6\left(1+\hat{x}_3\right)\left(1+w_3\hat{x}_3\right)} \\ \lambda_{34} = -\frac{w_8}{w_9} - w_{11} + w_7\hat{x}_3 \end{array}\right\}. \tag{25}$$

Direct computation shows that the eigenvalues λ_{31} and λ_{33} have negative real parts, and λ_{32} with λ_{34} are negative if the following conditions hold.

$$\hat{x}_3 < \frac{1}{c_1w_6\left(1+w_1\hat{x}_3\right)}. \tag{26}$$

$$\frac{w_{10}}{c_1w_6\left(1+\hat{x}_3\right)^2\left(1+w_1\hat{x}_3\right)} < \left(w_4\hat{x}_3 + w_4\left(1+\hat{x}_3\right) + \frac{w_1\left(c_1w_6\left(1+\hat{x}_3\right)-w_{10}\right)}{c_1w_6\left(1+\hat{x}_3\right)\left(1+w_1\hat{x}_3\right)^2}\right). \tag{27}$$

$$\frac{w_2w_{10}}{c_1w_6\left(1+\hat{x}_3\right)\left(1+w_3\hat{x}_3\right)} < w_5 + w_4\hat{x}_3\left(1+\hat{x}_3\right). \tag{28}$$

$$w_7\hat{x}_3 < \frac{w_8}{w_9} + w_{11}. \tag{29}$$

Therefore, the DFEP is a LAS if conditions 26–29 are satisfied. It becomes a non-hyperbolic point when any of the inequalities 26, 28, or 29 becomes equality. Moreover, it is a saddle point in case any of the inequalities 26, 28, or 29 are reflected.

Moreover, the JM that is given by Eq. (12) at PDFEP that is represented by E_4 can be written as:

$$J_{E_4} = \left[\check{s}_{ij}\right]_{4\times 4}, \tag{30}$$

Where

$$\check{s}_{11} = -\frac{\check{x}_1}{1+w_1\check{x}_3},\ \check{s}_{12} = -\check{x}_1\left(\frac{1}{1+w_1\check{x}_3} + \frac{w_2}{1+w_3\check{x}_3}\right),$$

$$\check{s}_{13} = \check{x}_1\left(-w_4\left(1+2\check{x}_3\right) + \frac{w_1\left(-1+\check{x}_1+\check{x}_2\right)}{\left(1+w_1\check{x}_3\right)^2} + \frac{w_2w_3\check{x}_2}{\left(1+w_3\check{x}_3\right)^2}\right),$$

$$\check{s}_{14} = \check{x}_1\left(-w_4\left(1+2\check{x}_3\right) + \frac{w_1\left(-1+\check{x}_1+\check{x}_2\right)}{\left(1+w_1\check{x}_3\right)^2} + \frac{w_2w_3\check{x}_2}{\left(1+w_3\check{x}_3\right)^2}\right),$$

$$\check{s}_{21} = \frac{w_2\check{x}_2}{1+w_3\check{x}_3},\ \check{s}_{22} = 0,\ \check{s}_{23} = -\check{x}_2\left(w_4\left(1+2\check{x}_3\right) + \frac{w_2w_3\check{x}_1}{\left(1+w_3\check{x}_3\right)^2}\right),$$

$$\check{s}_{24} = -\check{x}_2 \left(w_4 \left(1 + 2\check{x}_3\right) + \frac{w_2 w_3 \check{x}_1}{\left(1 + w_3 \check{x}_3\right)^2} \right),$$

$$\check{s}_{31} = c_1 w_6 \check{x}_3 \left(1 + \check{x}_3\right), \ \check{s}_{32} = c_2 w_6 \check{x}_3 \left(1 + \check{x}_3\right),$$

$$\check{s}_{33} = -w_{10} + w_6 \left(c_1 \check{x}_1 + c_2 \check{x}_2\right) \left(1 + 2\check{x}_3\right),$$

$$\check{s}_{34} = \frac{w_8}{w_9} - w_7 \check{x}_3 + w_6 \left(c_1 \check{x}_1 + c_2 \check{x}_2\right) \left(1 + 2\check{x}_3\right),$$

$$\check{s}_{41} = 0, \ \check{s}_{42} = 0, \ \check{s}_{43} = 0, \ \check{s}_{44} = -\frac{w_8}{w_9} - w_{11} + w_7 \check{x}_3.$$

Hence, the characteristic equation of J_{E_4} can be written as:

$$\left[\lambda^3 - A_{41}\lambda^2 + A_{42}\lambda + A_{43}\right] \left[\check{s}_{44} - \lambda\right] = 0, \tag{31}$$

Where

$$A_{41} = -\left(\check{s}_{11} + \check{s}_{33}\right),$$

$$A_{42} = -\check{s}_{12}\check{s}_{21} + \check{s}_{11}\check{s}_{33} - \check{s}_{13}\check{s}_{31} - \check{s}_{23}\check{s}_{32},$$

$$A_{43} = -\left[\check{s}_{23}\left(\check{s}_{12}\check{s}_{31} - \check{s}_{11}\check{s}_{32}\right) + \check{s}_{21}\left(\check{s}_{13}\check{s}_{32} - \check{s}_{12}\check{s}_{33}\right)\right],$$

$$\Delta = A_{41}A_{42} - A_{43} = -\left(\check{s}_{11} + \check{s}_{33}\right)\left[\check{s}_{11}\check{s}_{33} - \check{s}_{13}\check{s}_{31}\right] + \check{s}_{12}\left(\check{s}_{11}\check{s}_{21} + \check{s}_{23}\check{s}_{31}\right) + \check{s}_{32}\left(\check{s}_{23}\check{s}_{33} + \check{s}_{13}\check{s}_{21}\right).$$

According to the Routh-Hurwitz criterion,[1] the characteristic Eq. (31) has eigenvalues with negative real parts if the following conditions are satisfied:

$$\frac{w_1\left(-1 + \check{x}_1 + \check{x}_2\right)}{\left(1 + w_1\check{x}_3\right)^2} + \frac{w_2 w_3 \check{x}_2}{\left(1 + w_3\check{x}_3\right)^2} < w_4\left(1 + 2\check{x}_3\right). \tag{32}$$

$$w_6\left(c_1\check{x}_1 + c_2\check{x}_2\right)\left(1 + 2\check{x}_3\right) < w_{10}. \tag{33}$$

$$w_7\check{x}_3 < \frac{w_8}{w_9} + w_{11}. \tag{34}$$

$$\left(\frac{1}{1 + w_1\check{x}_3} + \frac{w_2}{1 + w_3\check{x}_3}\right)c_1 < \frac{c_2}{1 + w_1\check{x}_3}. \tag{35}$$

$$\check{x}_1\left(w_4\left(1 + 2\check{x}_3\right) - \frac{w_1\left(-1 + \check{x}_1 + \check{x}_2\right)}{\left(1 + w_1\check{x}_3\right)^2} - \frac{w_2 w_3 \check{x}_2}{\left(1 + w_3\check{x}_3\right)^2}\right)\frac{w_2\check{x}_2}{1 + w_3\check{x}_3} < \left[\check{x}_2\left(w_4\left(1 + 2\check{x}_3\right) + \frac{w_2 w_3 \check{x}_1}{\left(1 + w_3\check{x}_3\right)^2}\right)\right]\left[w_{10} - w_6\left(c_1\check{x}_1 + c_2\check{x}_2\right)\left(1 + 2\check{x}_3\right)\right]. \tag{36}$$

The JM that is given by Eq. (12) at the HPEP that is represented by E_5 can be written as:

$$J_{E_5} = \left[\tilde{s}_{ij}\right]_{4\times 4}, \tag{37}$$

Where

$$\tilde{s}_{11} = -\frac{\tilde{x}_1}{1 + w_1\left(\tilde{x}_3 + \tilde{x}_4\right)}, \ \tilde{s}_{12} = -\tilde{x}_1\left(\frac{1}{1 + w_1\left(\tilde{x}_3 + \tilde{x}_4\right)} + \frac{w_2}{1 + w_3\left(\tilde{x}_3 + \tilde{x}_4\right)}\right),$$

$$\tilde{s}_{13} = -\tilde{x}_1\left(w_4\left(1 + 2\tilde{x}_3 + 2\tilde{x}_4\right) + \frac{w_1\left(1 - \tilde{x}_1\right)}{\left(1 + w_1\left(\tilde{x}_3 + \tilde{x}_4\right)\right)^2}\right),$$

$$\tilde{s}_{14} = -\tilde{x}_1\left(w_4\left(1 + 2\tilde{x}_3 + 2\tilde{x}_4\right) + \frac{w_1\left(1 - \tilde{x}_1\right)}{\left(1 + w_1\left(\tilde{x}_3 + \tilde{x}_4\right)\right)^2}\right),$$

$$\tilde{s}_{21} = 0,\ \tilde{s}_{22} = -w_5 - w_4(\tilde{x}_3 + \tilde{x}_4)(1 + \tilde{x}_3 + \tilde{x}_4) + \frac{w_2\tilde{x}_1}{1 + w_3(\tilde{x}_3 + \tilde{x}_4)},\ \tilde{s}_{23} = 0,\ \tilde{s}_{24} = 0,$$

$$\tilde{s}_{31} = c_1 w_6(\tilde{x}_3 + \tilde{x}_4)(1 + \tilde{x}_3 + \tilde{x}_4),\ \tilde{s}_{32} = c_2 w_6(\tilde{x}_3 + \tilde{x}_4)(1 + \tilde{x}_3 + \tilde{x}_4),$$

$$\tilde{s}_{33} = -w_{10} - w_7\tilde{x}_4 + c_1 w_6\tilde{x}_1(1 + 2\tilde{x}_3 + 2\tilde{x}_4),$$

$$\tilde{s}_{34} = w_7\tilde{x}_3 + \frac{w_8 w_9}{(w_9 + \tilde{x}_4)^2} + c_1 w_6\tilde{x}_1(1 + 2\tilde{x}_3 + 2\tilde{x}_4),$$

$$\tilde{s}_{41} = 0,\ \tilde{s}_{42} = 0,\ \tilde{s}_{43} = w_7\tilde{x}_4,\ \tilde{s}_{44} = \frac{w_8\tilde{x}_4}{(w_9 + \tilde{x}_4)^2}.$$

Hence, the characteristic equation of J_{E_5} can be written as:

$$\left[\lambda^3 - A_{51}\lambda^2 + A_{52}\lambda + A_{53}\right][\tilde{s}_{22} - \lambda] = 0, \tag{38}$$

Where

$$A_{51} = -(\tilde{s}_{11} + \tilde{s}_{33} + \tilde{s}_{44}),$$

$$A_{52} = \tilde{s}_{11}\tilde{s}_{33} - \tilde{s}_{13}\tilde{s}_{31} + \tilde{s}_{33}\tilde{s}_{44} - \tilde{s}_{34}\tilde{s}_{43} + \tilde{s}_{11}\tilde{s}_{44},$$

$$A_{53} = -[\tilde{s}_{11}(\tilde{s}_{33}\tilde{s}_{44} - \tilde{s}_{34}\tilde{s}_{43}) + \tilde{s}_{31}(\tilde{s}_{14}\tilde{s}_{43} - \tilde{s}_{13}\tilde{s}_{44})],$$

With

$$\begin{aligned}\Delta = A_{51}A_{52} - A_{53} = &-(\tilde{s}_{11} + \tilde{s}_{33})[\tilde{s}_{11}\tilde{s}_{33} - \tilde{s}_{13}\tilde{s}_{31}] - \tilde{s}_{11}\tilde{s}_{44}(\tilde{s}_{11} + \tilde{s}_{44}) \\ &- (\tilde{s}_{33} + \tilde{s}_{44})[\tilde{s}_{33}\tilde{s}_{44} - \tilde{s}_{34}\tilde{s}_{43}] - 2\tilde{s}_{11}\tilde{s}_{33}\tilde{s}_{44} + \tilde{s}_{14}\tilde{s}_{31}\tilde{s}_{43}.\end{aligned}$$

With the following sufficient conditions that satisfy $A_{51} > 0;\ A_{53} > 0$, and $\Delta = A_{51}A_{52} - A_{53} > 0$, it is simple to confirm that the characteristic Eq. (38) has either $\lambda_{52} = \tilde{s}_{22}$, which is negative provided that condition 39 holds, or three eigenvalues, namely $\lambda_{51},\ \lambda_{53},\ and\ \lambda_{54}$, with negative real parts, due to Routh-Hurwitz criterion.

$$\frac{w_2\tilde{x}_1}{1 + w_3(\tilde{x}_3 + \tilde{x}_4)} < w_5 + w_4(\tilde{x}_3 + \tilde{x}_4)(1 + \tilde{x}_3 + \tilde{x}_4), \tag{39}$$

$$c_1 w_6\tilde{x}_1(1 + 2\tilde{x}_3 + 2\tilde{x}_4) < w_{10} + w_7\tilde{x}_4, \tag{40}$$

$$\tilde{s}_{11} + \tilde{s}_{33} + \tilde{s}_{44} < 0, \tag{41}$$

$$0 < \tilde{s}_{11}(\tilde{s}_{33}\tilde{s}_{44} - \tilde{s}_{34}\tilde{s}_{43}) < \tilde{s}_{31}(\tilde{s}_{13}\tilde{s}_{44} - \tilde{s}_{14}\tilde{s}_{43}), \tag{42}$$

$$max.\ \{-\tilde{s}_{11}, -\tilde{s}_{33}\} < \tilde{s}_{44}, \tag{43}$$

$$0 < 2\tilde{s}_{11}\tilde{s}_{33}\tilde{s}_{44} - \tilde{s}_{14}\tilde{s}_{31}\tilde{s}_{43} < -\tilde{s}_{11}\tilde{s}_{44}(\tilde{s}_{11} + \tilde{s}_{44}) - (\tilde{s}_{33} + \tilde{s}_{44})[\tilde{s}_{33}\tilde{s}_{44} - \tilde{s}_{34}\tilde{s}_{43}]. \tag{44}$$

Finally, the JM that is given by Eq. (12) at E_6 becomes:

$$J_{E_6} = \left[s^*_{ij}\right]_{4\times 4}, \tag{45}$$

Where

$$s^*_{11} = -\frac{x^*_1}{1 + w_1(x^*_3 + x^*_4)},\ s^*_{12} = -x^*_1\left(\frac{1}{1 + w_1(x^*_3 + x^*_4)} + \frac{w_2}{1 + w_3(x^*_3 + x^*_4)}\right),$$

$$s^*_{13} = x^*_1\left(-w_4(1 + 2x^*_3 + 2x^*_4) + \frac{w_1(-1 + x^*_1 + x^*_2)}{(1 + w_1(x^*_3 + x^*_4))^2} + \frac{w_2 w_3 x^*_2}{(1 + w_3(x^*_3 + x^*_4))^2}\right),$$

$$s^*_{14} = x^*_1\left(-w_4(1 + 2x^*_3 + 2x^*_4) + \frac{w_1(-1 + x^*_1 + x^*_2)}{(1 + w_1(x^*_3 + x^*_4))^2} + \frac{w_2 w_3 x^*_2}{(1 + w_3(x^*_3 + x^*_4))^2}\right)$$

$$s_{21}^* = \frac{w_2 x_2^*}{1 + w_3 \left(x_3^* + x_4^*\right)},\ s_{22}^* = 0,\ s_{23}^* = -x_2^* \left(w_4 \left(1 + 2x_3^* + 2x_4^*\right) + \frac{w_2 w_3 x_1^*}{\left(1 + w_3 \left(x_3^* + x_4^*\right)\right)^2} \right),$$

$$s_{24}^* = -x_2^* \left(w_4 \left(1 + 2x_3^* + 2x_4^*\right) + \frac{w_2 w_3 x_1^*}{\left(1 + w_3 \left(x_3^* + x_4^*\right)\right)^2} \right),$$

$$s_{31}^* = c_1 w_6 \left(x_3^* + x_4^*\right)\left(1 + x_3^* + x_4^*\right),\ s_{32}^* = c_2 w_6 \left(x_3^* + x_4^*\right)\left(1 + x_3^* + x_4^*\right),$$

$$s_{33}^* = -w_{10} - w_7 x_4^* + w_6 \left(c_1 x_1^* + c_2 x_2^*\right)\left(1 + 2x_3^* + 2x_4^*\right),$$

$$s_{34}^* = -w_7 x_3^* + \frac{w_8 w_9}{\left(w_9 + x_4^*\right)^2} + w_6 \left(c_1 x_1^* + c_2 x_2^*\right)\left(1 + 2x_3^* + 2x_4^*\right),$$

$$s_{41}^* = 0,\ s_{42}^* = 0,\ s_{43}^* = w_7 x_4^*,\ s_{44}^* = \frac{w_8 x_4^*}{\left(w_9 + x_4^*\right)^2}.$$

Hence, the characteristic equation of $J(E_6)$ can be written as:

$$\left[\lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4\right] = 0, \tag{46}$$

Where

$$A_1 = -\left(s_{11}^* + s_{33}^* + s_{44}^*\right),$$

$$A_2 = -s_{12}^* s_{21}^* + s_{11}^* s_{33}^* - s_{13}^* s_{31}^* + s_{11}^* s_{44}^* - s_{23}^* s_{32}^* + s_{33}^* s_{44}^* - s_{34}^* s_{43}^*,$$

$$A_3 = -\left(s_{12}^* \left[s_{23}^* s_{31}^* - s_{21}^* s_{33}^*\right] + s_{13}^* s_{21}^* s_{32}^* - s_{11}^* \left(s_{23}^* s_{32}^* + s_{34}^* s_{43}^*\right) + s_{24}^* s_{32}^* s_{43}^* + \left[\left(s_{11}^* s_{33}^* - s_{13}^* s_{31}^*\right) - \left(s_{23}^* s_{32}^* + s_{12}^* s_{21}^*\right)\right] s_{44}^* + s_{14}^* s_{31}^* s_{43}^*\right),$$

$$A_4 = -s_{43}^* \left(s_{12}^* s_{24}^* s_{31}^* + s_{14}^* s_{21}^* s_{32}^* - s_{11}^* s_{24}^* s_{32}^* - s_{12}^* s_{21}^* s_{34}^*\right) + s_{44}^* \left(s_{12}^* s_{23}^* s_{31}^* + s_{13}^* s_{21}^* s_{32}^* - s_{11}^* s_{23}^* s_{32}^* - s_{12}^* s_{21}^* s_{33}^*\right).$$

According to the Routh-Hurwitz criterion,[1] the characteristic Eq. (46) has four eigenvalues with negative real parts if the following conditions are satisfied.

$$\frac{w_1 \left(-1 + x_1^* + x_2^*\right)}{\left(1 + w_1 \left(x_3^* + x_4^*\right)\right)^2} + \frac{w_2 w_3 x_2^*}{\left(1 + w_3 \left(x_3^* + x_4^*\right)\right)^2} < w_4 \left(1 + 2x_3^* + 2x_4^*\right), \tag{47}$$

$$w_6 \left(c_1 x_1^* + c_2 x_2^*\right)\left(1 + 2x_3^* + 2x_4^*\right) < w_{10} + w_7 x_4^*, \tag{48}$$

$$\frac{w_8 w_9}{\left(w_9 + x_4^*\right)^2} + w_6 \left(c_1 x_1^* + c_2 x_2^*\right)\left(1 + 2x_3^* + 2x_4^*\right) < w_7 x_3^*, \tag{49}$$

$$s_{11}^* + s_{33}^* + s_{44}^* < 0, \tag{50}$$

$$s_{21}^* s_{33}^* < s_{23}^* s_{31}^*, \tag{51}$$

$$\left[\left(s_{11}^* s_{33}^* - s_{13}^* s_{31}^*\right) - \left(s_{23}^* s_{32}^* + s_{12}^* s_{21}^*\right)\right] s_{44}^* < s_{11}^* \left(s_{23}^* s_{32}^* + s_{34}^* s_{43}^*\right), \tag{52}$$

$$s_{12}^* s_{24}^* s_{31}^* + s_{14}^* s_{21}^* s_{32}^* - s_{11}^* s_{24}^* s_{32}^* - s_{12}^* s_{21}^* s_{34}^* < 0, \tag{53}$$

$$s_{12}^* s_{23}^* s_{31}^* + s_{13}^* s_{21}^* s_{32}^* - s_{11}^* s_{23}^* s_{32}^* - s_{12}^* s_{21}^* s_{33}^* > 0, \tag{54}$$

$$\left(A_1 A_2 - A_3\right) A_3 > A_1^2 A_4. \tag{55}$$

Local bifurcation

To ascertain whether local bifurcation may occur close to the system's 2 equilibrium points when the parameter crosses a particular value that turns the equilibrium point into a non-hyperbolic point, Sotomayor's bifurcation theorem[30] was utilized. For a known local bifurcation to occur, it is essential but not sufficient that the equilibrium point be non-hyperbolic. Because the parameters are dynamic and always changing based on the conditions of the environment in which the system's organisms exist, the study of the bifurcation of system 2 is essential. Now, rewrite system 2 in vector form as follows:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, \beta); \mathbf{X} = (x_1, x_2, x_3, x_4)^{\mathrm{T}}; \mathbf{F} = (F_1(\mathbf{X}, \beta), F_2(\mathbf{X}, \beta), F_3(\mathbf{X}, \beta), F_4(\mathbf{X}, \beta))^{\mathrm{T}}, \tag{56}$$

Where $\beta \in \mathbb{R}$ represents a bifurcation parameter. Hence the second and third directional derivatives for system 56 can be written respectively:

$$D^2\mathbf{F}(\boldsymbol{X}, \beta)(\mathbf{V}, \mathbf{V}) = [b_{i1}]_{4\times 1}, \tag{57}$$

Where $\mathbf{V} = (v_1, v_2, v_3, v_4)^T$ be any vector and b_{i1} was determined by:

$$\begin{aligned}
b_{11} = & -2w_4(v_3+v_4)[(v_3+v_4)x_1 + v_1(1+2x_3)] - 4v_1(v_3+v_4)w_4x_4 \\
& - \frac{2(v_3+v_4)^2 w_1^2 x_1(-1+x_1+x_2)}{(1+w_1(x_3+x_4))^3} + \frac{2(v_3+v_4)w_1(v_2x_1 + v_1(-1+2x_1+x_2))}{(1+w_1(x_3+x_4))^2} - \frac{2v_1(v_1+v_2)}{1+w_1(x_3+x_4)} \\
& - \frac{2(v_3+v_4)^2 w_2 w_3^2 x_1 x_2}{(1+w_3(x_3+x_4))^3} + \frac{2(v_3+v_4)w_2w_3(v_2x_1+v_1x_2)}{(1+w_3(x_3+x_4))^2} - \frac{2v_1v_2w_2}{1+w_3(x_3+x_4)}, \\
b_{21} = & -2w_4(v_3+v_4)[(v_3+v_4)x_2 + v_2(1+2x_3)] - 4v_2(v_3+v_4)w_4x_4 \\
& + \frac{2(v_3+v_4)^2 w_2 w_3^2 x_1 x_2}{(1+w_3(x_3+x_4))^3} - \frac{2(v_3+v_4)w_2w_3(v_2x_1+v_1x_2)}{(1+w_3(x_3+x_4))^2} + \frac{2v_1v_2w_2}{1+w_3(x_3+x_4)}, \\
b_{31} = & 2v_4\left(-v_3w_7 - \frac{v_4w_8w_9}{(w_9+x_4)^3}\right) + 2c_1w_6(v_3+v_4)[(v_3+v_4)x_1 + v_1(1+2x_3+2x_4)] \\
& + 2c_2w_6(v_3+v_4)[(v_3+v_4)x_2 + v_2(1+2x_3+2x_4)], \\
b_{41} = & 2v_4\left(v_3w_7 + \frac{v_4w_8w_9}{(w_9+x_4)^3}\right).
\end{aligned}$$

While:

$$D^3F(X, \beta)(V, V, V) = [c_{i2}]_{4\times 1}, \tag{58}$$

Where

$$\begin{aligned}
c_{12} = & 6(v_3+v_4)\Bigg[-v_1(v_3+v_4)w_4 + \frac{(v_3+v_4)^2 w_1^3 x_1(-1+x_1+x_2)}{(1+w_1(x_3+x_4))^4} \\
& - \frac{(v_3+v_4)w_1^2(v_2x_1 + v_1(-1+2x_1+x_2))}{(1+w_1(x_3+x_4))^3} + \frac{v_1(v_1+v_2)w_1}{(1+w_1(x_3+x_4))^2} + \frac{(v_3+v_4)^2 w_2 w_3^3 x_1 x_2}{(1+w_3(x_3+x_4))^4} \\
& - \frac{(v_3+v_4)w_2w_3^2(v_2x_1+v_1x_2)}{(1+w_3(x_3+x_4))^3} + \frac{v_1v_2w_2w_3}{(1+w_3(x_3+x_4))^2}\Bigg], \\
c_{22} = & \frac{6(v_3+v_4)^2 w_2 w_3^2 x_2[-(v_3+v_4)w_3x_1 + v_1(1+w_3(x_3+x_4))]}{(1+w_3(x_3+x_4))^4} - \frac{6(v_3+v_4)v_2v_1w_2w_3(1+w_3(x_3+x_4))^2}{(1+w_3(x_3+x_4))^4} \\
& - \frac{6(v_3+v_4)^2 v_2(1+w_3(x_3+x_4))\left(-w_2w_3^2x_1 + w_4(1+w_3(x_3+x_4))^3\right)}{(1+w_3(x_3+x_4))^4}, \\
c_{32} = & 6w_6(c_1v_1 + c_2v_2)(v_3+v_4)^2 + \frac{6v_4^3w_8w_9}{(w_9+x_4)^4}, \\
c_{42} = & -\frac{6v_4^3w_8w_9}{(w_9+x_4)^4}.
\end{aligned}$$

With the help of the above findings, the following theorem can investigate the occurrence of local bifurcation near the equilibrium points.

Theorem 3: *Assume that condition 17 is met, then system 2 undergoes a Transcritical bifurcation (TB) near AEP when the parameter w_{10} passes through the value $w_{10}^* = c_1 w_6$ provided that the following condition holds:*

$$w_4 \neq 1. \tag{59}$$

Otherwise, pitchfork bifurcation (PB) takes place.

Proof: From the Eq. (15) with $w_{10}^* = c_1 w_6$ the JM becomes

$$J_1^* = J\left(E_1, w_{10}^*\right) = \begin{bmatrix} -1 & -1-w_2 & -w_4 & -w_4 \\ 0 & w_2 - w_5 & 0 & 0 \\ 0 & 0 & 0 & c_1 w_6 + \frac{w_8}{w_9} \\ 0 & 0 & 0 & -\frac{w_8}{w_9} - w_{11} \end{bmatrix}.$$

Therefore, the eigenvalues of J_1^* are given by:

$$\lambda_{11}\left(w_{10}^*\right) = -1 < 0, \lambda_{12}\left(w_{10}^*\right) = w_2 - w_5, \lambda_{13}\left(w_{10}^*\right) = 0, \text{and} \lambda_{14}\left(w_{10}^*\right) = -\frac{w_8}{w_9} - w_{11} < 0.$$

Note that the eigenvalue $\lambda_{12}(w_{10}^*)$ is negative under the condition 17. Thus AEP is a non-hyperbolic point at $w_{10}^* = c_1 w_6$.

Let $\mathbf{V}_1 = (v_{11}, v_{12}, v_{13}, v_{14})^{\mathrm{T}}$ and $\mathbf{U}_1 = (u_{11}, u_{12}, u_{13}, u_{14})^{\mathrm{T}}$ are the eigenvectors conjugate with the eigenvalue $\lambda_{13}(w_{10}^*)$ of J_1^* and ${J_1^*}^T$ respectively. It is obtained after direct computation that:

$$\mathbf{V}_1 = (-w_4, 0, 1, 0)^{\mathrm{T}} \text{ and } \mathbf{U}_1 = (0, 0, 1, \delta)^T, \text{ with } \delta = \frac{c_1 w_6 w_9 + w_8}{w_8 + w_9 w_{11}} > 0.$$

Following Sotomayor's theorem, gives that:

$$\frac{\partial}{\partial w_{10}} \mathbf{F}\left(\mathbf{X}, w_{10}\right) = \begin{pmatrix} 0 \\ 0 \\ -x_3 \\ 0 \end{pmatrix}; \Rightarrow \frac{\partial}{\partial w_{10}} \mathbf{F}\left(E_1, w_{10}^*\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, ${\mathbf{U}_1}^T \mathbf{F}_{w_{10}}(E_1, w_{10}^*) = 0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$D\mathbf{F}_{w_{10}}\left(\mathbf{X}, w_{10}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow D\mathbf{F}_{w_{10}}\left(E_1, w_{10}^*\right) \mathbf{V}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Therefore

$${\mathbf{U}_1}^{\mathrm{T}} D\mathbf{F}_{w_{10}}\left(E_1, w_{10}^*\right) \mathbf{V}_1 = -1 \neq 0.$$

Also, by using Eq. (57), it is obtained that

$$D^2\mathbf{F}_{w_{10}}\left(\mathbf{X}, w_{10}\right)\left(\mathbf{V}_1, \mathbf{V}_1\right) = \begin{bmatrix} -2w_1 w_4 - 2\left(1-w_4\right) w_4 - 2w_4^2 \\ 0 \\ 2c_1\left(1-w_4\right) w_6 \\ 0 \end{bmatrix}.$$

Therefore, depending on the condition 59, the following is reached:

$\mathbf{U}_1{}^{\mathrm{T}} D^2 \mathbf{F}_{w_{10}} \left(E_1, w_{10}^*\right) (\mathbf{V}_1, \mathbf{V}_1) = 2c_1 w_6 (1 - w_4) \neq 0.$

Hence a TB takes place near AEP under condition 59. Otherwise, by using Eq. (58), it is obtained that

$$D^3\mathbf{F}\left(E_1, w_{10}^*\right)(\mathbf{V}_1, \mathbf{V}_1, \mathbf{V}_1, \mathbf{V}_1) = \begin{bmatrix} 6\left(w_1^2 w_4 + w_4^2 + w_1 w_4^2\right) \\ 0 \\ -6c_1 w_4 w_6 \\ 0 \end{bmatrix}.$$

Accordingly, it is obtained that

$\mathbf{U}_1{}^{T} D^3 \left(E_1, w_{10}^*\right) (\mathbf{V}_1, \mathbf{V}_1, \mathbf{V}_1, \mathbf{V}_1) = -6c_1 w_4 w_6 \neq 0.$

Therefore, PB takes place near AEP, and the proof is complete.

Due to the above theorem, a Transcritical bifurcation happens when a parameter passes a critical value, trading stability with an unstable equilibrium point. Transcritical bifurcation in biological systems can signify a qualitative shift in the system's dynamics. It might signify a transition from one stable equilibrium state to another in which there is a notable change in the species composition or population. It can also depict a situation in which a species or population is in danger of going extinct or being invaded. However, as a parameter crosses a critical value, pitchfork bifurcation happens when one stable equilibrium point gives rise to two additional stable equilibrium points. Pitchfork bifurcation is a common indicator of population or species splitting or diversification in biological systems. It may signify the coexistence of several species inside the system or the emergence of several stable states.

Theorem 4: *System 2 undergoes a TB near PFEP when the parameter w_{10} passes through the value $w_{10}^{**} = \frac{w_6[c_2(w_2 - w_5) + c_1(1+w_2)w_5]}{w_2(1+w_2)}$ provided that the following condition holds:*

$$c_2\left(n_2 + \frac{w_2 - w_5}{w_2(1+w_2)}\right) + c_1\left(n_1 + \frac{w_5}{w_2}\right) \neq 0, \tag{60}$$

Where all the new symbols will be defined in the proof. Otherwise, pitchfork bifurcation (PB) takes place.

Proof: From Eq. (19) with $w_{10} = w_{10}^{**}$ the JM becomes

$J_2^* = J\left(E_2, w_{10}^*\right) = \left[a_{ij}\right],$

Where a_{ij} for all $i, j = 1, 2, 3, 4$, are given in Eq. (19) with $a_{33} = 0$. Thus, J_2^* has the negative real parts eigenvalues given in Eq. (21), with $\lambda_{23}(w_{10}^{**}) = 0$. Hence, PFEP is a non-hyperbolic point at $w_{10} = w_{10}^{**}$.

Let $\mathbf{V}_2 = (v_{21}, v_{22}, v_{23}, v_{24})^{\mathrm{T}}$, and $\mathbf{U}_2 = (u_{21}, u_{22}, u_{23}, u_{24})^{\mathrm{T}}$ are the eigenvectors conjugate with the eigenvalue $\lambda_{33}(w_{10}^{**})$ of J_2^* and J_2^{*T} respectively. Direct computation shows that:

$$\mathbf{V}_2 = (n_1, n_2, 1, 0)^{\mathrm{T}}, n_1 = -\frac{a_{23}}{a_{21}} > 0,\ n_2 = \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{12}a_{21}}.$$

$$\mathbf{U}_2 = (0, 0, 1, n_3)^T, n_3 = -\frac{a_{34}}{a_{44}} > 0.$$

Note that the sign of n_2 can be positive or negative depending on the value of a_{13}. Now, applying Sotomayor's theorem gives that:

$$\frac{\partial}{\partial w_{10}}\mathbf{F}(\mathbf{X}, w_{10}) = \begin{pmatrix} 0 \\ 0 \\ -x_3 \\ 0 \end{pmatrix}; \Rightarrow \frac{\partial}{\partial w_{10}}\mathbf{F}\left(E_2, w_{10}^{**}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, ${\mathbf{U}_2}^{T}\mathbf{F}_{w_{10}}(E_2, w_{10}^{**}) = 0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$D\mathbf{F}_{w_{10}}(\mathbf{X}, w_{10}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow D\mathbf{F}_{w_{10}}\left(E_2, w_{10}^{**}\right)\mathbf{V}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Therefore

$${\mathbf{U}_2}^{T} D\mathbf{F}_{w_{10}}\left(E_2, w_{10}^{**}\right)\mathbf{V}_2 = -1 \neq 0.$$

Also, by using Eq. (57), it is obtained that

$$D^2\mathbf{F}\left(E_2, w_{10}^{**}\right)(\mathbf{V}_2, \mathbf{V}_2) = [\sigma_{i1}]_{4\times 1},$$

where

$$\begin{aligned}\sigma_{11} &= -2n_1(n_1+n_2) - 2n_1n_2w_2 - \frac{2w_3^2(w_2-w_5)w_5}{w_2(1+w_2)} - 2w_4\left(n_1 + \frac{w_5}{w_2}\right) \\ &\quad - \frac{2w_1^2w_5\left(-1 + \frac{w_2-w_5}{w_2(1+w_2)} + \frac{w_5}{w_2}\right)}{w_2} + 2w_2w_3\left(\frac{n_1(w_2-w_5)}{w_2(1+w_2)} + \frac{n_2w_5}{w_2}\right) \\ &\quad + 2w_1\left[\frac{n_2w_5}{w_2} + n_1\left(-1 + \frac{w_2-w_5}{w_2(1+w_2)} + \frac{2w_5}{w_2}\right)\right].\end{aligned}$$

$$\sigma_{21} = 2n_1n_2w_2 - 2w_4\left(n_2 + \frac{w_2-w_5}{w_2(1+w_2)}\right) + \frac{2w_3^2(w_2-w_5)w_5}{w_2(1+w_2)} - 2w_2w_3\left(\frac{n_1(w_2-w_5)}{w_2(1+w_2)} + \frac{n_2w_5}{w_2}\right).$$

$$\sigma_{31} = 2w_6\left[c_2\left(n_2 + \frac{w_2-w_5}{w_2(1+w_2)}\right) + c_1\left(n_1 + \frac{w_5}{w_2}\right)\right].$$

$$\sigma_{41} = 0.$$

Then, when condition 60 is met, it is obtained that

$${\mathbf{U}_2}^{T} D^2\mathbf{F}\left(E_2, w_{10}^{**}\right)(\mathbf{V}_2, \mathbf{V}_2) = 2w_6\left[c_2\left(n_2 + \frac{w_2-w_5}{w_2(1+w_2)}\right) + c_1\left(n_1 + \frac{w_5}{w_2}\right)\right] \neq 0.$$

Thus a TB near PFEP takes place. Otherwise, by using Eq. (58), it is obtained that

$$D^3\mathbf{F}\left(E_2, w_{10}^{**}\right)(\mathbf{V}_2, \mathbf{V}_2, \mathbf{V}_2) = [\rho_{i1}]_{4\times 1},$$

Where

$$\begin{aligned}\rho_{11} &= 6\left[n_1(n_1+n_2)w_1 + n_1n_2w_2w_3 - n_1w_4 + \frac{w_3^3(w_2-w_5)w_5}{w_2(1+w_2)}\right. \\ &\quad + \frac{w_1^3w_5\left(-1 + \frac{w_2-w_5}{w_2(1+w_2)} + \frac{w_5}{w_2}\right)}{w_2} - w_2w_3^2\left(\frac{n_1(w_2-w_5)}{w_2(1+w_2)} + \frac{n_2w_5}{w_2}\right) \\ &\quad \left. - w_1^2\left(\frac{n_2w_5}{w_2} + n_1\left(-1 + \frac{w_2-w_5}{w_2(1+w_2)} + \frac{2w_5}{w_2}\right)\right)\right].\end{aligned}$$

$$\rho_{21} = 6\left[\frac{w_3^2(w_2-w_5)\left(n_1 - \frac{w_3w_5}{w_2}\right)}{1+w_2} - n_2\left(n_1w_2w_3 + w_4 - w_3^2w_5\right)\right].$$

$\rho_{31} = 6\left(c_1 n_1 + c_2 n_2\right) w_6.$

$\rho_{41} = 0.$

Hence direct computation yields that $\mathbf{U}_2{}^T D^3 \mathbf{F}(E_2, w_{10}^{**})(\mathbf{V}_2, \mathbf{V}_2, \mathbf{V}_2) = 6(c_1 n_1 + c_2 n_2) w_6 \neq 0$ due to condition 60.

Thus, PB takes place, and then the proof is complete.

Note that, according to the form of n_2 in the above theorem, it is simply to specify that for $n_2 > 0$, then system 2 has a TB only. However, for $n_2 < 0$, system 2, under condition 60, undergoes a TB or otherwise it has a PB.

Theorem 8: *Assume that conditions 26, 27, and 29 are met, then system 2 undergoes a TB near DFEP when the parameter* w_5 *passes through the value* $w_5^* = -w_4 \hat{x}_3 (1 + \hat{x}_3) + \frac{w_2 w_{10}}{c_1 w_6 (1+\hat{x}_3)(1+w_3\hat{x}_3)}$ *provided that the following condition holds*

$$-2n_5 w_4\left(1 + 2\hat{x}_3\right) - \frac{2n_5 w_2 w_3 w_{10}}{c_1 w_6\left(1 + \hat{x}_3\right)\left(1 + w_3 \hat{x}_3\right)^2} + \frac{2n_4 w_2}{1 + w_3 \hat{x}_3} \neq 0. \quad (61)$$

Where all the new symbols will be defined in the proof. Otherwise, pitchfork bifurcation (PB) takes place provided that the following condition holds.

$$n_4 w_2 w_3\left(1 + w_3 \hat{x}_3\right) + n_5\left(-\frac{w_2 w_3^2 w_{10}}{c_1 w_6\left(1 + \hat{x}_3\right)} + w_4 (1 + w_3 \hat{x}_3)^3\right) \neq 0. \quad (62)$$

Proof: From Eq. (23) with $w_5 = w_5^*$ the JM becomes

$J_3^* = J\left(E_3, w_5^*\right) = \left(\hat{s}_{ij}\right)_{4\times 4},$

Where $\hat{s}_{ij}$ for all $i, j = 1, 2, 3, 4$, are given in Eq. (23) with $\hat{s}_{22}(w_5^*) = 0$. Thus, due to the above-mentioned conditions, J_3^* has the negative real parts eigenvalues given in Eq. (25), with $\lambda_{32}(w_5^*) = 0$. Hence, DFEP is a non-hyperbolic point at $w_5 = w_5^*$.

Let $\mathbf{V}_3 = (v_{31}, v_{32}, v_{33}, v_{34})^{\mathrm{T}}$ and $\mathbf{U}_3 = (u_{31}, u_{32}, u_{33}, u_{34})^{\mathrm{T}}$ are the eigenvectors conjugate with the eigenvalue $\lambda_{32}(w_5^*)$ of J_3^* and J_3^{*T} respectively. Now, direct computation determined that

$$\mathbf{V}_3 = (n_4, 1, n_5, 0)^{\mathrm{T}},\ n_4 = \frac{\hat{s}_{13}\hat{s}_{32} - \hat{s}_{12}\hat{s}_{33}}{\hat{s}_{11}\hat{s}_{33} - \hat{s}_{13}\hat{s}_{31}},\ n_5 = \frac{\hat{s}_{12}\hat{s}_{31} - \hat{s}_{11}\hat{s}_{32}}{\hat{s}_{11}\hat{s}_{33} - \hat{s}_{13}\hat{s}_{31}}.$$

$\mathbf{U}_3 = (0, 1, 0, 0)^T.$

Now, applying Sotomayor's theorem gives that:

$$\frac{\partial}{\partial w_5}\mathbf{F}\left(\mathbf{X}, w_5\right) = \begin{pmatrix} 0 \\ -x_2 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \frac{\partial}{\partial w_5}\mathbf{F}\left(E_3, w_5^*\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, $\mathbf{U}_3{}^{\mathrm{T}}\mathbf{F}_{w_5}(E_3, w_5^*) = 0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$D\mathbf{F}_{w_5}\left(\mathbf{X}, w_5\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow D\mathbf{F}_{w_5}\left(E_3, w_5^*\right)\mathbf{V}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\mathbf{U_3}^{\mathrm{T}} D\mathbf{F}_{w_5}\left(E_3, w_5^*\right)\mathbf{V}_3 = -1 \neq 0.$$

Also, by using Eq. (57), it is obtained that

$$D^2\mathbf{F}\left(E_3, w_5^*\right)(\mathbf{V}_3, \mathbf{V}_3) = [\vartheta_{i1}]_{4\times1},$$

Where

$$\begin{aligned}
\vartheta_{11} &= -\frac{2n_4(1+n_4)}{1+w_1\hat{x}_3} + \frac{2n_5w_2w_3w_{10}}{c_1w_6(1+\hat{x}_3)(1+w_3\hat{x}_3)^2} - \frac{2n_4w_2}{1+w_3\hat{x}_3} - \frac{2n_5^2w_1^2w_{10}\left(-1+\frac{w_{10}}{c_1w_6(1+\hat{x}_3)}\right)}{c_1w_6(1+\hat{x}_3)(1+w_1\hat{x}_3)^3} \\
&\quad - 2n_5w_4\left(\frac{n_5w_{10}}{c_1w_6(1+\hat{x}_3)} + n_4(1+2\hat{x}_3)\right) + \frac{2n_5w_1\left(\frac{w_{10}}{c_1w_6(1+\hat{x}_3)} + n_4\left(-1+\frac{2w_{10}}{c_1w_6(1+\hat{x}_3)}\right)\right)}{(1+w_1\hat{x}_3)^2}. \\
\vartheta_{21} &= -2n_5w_4(1+2\hat{x}_3) - \frac{2n_5w_2w_3w_{10}}{c_1w_6(1+\hat{x}_3)(1+w_3\hat{x}_3)^2} + \frac{2n_4w_2}{1+w_3\hat{x}_3}. \\
\vartheta_{31} &= 2\left[c_2n_5w_6(1+2\hat{x}_3) + c_1n_5w_6\left(\frac{n_5w_{10}}{c_1w_6(1+\hat{x}_3)} + n_4(1+2\hat{x}_3)\right)\right]. \\
\vartheta_{41} &= 0.
\end{aligned}$$

Then, when the condition 61 is met, it is obtained that

$$\mathbf{U_3}^{T} D^2\mathbf{F}\left(E_3, w_5^*\right)(\mathbf{V}_3, \mathbf{V}_3) = -2n_5w_4(1+2\hat{x}_3) - \frac{2n_5w_2w_3w_{10}}{c_1w_6(1+\hat{x}_3)(1+w_3\hat{x}_3)^2} + \frac{2n_4w_2}{1+w_3\hat{x}_3} \neq 0.$$

Thus, a TB near PFEP takes place. Otherwise, by using Eq. (58), it is obtained that

$$D^3\mathbf{F}\left(E_3, w_5^*\right)(\mathbf{V}_3, \mathbf{V}_3, \mathbf{V}_3) = [\mu_{i1}]_{4\times1},$$

Where

$$\begin{aligned}
\mu_{11} &= 6n_5\left[-n_4n_5w_4 + \frac{n_4(1+n_4)w_1}{(1+w_1\hat{x}_3)^2} - \frac{n_5w_2w_3^2w_{10}}{c_1w_6(1+\hat{x}_3)(1+w_3\hat{x}_3)^3} + \frac{n_4w_2w_3}{(1+w_3\hat{x}_3)^2}\right. \\
&\quad \left. + \frac{n_5^2w_1^3w_{10}\left(-1+\frac{w_{10}}{c_1w_6(1+\hat{x}_3)}\right)}{c_1w_6(1+\hat{x}_3)(1+w_1\hat{x}_3)^4} - \frac{n_5w_1^2\left(\frac{w_{10}}{c_1w_6(1+\hat{x}_3)} + n_4\left(-1+\frac{2w_{10}}{c_1w_6(1+\hat{x}_3)}\right)\right)}{(1+w_1\hat{x}_3)^3}\right]. \\
\mu_{21} &= -\frac{6n_5\left(n_4w_2w_3(1+w_3\hat{x}_3) + n_5\left(-\frac{w_2w_3^2w_{10}}{c_1w_6(1+\hat{x}_3)} + w_4(1+w_3\hat{x}_3)^3\right)\right)}{(1+w_3\hat{x}_3)^3}. \\
\mu_{31} &= 6(c_2+c_1n_4)n_5^2w_6. \\
\mu_{41} &= 0.
\end{aligned}$$

According to condition 62, it is obtained that

$$\mathbf{U_3}^{T} D^3\mathbf{F}\left(E_3, w_5^*\right)(\mathbf{V}_3, \mathbf{V}_3, \mathbf{V}_3) = \mu_{21} \neq 0$$

Thus, PB takes place, and then the proof is complete.

Theorem 9: *Assume that conditions 32, 33, 35, and 36 are met, then system 2 undergoes a TB near PDFEP when the parameter w_7 passes through the value $w_7^* = \frac{w_8+w_9w_{11}}{w_9\check{x}_3}$ provided that the following condition is met.*

$$n_8 w_7^* + \frac{w_8}{w_9^2} \neq 0, \tag{63}$$

Where all the new symbols will be defined in the proof. Otherwise, pitchfork bifurcation (PB) takes place.

Proof: From Eq. (30) with $w_7 = w_7^*$ the JM becomes

$$J_4^* = J\left(E_4, w_7^*\right) = \left(\check{s}_{ij}\right)_{4\times4},$$

Where $\check{s}_{ij}$ for all $i, j = 1, 2, 3, 4$, are given in Eq. (30) with $\check{s}_{34}(w_7^*) = -w_{11} + w_6(c_1\check{x}_1 + c_2\check{x}_2)(1 + 2\check{x}_3)$ and $\hat{s}_{44}(w_7^*) = 0$. Thus, depending on the above-mentioned conditions, J_4^* has three negative real parts roots (eigenvalues) of the characteristic equation that is given in Eq. (31) due to the Routh-Hurwitz criterion, with $\lambda_{44}(w_7^*) = 0$. Hence, PDFEP is a non-hyperbolic point at $w_7 = w_7^*$.

Let $\mathbf{V}_4 = (v_{41}, v_{42}, v_{43}, v_{44})^{\mathrm{T}}$ and $\mathbf{U}_4 = (u_{41}, u_{42}, u_{43}, u_{44})^{\mathrm{T}}$ are the eigenvectors conjugate with the eigenvalue $\lambda_{44}(w_7^*)$ of J_4^* and J_4^{*T} respectively. Now, direct computation determined that

$$\mathbf{V}_4 = (n_6, n_7, n_8, 1)^{\mathrm{T}},$$

Where

$$n_6 = \frac{-\check{s}_{14}\check{s}_{23}\check{s}_{32} + \check{s}_{13}\check{s}_{24}\check{s}_{32} - \check{s}_{12}\check{s}_{24}\check{s}_{33} + \check{s}_{12}\check{s}_{23}\check{s}_{34}}{-\check{s}_{12}\check{s}_{23}\check{s}_{31} - \check{s}_{13}\check{s}_{21}\check{s}_{32} + \check{s}_{11}\check{s}_{23}\check{s}_{32} + \check{s}_{12}\check{s}_{21}\check{s}_{33}}.$$
$$n_7 = \frac{\check{s}_{14}\check{s}_{23}\check{s}_{31} - \check{s}_{13}\check{s}_{24}\check{s}_{31} - \check{s}_{14}\check{s}_{21}\check{s}_{33} + \check{s}_{11}\check{s}_{24}\check{s}_{33} + \check{s}_{13}\check{s}_{21}\check{s}_{34} - \check{s}_{11}\check{s}_{23}\check{s}_{34}}{-\check{s}_{12}\check{s}_{23}\check{s}_{31} - \check{s}_{13}\check{s}_{21}\check{s}_{32} + \check{s}_{11}\check{s}_{23}\check{s}_{32} + \check{s}_{12}\check{s}_{21}\check{s}_{33}}.$$
$$n_8 = \frac{\check{s}_{12}\check{s}_{24}\check{s}_{31} + \check{s}_{14}\check{s}_{21}\check{s}_{32} - \check{s}_{11}\check{s}_{24}\check{s}_{32} - \check{s}_{12}\check{s}_{21}\check{s}_{34}}{-\check{s}_{12}\check{s}_{23}\check{s}_{31} - \check{s}_{13}\check{s}_{21}\check{s}_{32} + \check{s}_{11}\check{s}_{23}\check{s}_{32} + \check{s}_{12}\check{s}_{21}\check{s}_{33}}.$$

While

$$\mathbf{U}_4 = (0, 0, 0, 1)^T.$$

Now, applying Sotomayor's theorem gives that:

$$\frac{\partial}{\partial w_7}\mathbf{F}\left(\mathbf{X}, w_7\right) = \begin{pmatrix} 0 \\ 0 \\ -x_3x_4 \\ x_3x_4 \end{pmatrix} \Rightarrow \frac{\partial}{\partial w_7}\mathbf{F}\left(E_4, w_7^*\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, $\mathbf{U}_4{}^{\mathrm{T}}\mathbf{F}_{w_7}(E_4, w_7^*) = 0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$D\mathbf{F}_{w_7}\left(\mathbf{X}, w_7\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x_4 & -x_3 \\ 0 & 0 & x_4 & x_3 \end{bmatrix} \Rightarrow D\mathbf{F}_{w_7}\left(E_4, w_7^*\right)\mathbf{V}_4 = \begin{bmatrix} 0 \\ 0 \\ -\check{x}_3 \\ \check{x}_3 \end{bmatrix}.$$

Therefore

$$\mathbf{U}_4{}^{\mathrm{T}}D\mathbf{F}_{w_7}\left(E_4, w_7^*\right)\mathbf{V}_4 = \check{x}_3 \neq 0.$$

Also, by using Eq. (57), it is obtained that

$$D^2\mathbf{F}\left(E_4, w_7^*\right)(\mathbf{V}_4, \mathbf{V}_4) = [\tau_{i1}]_{4\times 1},$$

Where

$$\tau_{11} = -\frac{2(1+n_8)^2 w_1^2 \check{x}_1(-1+\check{x}_1+\check{x}_2)}{(1+w_1\check{x}_3)^3} + \frac{2(1+n_8)w_1[n_7\check{x}_1+n_6(-1+2\check{x}_1+\check{x}_2)]}{(1+w_1\check{x}_3)^2} - \frac{2n_6(n_6+n_7)}{1+w_1\check{x}_3}$$
$$-\frac{2(1+n_8)^2 w_2 w_3^2 \check{x}_1\check{x}_2}{(1+w_3\check{x}_3)^3} + \frac{2(1+n_8)w_2w_3(n_7\check{x}_1+n_6\check{x}_2)}{(1+w_3\check{x}_3)^2} - \frac{2n_6n_7w_2}{1+w_3\check{x}_3}$$
$$-2(1+n_8)w_4[(1+n_8)\check{x}_1+n_6(1+2\check{x}_3)].$$

$$\tau_{21} = \frac{2(1+n_8)^2 w_2 w_3^2 \check{x}_1\check{x}_2}{(1+w_3\check{x}_3)^3} - \frac{2(1+n_8)w_2w_3(n_7\check{x}_1+n_6\check{x}_2)}{(1+w_3\check{x}_3)^2} + \frac{2n_6n_7w_2}{1+w_3\check{x}_3}$$
$$-2(1+n_8)w_4[(1+n_8)\check{x}_2+n_7(1+2\check{x}_3)].$$

$$\tau_{31} = 2\left[-n_8w_7^* - \frac{w_8}{w_9^2} + c_1(1+n_8)w_6[(1+n_8)\check{x}_1+n_6(1+2\check{x}_3)]\right.$$
$$\left.+c_2(1+n_8)w_6[(1+n_8)\check{x}_2+n_7(1+2\check{x}_3)]\right].$$

$$\tau_{41} = 2\left(n_8w_7^* + \frac{w_8}{w_9^2}\right).$$

Then, when the condition 63 is met, it is obtained that

$$\mathbf{U}_4{}^T D^2\mathbf{F}\left(E_4, w_7^*\right)(\mathbf{V}_4, \mathbf{V}_4) = \tau_{41} \neq 0.$$

Thus, a TB near PDFEP takes place. Otherwise, by using Eq. (58), it is obtained that:

$$D^3\mathbf{F}\left(E_4, w_7^*\right)(\mathbf{V}_4, \mathbf{V}_4, \mathbf{V}_4) = [\theta_{i1}]_{4\times 1},$$

Where

$$\theta_{11} = 6(1+n_8)\left[-n_6(1+n_8)w_4 + \frac{(1+n_8)^2 w_1^3\check{x}_1(-1+\check{x}_1+\check{x}_2)}{(1+w_1\check{x}_3)^4}\right.$$
$$-\frac{(1+n_8)w_1^2[n_7\check{x}_1+n_6(-1+2\check{x}_1+\check{x}_2)]}{(1+w_1\check{x}_3)^3} + \frac{n_6(n_6+n_7)w_1}{(1+w_1\check{x}_3)^2}$$
$$\left.+\frac{(1+n_8)^2 w_2 w_3^3\check{x}_1\check{x}_2}{(1+w_3\check{x}_3)^4} - \frac{(1+n_8)w_2w_3^2(n_7\check{x}_1+n_6\check{x}_2)}{(1+w_3\check{x}_3)^3} + \frac{n_6n_7w_2w_3}{(1+w_3\check{x}_3)^2}\right].$$

$$\theta_{21} = \frac{6(1+n_8)}{(1+w_3\check{x}_3)^4}\left[(1+n_8)w_2w_3^2\check{x}_2((-1-n_8)w_3\check{x}_1+n_6(1+w_3\check{x}_3))\right.$$
$$\left.-n_7(1+w_3\check{x}_3)\left(n_6w_2w_3(1+w_3\check{x}_3)+(1+n_8)\left[-w_2w_3^2\check{x}_1+w_4(1+w_3\check{x}_3)^3\right]\right)\right].$$

$$\theta_{31} = 6(c_1n_6+c_2n_7)(1+n_8)^2 w_6 + \frac{6w_8}{w_9^3}.$$

$$\theta_{41} = -\frac{6w_8}{w_9^3}.$$

Accordingly, it is obtained that:

$$\mathbf{U}_4{}^T D^3 \mathbf{F}\left(E_4, w_7^*\right)(\mathbf{V}_4, \mathbf{V}_4, \mathbf{V}_4) = -\frac{6w_8}{w_9^3} \neq 0.$$

Thus, PB takes place, and then the proof is complete.

Theorem 10: *Assume that conditions 40, 41, 42, 43, and 44 are met, then system 2 undergoes a TB near HPEP when the parameter w_2 passes through the value $w_2^* = \frac{1+w_3(\tilde{x}_3+\tilde{x}_4)}{\tilde{x}_1}[w_5 + w_4(\tilde{x}_3+\tilde{x}_4)(1+\tilde{x}_3+\tilde{x}_4)]$ provided that the following condition is met.*

$$\begin{aligned} &-2n_{10}(1+n_{11})w_4(1+2\tilde{x}_3) - 4n_{10}(1+n_{11})w_4\tilde{x}_4 \\ &\quad -\frac{2n_{10}(1+n_{11})w_2w_3\tilde{x}_1}{(1+w_3(\tilde{x}_3+\tilde{x}_4))^2} + \frac{2n_9n_{10}w_2}{1+w_3(\tilde{x}_3+\tilde{x}_4)} \neq 0, \end{aligned} \tag{64}$$

Where all the new symbols will be defined in the proof. Otherwise, pitchfork bifurcation (PB) takes place if the following condition holds.

$$\begin{aligned} &6n_{10}(1+n_{11})[n_9w_2w_3(1+w_3(\tilde{x}_3+\tilde{x}_4)) + (1+n_{11}) \\ &\quad \left(-w_2w_3^2\tilde{x}_1 + w_4(1+w_3(\tilde{x}_3+\tilde{x}_4))^3\right)] \neq 0 \end{aligned}. \tag{65}$$

Proof: From the Eq. (37) with $w_2 = w_2^*$ the JM becomes

$$J_5^* = J\left(E_5, w_2^*\right) = \left(\tilde{s}_{ij}\right)_{4\times 4},$$

Where $\tilde{s}_{ij}$ for all $i, j = 1, 2, 3, 4$, are given in Eq. (37) and $\tilde{s}_{22}(w_2^*) = 0$. Thus, depending on the above-mentioned conditions, J_5^* has three negative real parts roots (eigenvalues) of the characteristic equation that is given in Eq. (38) due to the Routh-Hurwitz criterion, with $\lambda_{52}(w_2^*) = 0$. Hence, HPEP is a non-hyperbolic point at $w_2 = w_2^*$.

Let $\mathbf{V}_5 = (v_{51}, v_{52}, v_{53}, v_{54})^{\mathrm{T}}$ and $\mathbf{U}_5 = (u_{51}, u_{52}, u_{53}, u_{54})^{\mathrm{T}}$ are the eigenvectors conjugate with the eigenvalue $\lambda_{52}(w_2^*)$ of J_5^* and J_5^{*T} respectively. Now, direct computation determined that

$$\mathbf{V}_5 = (n_9, n_{10}, n_{11}, 1)^{\mathrm{T}},$$

Where

$$n_9 = \frac{\tilde{s}_{14}\tilde{s}_{32}\tilde{s}_{43} - \tilde{s}_{12}\tilde{s}_{34}\tilde{s}_{43} - \tilde{s}_{13}\tilde{s}_{32}\tilde{s}_{44} + \tilde{s}_{12}\tilde{s}_{33}\tilde{s}_{44}}{(\tilde{s}_{12}\tilde{s}_{31} - \tilde{s}_{11}\tilde{s}_{32})\tilde{s}_{43}}.$$

$$n_{10} = \frac{-\tilde{s}_{14}\tilde{s}_{31}\tilde{s}_{43} + \tilde{s}_{11}\tilde{s}_{34}\tilde{s}_{43} + \tilde{s}_{13}\tilde{s}_{31}\tilde{s}_{44} - \tilde{s}_{11}\tilde{s}_{33}\tilde{s}_{44}}{(\tilde{s}_{12}\tilde{s}_{31} - \tilde{s}_{11}\tilde{s}_{32})\tilde{s}_{43}}.$$

$$n_{11} = -\frac{\tilde{s}_{44}}{\tilde{s}_{43}} < 0.$$

While

$$\mathbf{U}_5 = (0, 1, 0, 0)^T.$$

Now, applying Sotomayor's theorem gives that:

$$\frac{\partial}{\partial w_2}\mathbf{F}(\mathbf{X}, w_2) = \begin{pmatrix} 0 \\ \frac{x_1x_2}{1+w_3(x_3+x_4)} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \frac{\partial}{\partial w_2}\mathbf{F}(E_5, w_2^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $\mathbf{U}_5{}^{\mathrm{T}}\mathbf{F}_{w_2}(E_5, w_2^*) = 0$, as a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, since

$$D\mathbf{F}_{w_2}(\mathbf{X}, w_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{x_2}{1+w_3(x_3+x_4)} & \frac{x_1}{1+w_3(x_3+x_4)} & \frac{-w_3x_1x_2}{(1+w_3(x_3+x_4))^2} & \frac{-w_3x_1x_2}{(1+w_3(x_3+x_4))^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Which gives:

$$D\mathbf{F}_{w_2}\left(E_5, w_2^*\right)\mathbf{V}_5 = \begin{bmatrix} 0 \\ \frac{\tilde{x}_1 n_{10}}{1+w_3(\tilde{x}_3+\tilde{x}_4)} \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\mathbf{U}_5{}^{\mathrm{T}}D\mathbf{F}_{w_2}\left(E_5, w_2^*\right)\mathbf{V}_5 = \frac{\tilde{x}_1 n_{10}}{1 + w_3(\tilde{x}_3 + \tilde{x}_4)} \neq 0.$$

Also, by using Eq. (57), it is obtained that

$$D^2\mathbf{F}\left(E_5, w_2^*\right)(\mathbf{V}_5, \mathbf{V}_5) = [\epsilon_{i1}]_{4\times 1},$$

Where

$$\begin{aligned}\epsilon_{11} &= -2(1+n_{11})w_4[(1+n_{11})\tilde{x}_1 + n_9(1+2\tilde{x}_3)] - 4n_9(1+n_{11})w_4\tilde{x}_4 \\ &\quad - \frac{2(1+n_{11})^2 w_1^2(-1+\tilde{x}_1)\tilde{x}_1}{(1+w_1(\tilde{x}_3+\tilde{x}_4))^3} + \frac{2(1+n_{11})w_1(n_{10}\tilde{x}_1 + n_9(-1+2\tilde{x}_1))}{(1+w_1(\tilde{x}_3+\tilde{x}_4))^2} - \frac{2n_9(n_9+n_{10})}{1+w_1(\tilde{x}_3+\tilde{x}_4)} \\ &\quad + \frac{2n_{10}(1+n_{11})w_2w_3\tilde{x}_1}{(1+w_3(\tilde{x}_3+\tilde{x}_4))^2} - \frac{2n_9n_{10}w_2}{1+w_3(\tilde{x}_3+\tilde{x}_4)}.\\ \epsilon_{21} &= -2n_{10}(1+n_{11})w_4(1+2\tilde{x}_3) - 4n_{10}(1+n_{11})w_4\tilde{x}_4 - \frac{2n_{10}(1+n_{11})w_2w_3\tilde{x}_1}{(1+w_3(\tilde{x}_3+\tilde{x}_4))^2} + \frac{2n_9n_{10}w_2}{1+w_3(\tilde{x}_3+\tilde{x}_4)}.\\ \epsilon_{31} &= 2\left[-n_{11}w_7 - \frac{w_8w_9}{(w_9+\tilde{x}_4)^3} + c_2n_{10}(1+n_{11})w_6(1+2\tilde{x}_3+2\tilde{x}_4)\right.\\ &\quad \left. + c_1(1+n_{11})w_6[(1+n_{11})\tilde{x}_1 + n_9(1+2\tilde{x}_3+2\tilde{x}_4)]\right].\\ \epsilon_{41} &= 2\left(n_{11}w_7 + \frac{w_8w_9}{(w_9+\tilde{x}_4)^3}\right).\end{aligned}$$

Then, when condition 64 is met, it is obtained that

$$\mathbf{U}_5{}^{T}D^2\mathbf{F}\left(E_5, w_2^*\right)(\mathbf{V}_5, \mathbf{V}_5) = \epsilon_{21} \neq 0.$$

Thus, a TB near PDFEP takes place. Otherwise, by using Eq. (58), it is obtained that

$$D^3\mathbf{F}\left(E_5, w_2^*\right)(\mathbf{V}_5, \mathbf{V}_5, \mathbf{V}_5) = [\mu_{i1}]_{4\times 1},$$

Where

$$\mu_{11} = 6(1+n_{11})\Bigg[-n_9(1+n_{11})w_4 + \frac{(1+n_{11})^2 w_1^3(-1+\tilde{x}_1)\tilde{x}_1}{(1+w_1(\tilde{x}_3+\tilde{x}_4))^4} - \frac{(1+n_{11})w_1^2(n_{10}\tilde{x}_1+n_9(-1+2\tilde{x}_1))}{(1+w_1(\tilde{x}_3+\tilde{x}_4))^3}$$
$$+\frac{n_9(n_9+n_{10})w_1}{(1+w_1(\tilde{x}_3+\tilde{x}_4))^2} - \frac{n_{10}(1+n_{11})w_2w_3^2\tilde{x}_1}{(1+w_3(\tilde{x}_3+\tilde{x}_4))^3} + \frac{n_9n_{10}w_2w_3}{(1+w_3(\tilde{x}_3+\tilde{x}_4))^2}\Bigg].$$
$$\mu_{21} = -\frac{6n_{10}(1+n_{11})\lceil n_9w_2w_3(1+w_3(\tilde{x}_3+\tilde{x}_4)) + (1+n_{11})\left(-w_2w_3^2\tilde{x}_1 + w_4(1+w_3(\tilde{x}_3+\tilde{x}_4))^3\right)\rceil}{(1+w_3(\tilde{x}_3+\tilde{x}_4))^3}.$$
$$\mu_{31} = 6(c_1n_9+c_2n_{10})(1+n_{11})^2w_6 + \frac{6w_8w_9}{(w_9+\tilde{x}_4)^4}.$$
$$\mu_{41} = -\frac{6w_8w_9}{(w_9+\tilde{x}_4)^4}.$$

Then, when condition 65 is met, it is obtained that

$$\mathbf{U}_5{}^T D^3\mathbf{F}\left(E_5, w_2^*\right)(\mathbf{V}_5, \mathbf{V}_5, \mathbf{V}_5) = \mu_{21} \neq 0.$$

Thus, a PB near HPEP takes place.

Theorem 11: *Assume that conditions 47–52 are met, and then system 2 undergoes a saddle-node bifurcation (SNB) near CEP when the parameter* w_8 *passes through the value* w_8^* *that is given below provided that the following conditions are met.*

$$n_{17} - 1 \neq 0. \tag{66}$$
$$n_{15}b_{11}^* + n_{16}b_{21}^* + n_{17}b_{31}^* + b_{41}^* \neq 0. \tag{67}$$

Where all the new symbols are defined in the proof and that:

$$w_8^* = \frac{s_{43}^*(w_9+x_4^*)^2\left[\left(s_{12}^*s_{24}^*s_{31}^* + s_{14}^*s_{21}^*s_{32}^* - s_{11}^*s_{24}^*s_{32}^*\right) - s_{12}^*s_{21}^*\left(-w_7x_3^* + w_6\left(c_1x_1^*+c_2x_2^*\right)\left(1+2x_3^*+2x_4^*\right)\right)\right]}{\left(s_{12}^*s_{23}^*s_{31}^* + s_{13}^*s_{21}^*s_{32}^* - s_{11}^*s_{23}^*s_{32}^* - s_{12}^*s_{21}^*s_{33}^*\right)x_4^* + w_9s_{12}^*s_{21}^*s_{43}^*}.$$

Proof: From Eq. (45) with $w_8 = w_8^*$ the JM becomes

$$J_6^* = J\left(E_6, w_8^*\right) = \left[s_{ij}^*\right]_{4\times4},$$

Where s_{ij}^* for all $i, j = 1, 2, 3, 4$, are given in Eq. (45) with $s_{34}^*(w_8^*) = -w_7x_3^* + \frac{w_8^*w_9}{(w_9+x_4^*)^2} + w_6(c_1x_1^*+c_2x_2^*)(1+2x_3^*+2x_4^*)$ and $s_{44}^*(w_8^*) = \frac{w_8^*x_4^*}{(w_9+x_4^*)^2}$. Thus, depending on the above-mentioned conditions, the determinant of J_6^* that is given by A_4 in the Eq. (46) becomes $A_4 = 0$. Thus, J_6^* has a zero eigenvalue represented by $\lambda_{64}(w_8^*) = 0$. Hence, CEP is a non-hyperbolic point at $w_8 = w_8^*$.

Let $\mathbf{V}_6 = (v_{61}, v_{62}, v_{63}, v_{64})^{\mathrm{T}}$ and $\mathbf{U}_6 = (u_{61}, u_{62}, u_{63}, u_{64})^{\mathrm{T}}$ are the eigenvectors conjugate with the eigenvalue $\lambda_{64}(w_8^*)$ of J_6^* and J_6^{*T} respectively. Now, direct computation determined that

$$\mathbf{V}_6 = (n_{12}, n_{13}, n_{14}, 1)^{\mathrm{T}},$$

Where

$$n_{12} = \frac{s_{12}^*s_{23}^*s_{34}^* - s_{12}^*s_{24}^*s_{33}^* + s_{13}^*s_{24}^*s_{32}^* - s_{14}^*s_{23}^*s_{32}^*}{s_{11}^*s_{23}^*s_{32}^* - s_{12}^*s_{23}^*s_{31}^* - s_{13}^*s_{21}^*s_{32}^* + s_{12}^*s_{21}^*s_{33}^*}.$$
$$n_{13} = \frac{-s_{11}^*\left(s_{23}^*s_{34}^* - s_{24}^*s_{33}^*\right) - s_{31}^*\left(s_{13}^*s_{24}^* - s_{14}^*s_{23}^*\right) + s_{21}^*\left(s_{13}^*s_{34}^* - s_{14}^*s_{33}^*\right)}{s_{11}^*s_{23}^*s_{32}^* - s_{12}^*s_{23}^*s_{31}^* - s_{13}^*s_{21}^*s_{32}^* + s_{12}^*s_{21}^*s_{33}^*}.$$

$$n_{14} = \frac{-s_{11}^* s_{24}^* s_{32}^* + s_{12}^* s_{24}^* s_{31}^* + s_{14}^* s_{21}^* s_{32}^* - s_{12}^* s_{21}^* s_{34}^*}{s_{11}^* s_{23}^* s_{32}^* - s_{12}^* s_{23}^* s_{31}^* - s_{13}^* s_{21}^* s_{32}^* + s_{12}^* s_{21}^* s_{33}^*}.$$

While

$$\mathbf{U}_6 = (n_{15}, n_{16}, n_{17}, 1)^T,$$

Where

$$n_{15} = \frac{s_{21}^* s_{32}^* s_{43}^*}{s_{11}^* s_{23}^* s_{32}^* - s_{12}^* s_{23}^* s_{31}^* - s_{13}^* s_{21}^* s_{32}^* + s_{12}^* s_{21}^* s_{33}^*}.$$
$$n_{16} = \frac{s_{43}^* \left(s_{12}^* s_{31}^* - s_{11}^* s_{32}^*\right)}{s_{11}^* s_{23}^* s_{32}^* - s_{12}^* s_{23}^* s_{31}^* - s_{13}^* s_{21}^* s_{32}^* + s_{12}^* s_{21}^* s_{33}^*}.$$
$$n_{17} = -\frac{s_{12}^* s_{21}^* s_{43}^*}{s_{11}^* s_{23}^* s_{32}^* - s_{12}^* s_{23}^* s_{31}^* - s_{13}^* s_{21}^* s_{32}^* + s_{12}^* s_{21}^* s_{33}^*}.$$

Now, applying Sotomayor's theorem gives that:

$$\frac{\partial}{\partial w_8}\mathbf{F}(\mathbf{X}, w_8) = \begin{pmatrix} 0 \\ 0 \\ \frac{x_4}{w_9+x_4} \\ -\frac{x_4}{w_9+x_4} \end{pmatrix} \Rightarrow \frac{\partial}{\partial w_8}\mathbf{F}\left(E_6, w_8^*\right) = \begin{pmatrix} 0 \\ 0 \\ \frac{x_4^*}{w_9+x_4^*} \\ -\frac{x_4^*}{w_9+x_4^*} \end{pmatrix}$$

Therefore, due to condition 66, ${\mathbf{U}_6}^{\mathrm{T}}\mathbf{F}_{w_8}(E_6, w_8^*) = \frac{x_4^*(n_{17}-1)}{w_9+x_4^*} \neq 0$.

As a result, the first condition for the occurrence of transcritical bifurcation is met. Moreover, following the Eq. (57), it yields that

$$D^2\mathbf{F}\left(E_6, w_8^*\right)(\mathbf{V}_6, \mathbf{V}_6) = \left[b_{i1}^*\right]_{4\times1},$$

Where $b_{i1}^* = b_{i1}(E_6, w_8^*)$ for $i = 1, 2, 3, 4$. Hence, direct computation using condition 67 gives that:

$${\mathbf{U}_6}^{\mathrm{T}}\left[D^2\mathbf{F}\left(E_6, w_8^*\right)(\mathbf{V}_6, \mathbf{V}_6)\right] \neq 0.$$

Hence, SNB near CEP when $w_8 = w_8^*$, which completes the proof.

From the above theorem it can be concluded that, in biological systems, a saddle-node bifurcation denotes a critical threshold at which a significant shift in the system's behavior occurs. This kind of bifurcation occurs when a parameter is changed, causing two equilibrium points an unstable saddle point and a stable node to clash and eventually annihilate one another. This may cause abrupt changes in the system's condition. Saddle-node bifurcations are frequently used to characterize circumstances in which slow alterations in environmental characteristics or conditions can result in sudden shifts in ecosystem states or population sizes.

Numerical solution

In this section, system 2 is solved numerically to confirm the previously obtained findings and predict the control set of parameters. Several sets of initial conditions are adopted in this investigation which are given by $p_{01} = (0.9, 0.9, 0.9, 0.9)$, $p_{02} = (0.75, 0.75, 0.75, 0.75)$, $p_{03} = (0.5, 0.5, 0.5, 0.5)$, $p_{04} = (0.25, 0.25, 0.25, 0.25)$, and $p_{05} = (0.1, 0.1, 0.1, 0.1)$ with the following set of hypothetical biologically feasible parameters.

To predict the control set of parameters and validate the previously obtained results, system 2 is numerically solved in this section. In this inquiry, many initial condition sets are used, with the following set of hypothetical biologically feasible parameters assigned to p_{01}, p_{02}, p_{03}, p_{04}, and p_{05}. MATLAB R2021a is used to solve and present the results in projections of phase portraits and the time series of the obtained trajectories. The blue

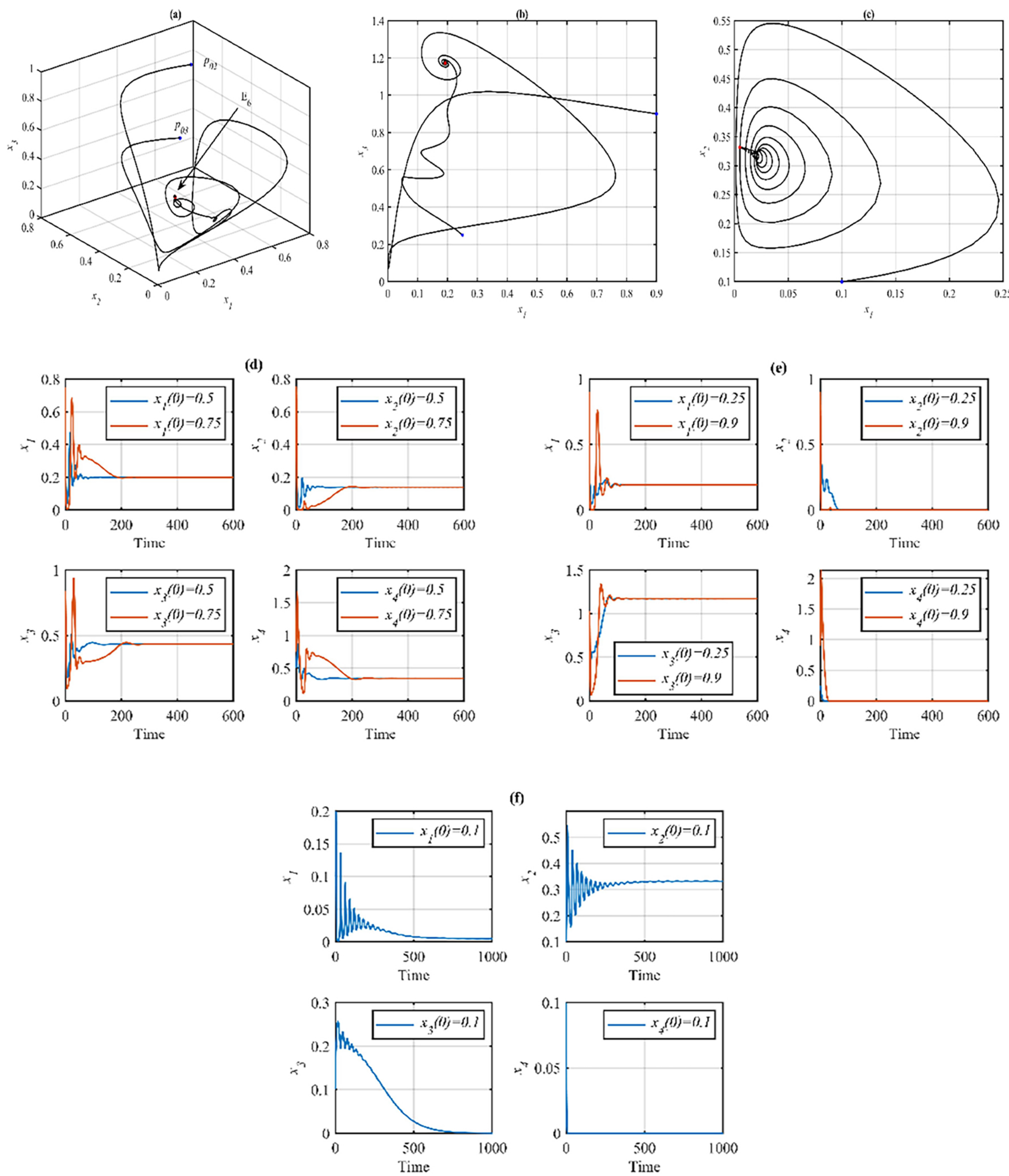

Fig. 1. The trajectories of system 2 using dataset 68. (a) 3D projection of phase portrait for trajectories starting from p_{02} and p_{03}, which are approaching E_6. (b) 2D projection of phase portrait for trajectories starting from p_{01} and p_{04}, which are approaching E_3. (c) 2D projection of phase portrait for trajectory starting from p_{05} that approaches E_2. (d) Time series of trajectories starting from p_{02} and p_{03}. (e) Time series of trajectories starting from p_{01} and p_{04}. (f) Time series of trajectory starting from p_{05}.

dots refer to the initial conditions, while the red dots refer to the attracting equilibrium points.

$$\begin{aligned} &w_1 = 0.5, w_2 = 2,\ w_3 = 0.5, w_4 = 0.2, w_5 = 0.01, w_6 = 0.4,\ w_7 = 0.75 \\ &\quad w_8 = 0.1, w_9 = 0.1, w_{10} = 0.05,\ w_{11} = 0.1,\ c_1 = 0.3,\ c_2 = 0.3. \end{aligned} \tag{68}$$

It is observed that the trajectories of system 2 converge asymptotically to three different equilibrium points, which are given by $E_2 = (0.005, 0.331, 0, 0)$, $E_3 = (0.19, 0, 1, 17, 0)$, and $E_6 = (0.199, 0.139, 0, 433, 0.343)$, simultaneously, indicating the existence of tri-stable dynamic behavior, see Fig. 1.

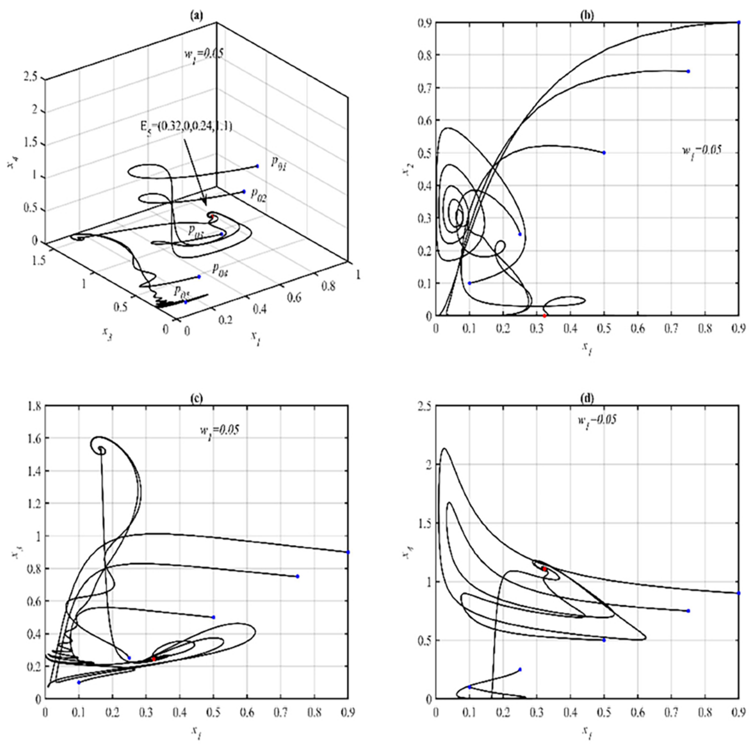

Fig. 2. The trajectories of system 2 using the dataset 68 with $w_1 = 0.05$. (a) 3D projection of phase portrait for trajectories starting from $p_{0i}; i = 1, 2, \ldots, 5$, which are approaching E_5. (b) 2D projection of phase portrait in the x_1x_2-plane. (c) 2D projection of phase portrait in the x_1x_3-plane. (d) 2D projection of phase portrait in the x_1x_4-plane.

According to Fig. 1, system 2 has no globally stable behavior and persistence. There is a basin of attraction for each equilibrium point, and the attracting point is dependent on the initial point. This is due to the possibility of the existence of multiple equilibrium points in the 3D and 4D spaces, which complicates the dynamic behavior around them.

Now, the influence of varying the parameter value on the dynamic behavior of system 2 is investigated. It is observed that for the values $w_1 < 0.1$ all the trajectories starting from the above different initial values approach E_5 as shown in Fig. 2. For the range $0.1 \le w_1 < 0.37$ the trajectories approaches E_3, and E_5 that is indicating bi-stable behavior as presented in Fig. 3. For, the values $0.37 \le w_1 < 0.4$ the system approaches E_3, E_5, and E_6 as given in Fig. 4, which indicates tri-stable behavior. Now, for the range $0.4 \le w_1 < 0.43$ the solutions of the system 2 approach asymptotically E_2, E_3, E_5, and E_6, which refers to four-stable behavior as explained in Fig. 5. However, the system 2 undergoes a tri-stable behavior as their solutions starting from different initial points approach asymptotically to E_2, E_3, and E_6, for $0.43 \le w_1 < 0.6$ see Fig. 1. It is observed that the CEP becomes unstable and system 2 undergoes tri-stable behavior among E_2, E_3, and 4D-periodic for the range $0.6 \le w_1 < 0.64$ as shown in Fig. 6. The system 2 loses its possibility of persistence at the 4D-periodic attractor and undergoes a bi-stable behavior between E_2, and E_3 for the range $0.64 \le w_1 < 1.53$ as explored in Fig. 7. Finally, system 2 approaches asymptotically to E_2 starting from different initial points as explored in Fig. 8 for the range $1.53 \le w_1$.

Similarly, the influence of varying other parameters is investigated and the obtained results are summarized in Table 2.

Results and discussion

In this paper, a prey-predator model that considers hunting cooperation and fear is created when infectious diseases exist in both populations. The idea of applying these biological factors in the case of the existence of the disease in both species is biologically realistic and new to our knowledge. The goals are to identify how hunting cooperation contributes to the anxiety that is created by the predation process and to comprehend the

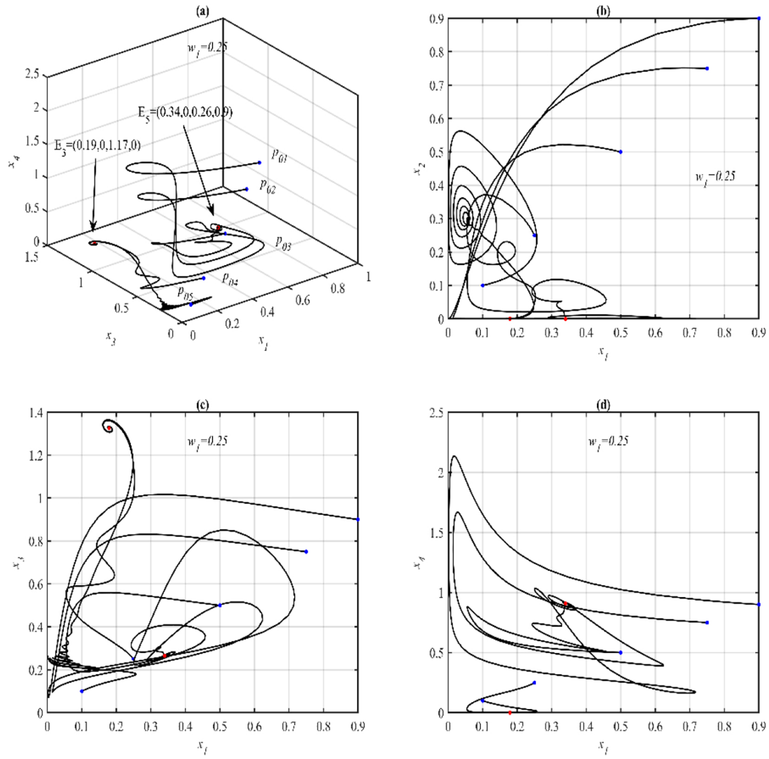

Fig. 3. The trajectories of system 2 using the dataset 68 with $w_1 = 0.25$. (a) 3D projection of phase portrait for trajectories starting from $p_{0i}; i = 1, 2, \ldots, 5$, which are approaching E_3 and E_5. (b) 2D projection of phase portrait in the x_1x_2-plane. (c) 2D projection of phase portrait in the x_1x_3-plane. (d) 2D projection of phase portrait in the x_1x_4-plane.

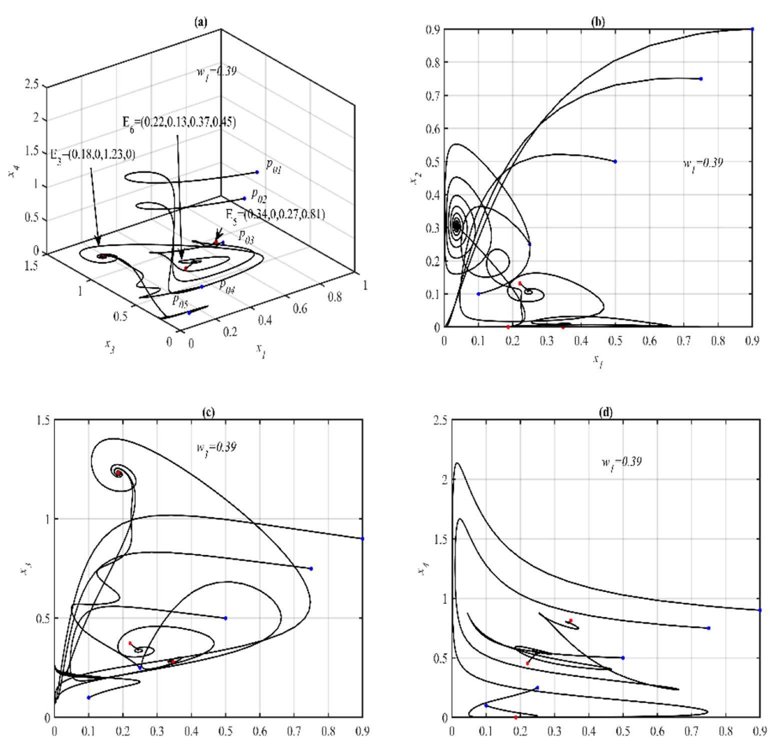

Fig. 4. The trajectories of system 2 using the dataset 68 with $w_1 = 0.39$. (a) 3D projection of phase portrait for trajectories starting from $p_{0i}; i = 1, 2, \ldots, 5$, which are approaching E_3, E_5 and E_6. (b) 2D projection of phase portrait in the x_1x_2-plane. (c) 2D projection of phase portrait in the x_1x_3-plane. (d) 2D projection of phase portrait in the x_1x_4-plane.

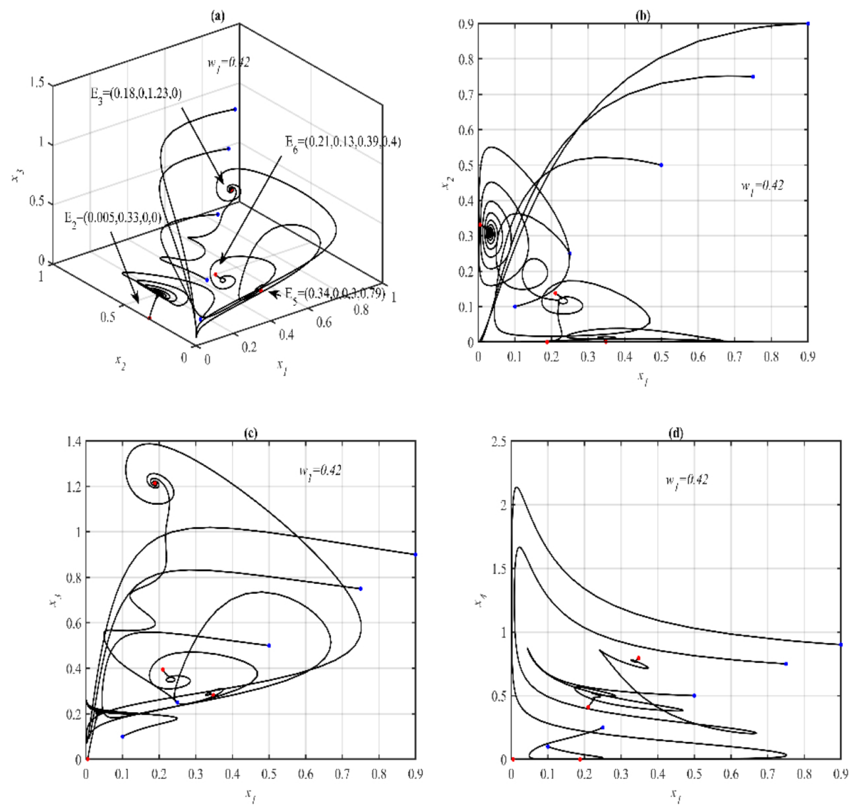

Fig. 5. The trajectories of system 2 using the dataset 68 with $w_1 = 0.42$. (a) 3D projection of phase portrait for trajectories starting from p_{0i}; $i = 1, 2, \ldots, 5$, which are approaching E_2, E_3, E_5 and E_6. (b) 2D projection of phase portrait in the x_1x_2−plane. (c) 2D projection of phase portrait in the x_1x_3−plane. (d) 2D projection of phase portrait in the x_1x_4−plane.

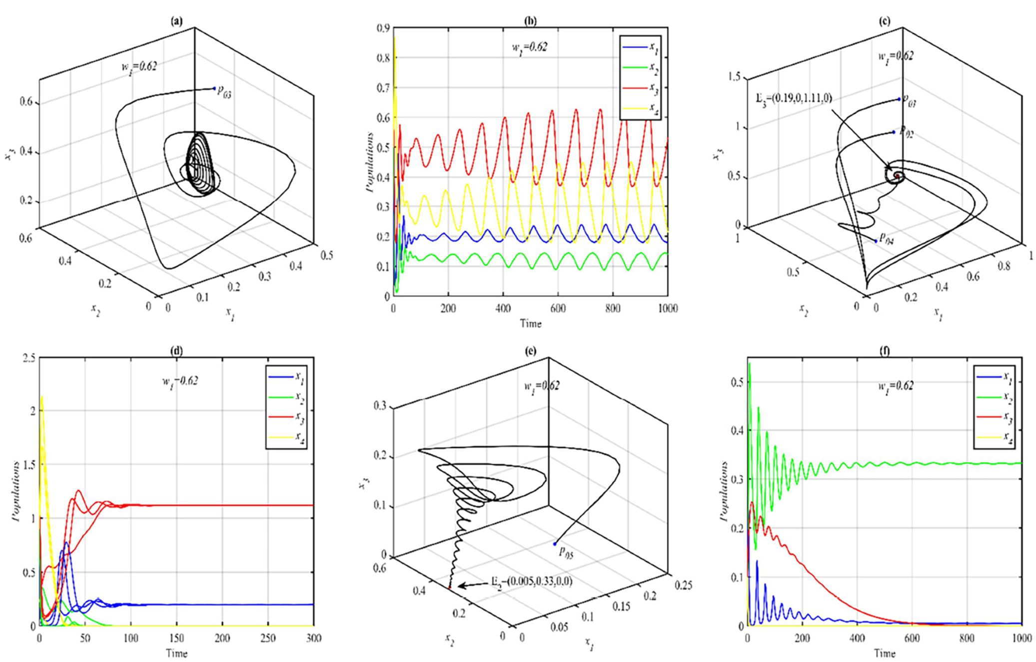

Fig. 6. The trajectories of system 2 using the dataset 68 with $w_1 = 0.62$. (a) 3D projection of phase portrait for trajectory starting from p_{03} that approaches 4D-periodic attractor. (b) Time series of the trajectory starting at p_{03}. (c) 3D projection of phase portrait for trajectories starting from p_{01}, p_{02} and p_{04}, which are approaching E_3. (d) Time series of the trajectories starting from p_{01}, p_{02} and p_{04}. (e) 3D projection of phase portrait for trajectory starting at p_{05} that approaches E_2. (f) Time series of the trajectories starting at p_{05}.

Table 2. Dynamical behavior regarding to parameters ranges using dataset 68.

Parameter varying	Ranges	Approaches to the attractors
w_2	$w_2 < 1.84$	E_3, E_5
	$1.84 \leq w_2 < 1.89$	E_3, E_5, E_6
	$1.89 \leq w_2 < 1.97$	E_3, E_6
	$1.97 \leq w_2 < 2.1$	E_2, E_3, E_6
	$2.1 \leq w_2 < 2.15$	E_2, E_3, 4D-periodic
	$2.15 \leq w_2$	E_2, E_3
w_3	$w_3 < 0.38$	E_2, E_3
	$0.38 \leq w_3 < 0.42$	E_2, E_3, 4D-periodic
	$0.42 \leq w_3 < 0.60$	E_2, E_3, E_6
	$0.60 \leq w_3 < 0.62$	E_2, E_3, E_5, E_6
	$0.62 \leq w_3 < 0.65$	E_3, E_5, E_6
	$0.65 \leq w_3$	E_3, E_5
w_4	$w_4 < 0.03$	3D-periodic, E_5, E_6
	$0.03 \leq w_4 < 0.06$	E_5
	$0.06 \leq w_4 < 0.11$	E_5, E_6
	$0.11 \leq w_4 < 0.14$	E_2, E_5, E_6
	$0.14 \leq w_4 < 0.16$	E_2, E_3, E_5, E_6
	$0.16 \leq w_4 < 0.3$	E_2, E_3, E_6
	$0.3 \leq w_4$	E_2, E_3
w_5	$w_5 < 0.03$	E_2, E_3, E_5, E_6
	$0.03 \leq w_5$	E_3, E_5
w_6	$w_6 < 0.29$	E_2
	$0.29 \leq w_6 < 0.39$	E_2, E_3
	$0.39 \leq w_6 < 0.42$	E_2, E_3, E_6
	$0.42 \leq w_6 < 0.43$	E_3, E_5, E_6
	$0.43 \leq w_6$	E_3, E_5
c_1	$c_1 < 0.13$	E_2
	$0.13 \leq c_1 < 0.28$	E_2, E_3
	$0.28 \leq c_1 < 0.29$	E_2, E_3, 4D periodic
	$0.29 \leq c_1 < 0.34$	E_2, E_3, E_6
	$0.34 \leq c_1$	E_3, E_5
c_2	$c_2 < 0.17$	E_2
	$0.17 \leq c_2 < 0.27$	E_2, E_3
	$0.27 \leq c_2 < 0.31$	E_2, E_3, 4D periodic
	$0.31 \leq c_2$	E_3, E_6
w_7	$w_7 < 0.54$	E_2, E_3
	$0.54 \leq w_7 < 0.64$	E_2, E_3, E_5
	$0.64 \leq w_7 < 0.85$	E_2, E_3, E_6
	$0.85 \leq w_7$	E_2, E_3, 4D periodic
w_8	$w_8 < 0.03$	E_2
	$0.03 \leq w_8 < 0.07$	E_2, 4D periodic
	$0.07 \leq w_8 < 0.09$	E_2, E_3, 4D periodic
	$0.09 \leq w_8 < 0.14$	E_2, E_3, E_6
	$0.14 \leq w_8$	E_2, E_3
w_9	$w_9 < 0.02$	E_2, E_3
	$0.02 \leq w_9 < 0.13$	E_2, E_3, E_6
	$0.13 \leq w_9 < 0.7$	E_2, E_6
	$0.7 \leq w_9 < 0.83$	E_2, 4D periodic
	$0.83 \leq w_9$	E_2
w_{10}	$w_{10} < 0.04$	E_2, E_5
	$0.04 \leq w_{10} < 0.05$	E_3, E_6
	$0.05 \leq w_{10} < 0.06$	E_2, E_3, E_6
	$0.06 \leq w_{10} < 0.09$	E_2, E_3
	$0.09 \leq w_{10}$	E_2
w_{11}	$w_{11} < 0.1$	E_2, E_3, E_5
	$0.1 \leq w_{11} < 0.12$	E_2, E_3, E_6
	$0.12 \leq w_{11}$	E_2, E_3

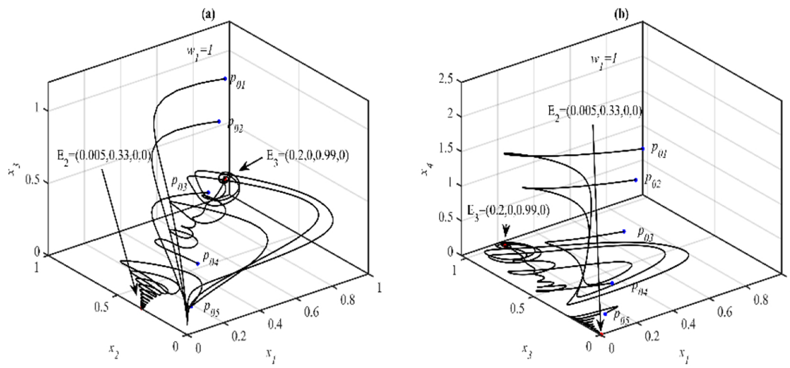

Fig. 7. The trajectories of system 2 using the dataset 68 with $w_1 = 1$. (a) 3D projection on the space $x_1x_2x_3$ of phase portrait for trajectories starting from $p_{0i}; i = 1, 2, \ldots, 5$, which are approaching E_2, and E_3. (b) 3D projection on the space $x_1x_3x_4$ of phase portrait for trajectories starting from $p_{0i}; i = 1, 2, \ldots, 5$ which are approaching E_2, and E_3.

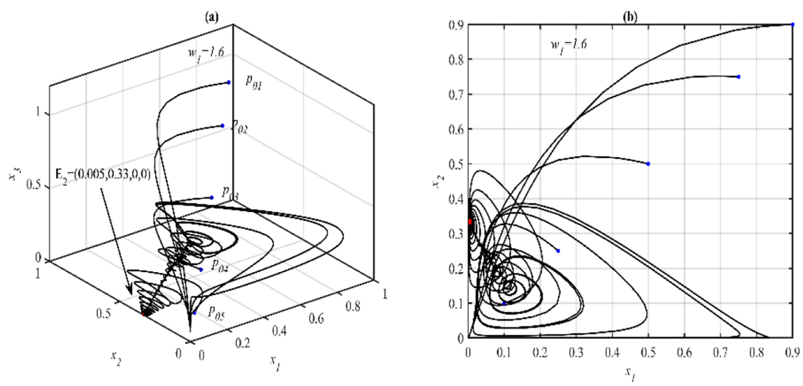

Fig. 8. The trajectories of system 2 using the dataset 68 with $w_1 = 1.6$. (a) 3D projection of phase portrait for trajectories starting from $p_{0i}; i = 1, 2, \ldots, 5$, which are approaching E_2. (b) 2D projection of phase portrait in the x_1x_2-plane.

dynamic behavior of such a system. The suggested system contains six boundary equilibrium points, whilst the coexistence point may or may not exist. All potential equilibrium points were computed together with the parameters governing their presence. The coexistence point's multiplicity of existence forced us to focus only on its local stability. Consequently, its global stability cannot be studied because its number is not known. As a result, each of them will have a basin of attraction, which precludes the existence of global stability. Furthermore, it was not possible to investigate the permanence of the suggested system because the conditions for the cohabitation and border points' existence overlapped with the prerequisites for those points' stability. Finally, as a future work, it can be thought about the role of delayed disease transition on the dynamic behavior and persistence of the system.

Conclusion

The presence of cooperative hunting and induced fear in the prey-predator system, coupled with the presence of infectious disease in both species greatly complicated the dynamics of the system. As a result of this complexity, there is a loss of global stability in the system, as well as persistence, since the dynamic behavior changes rapidly with changing parameter values. According to the numerical results shown in Fig. 1 to Fig. 8 and those presented in Table 2, the system has very multiple dynamics, including locally stable equilibrium points, periodic dynamics, and multiple stability. Also, the conversion rate from infected prey to healthy predator is the only parameter that keeps the system locally stable at the point of coexistence, while all other parameters lead to a loss of persistence in the system and the dynamic behavior prevailing over the system is multiple stability.

Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

Authors' contribution statement

R.K. N. developed the proposed idea. N.H. F. formulated the theories and carried out the computations. Both authors confirmed the analytical methodology. R.K. N. advised N.H. F. to conduct an investigation [The Influence of Hunting Cooperation and Fear on the Dynamics of the Eco-Epidemiological Model with Disease in Both Populations], and supervised the outcomes of this work. Both authors reviewed their results and contributed to the research process, including organizing and carrying out the study, data analysis, and manuscript writing.

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تأثير التعاون في الصيد والخوف على ديناميكيات النموذج الوبائي البيئي مع المرض في كلا المجموعتين السكانيتين

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[3] *هيئة البحث العلمي، بغداد، العراق.*

المستخلص

يتطلب التنبؤ بالديناميكيات البيئية والتحكم فيها فهمًا للتفاعل المعقد بين التعاون في الصيد، والخوف، والعناصر البيولوجية الأخرى في النماذج الوبائية البيئية. وتعد دراسة هذه النماذج أمرا بالغ الأهمية لتحقيق الاستدامة البيئية والحفاظ عليها، كما يتضح من ارتفاع الأمراض المعدية الناجمة عن النمو السكاني والتفاعلات بين الكائنات الحية. الهدف من هذه الدراسة هو إنشاء نموذج رياضي جديد يأخذ في الاعتبار الأمراض المعدية التي تؤثر على كل من الحيوانات المفترسة والفرائس، وكذلك كيف يسبب سلوك الصيد التعاوني من جانب الحيوانات المفترسة القلق لدى سكان الفرائس. تم اختبار السمات الهامة للنموذج، بما في ذلك وجود الحلول وحدودها وتفردها وإيجابيتها، بالإضافة إلى تحديد مواقع التوازن ومعايير الاستقرار المحلي التي تدعمها. تم إجراء تحليلات التشعب حول نقاط التوازن، وكشف مجموعة متنوعة من السلوكيات الديناميكية، بما في ذلك أحداث الاستقرار المتعدد. تم تأكيد الاستنتاجات النظرية وتحديد إعدادات التحكم من خلال عمليات المحاكاة العددية باستخدام طريقة رانكا-كوتا من الدرجة الرابعة في MATLAB R2021a.

الكلمات المفتاحية: نموذج بيئي-وبائي، خوف، التعاون في الصيد، الاستقرار، التشعب.