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Extended Inverse Lomax Distribution with Simulation and Application to Crude Birth Rate Data

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Abstract

This paper presents a flexible extended version of the inverse Lomax (EVIL) distribution with four parameters and different shapes of the hazard function. It is developed within the framework of the truncated Rayleigh odd Weibull generator family. Various statistical properties are discussed, including reliability measures, density mixture representation, moments, characteristic and moment-generating functions, quantile function, order statistics, and residual life function with its reversal. The maximum likelihood estimates of unknown parameters are assessed via simulation. The empirical results show that the estimates are reliable and flexible. Furthermore, the EVIL is employed with some competitive distributions to model real-life data concerning the crude birth rate in Iraq. The outcomes of the real data demonstrate that the EVIL outperforms existing inverse Lomax-based models in terms of flexibility and goodness of fit.

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1. Introduction

Statistical distributions are quite important in explaining and identifying actual data analysis. Lifetime distribution modeling has received more attention over the years, and this interest has grown with time. The literature contains many strategies for constructing more flexible distributions to fit real-life data, and many new distributions and generator families of distributions, especially continuous univariate ones, have been produced and examined with various procedures that include incorporating one or more additional parameters into the baseline/existing distribution. The extra parameters were helpful to assess tail features and improve the distribution's goodness of fit. Several generator (or G) families have been described in the literature using various probability distributions such as Beta [1], Kumaraswamy [2], exponentiated generalized [3], Weibull [4], Gompertz [5], odd Fréchet [6], Marshall-Olkin [7], and Topp-Leone [8]. These and other generators were used to create numerous extended and modified versions of traditional baseline distributions, including the inverse Lomax (IL) distribution.

The IL distribution is defined as the reciprocal of the Lomax distribution. In some cases, it is a viable alternative to popular distributions such as Weibull, inverse Weibull, and gamma. Further, it can be regarded as a member that belongs to the generalized beta family [9]. Due to its decreasing hazard rate and upside-down bathtub form, it is useful for modeling several data types, including economics and actuarial science, geophysical databases, and reliability analysis (see [10-12]). Several researchers, such as [13-16], have examined the IL's statistical inferences. However, to increase its flexibility for modeling real-life data, numerous versions and modifications of IL distribution have recently been developed particularly regarding reliability and biological studies, such as Weibull IL [17], odd Frechet IL [18], logarithmic IL [19], half logistic IL [20], modified IL [21], Kumaraswamy generalized IL [22], and exponentiated power IL [23]. Even with these modifications, new versions of IL are required to handle diverse data types.

This paper's major purpose is to introduce a new flexible version of IL with two extra parameters. The remainder of this paper is structured as follows: Section 2 discusses statistical descriptions of the suggested distribution, including its cumulative distribution function (CDF), probability density function (PDF), and reliability measures. Section 3 offered mixture representations of the density and CDF. In section 4, the most useful statistical aspects are discussed. Section 5 considered the maximum likelihood estimation method to estimate the unknown parameters. In section 6, numerical illustrations are conducted, including a simulation study to investigate the performance of the estimates and as well as a real-life application is offered to evaluate the applicability of the suggested distribution through different information criteria. Finally, conclusions are listed in section 7.

2. Extended Inverse Lomax Distribution

The CDF and PDF of the considered truncated Rayleigh odd Weibull generator family for a random variable $\boldsymbol{X}$ ($\boldsymbol{x > 0}$) are

$$\boldsymbol{F(x) = \frac{1}{1-e^{-\theta/2}}\left(1 - e^{-\frac{\theta}{2}\left(1-e^{-\tau(x)^{\beta}}\right)^{2}}\right)} \quad \dots(1)$$

$$\boldsymbol{f(x) = \frac{\theta\beta m(x) M(x)^{\beta-1}}{(1-e^{-\theta/2})\,\bar{M}(x)^{\beta+1}} e^{-\tau(x)^{\beta}}\left(1 - e^{-\tau(x)^{\beta}}\right) e^{-\frac{\theta}{2}\left(1-e^{-\tau(x)^{\beta}}\right)^{2}}} \quad \dots(2)$$

where $\boldsymbol{\theta > 0}$ and $\boldsymbol{\beta > 0}$ are the scale and shape parameters, $\boldsymbol{M(x)}$ and $\boldsymbol{m(x)}$ are the CDF and PDF of any baseline distribution, and $\boldsymbol{\tau(x) = \frac{M(x)}{\bar{M}(x)} = \frac{M(x)}{1-M(x)}}$ (see [24]).

Consider $\boldsymbol{M(x)}$ and $\boldsymbol{m(x)}$ as the traditional IL distribution with shape parameter ($\boldsymbol{\alpha > 0}$) and scale parameter ($\boldsymbol{\delta > 0}$) that provided by [22]:

$$\boldsymbol{M(x) = \left(1+\frac{\delta}{x}\right)^{-\alpha}} \quad \dots(3)$$

$$\boldsymbol{m(x) = \frac{\alpha\delta}{x^2}\left(1+\frac{\delta}{x}\right)^{-(\alpha+1)}} \quad \dots(4)$$

A new extended version of IL (EVIL) distribution with four parameters can be proposed by inserting (3) and (4) in (1) and (2) with $\boldsymbol{\tau(x) = \frac{1}{(1+\delta/x)^{\alpha}-1}}$. The CDF and PDF of EVIL distribution with overall vector parameters $\boldsymbol{v = (\theta, \beta, \alpha, \delta)}$ are given by

$$\boldsymbol{F(x; v) = \frac{1}{1-e^{-\theta/2}}\left(1 - e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta/x)^{\alpha}-1)^{-\beta}}\right)^{2}}\right)} \quad \dots(5)$$

$$f(x;v) = \frac{\theta\beta\alpha\delta(1+\delta/x)^{-(\alpha\beta+1)}}{(1-e^{-\theta/2})(1-(1+\delta/x)^{-\alpha})^{\beta+1}\, x^2} e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta/x)^{\alpha}-1)^{-\beta}}\right)^{2} - ((1+\delta/x)^{\alpha}-1)^{-\beta}} \left(1 - e^{-((1+\delta/x)^{\alpha}-1)^{-\beta}}\right) \quad \dots(6)$$

Further, the most commonly employed reliability measures in reliability engineering and life data analysis to evaluate the operational performance of a component/item/or system are the reliability function $\boldsymbol{R(x) = 1 - F(x)}$ and hazard function $\boldsymbol{h(x) = \frac{f(x)}{R(x)}}$. The mentioned reliability measures of the EVIL random variable with vector parameters $\boldsymbol{v = (\theta, \beta, \alpha, \delta)}$ are given respectively by

$$\boldsymbol{R(x; v) = 1 - \frac{1 - e^{-\frac{\theta}{2}\left(1-e^{-\left(\left(1+\frac{\delta}{x}\right)^{\alpha}-1\right)^{-\beta}}\right)^{2}}}{1-e^{-\frac{\theta}{2}}}} \quad \dots(7)$$

$$\boldsymbol{h(x; v) = \frac{\theta\beta\alpha\delta(1+\delta/x)^{-(\alpha\beta+1)}}{\left(e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta/x)^{\alpha}-1)^{-\beta}}\right)^{2}} - e^{-\frac{\theta}{2}}\right)x^2} (1-(1+\delta/x)^{-\alpha})^{-(\beta+1)} \left(1 - e^{-((1+\delta/x)^{\alpha}-1)^{-\beta}}\right) e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta/x)^{\alpha}-1)^{-\beta}}\right)^{2} - ((1+\delta/x)^{\alpha}-1)^{-\beta}}} \quad \dots(8)$$

Figure 1 clearly shows several PDF shapes that highlight the new distribution seems particularly well-suited to modeling various positive data. Furthermore, the various shapes of the hazard function, including decreasing, increasing, skewed, bathtub, $\boldsymbol{J}$, and reversed-$\boldsymbol{J}$, are clearly shown in Figure 2. It is thus once more emphasized that this novel distribution appears especially well-suited to representing various life data.

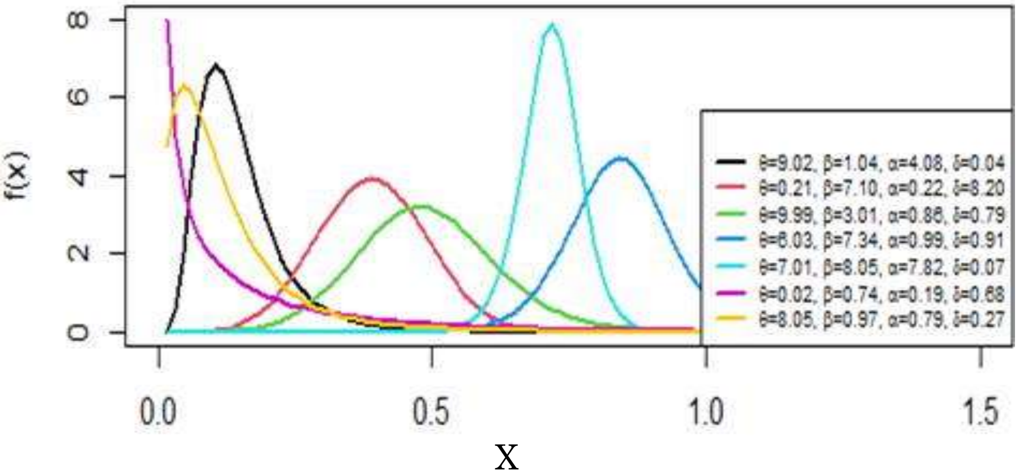

Figure 1. PDF plot for different parameter's values.

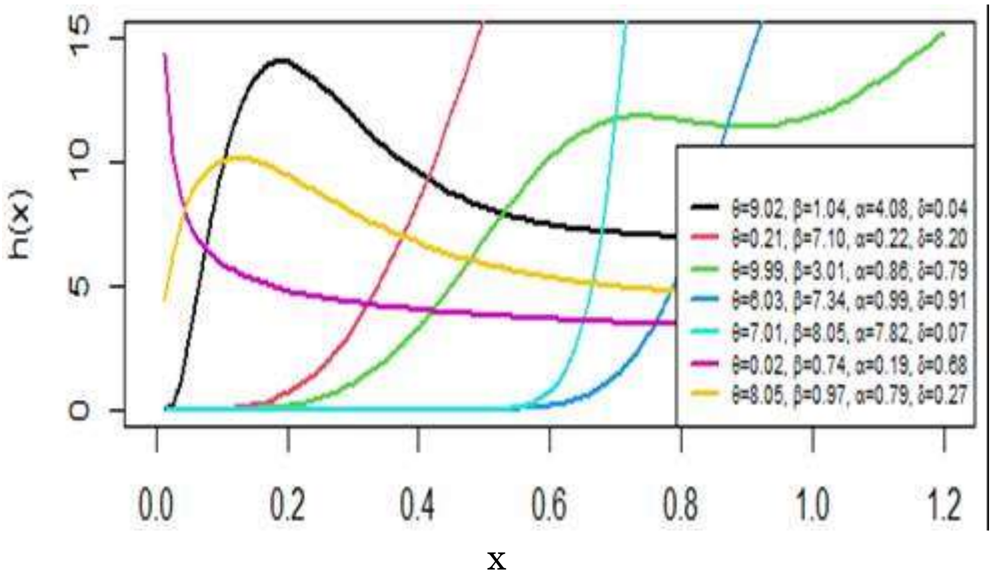

Figure 2. Hazard function plot for different parameter values.

3. Important Linear Representation

The PDF linear representation plays a crucial role and can be used to obtain various statistical features, including moments, inverse moments, characteristic functions, moment generating functions, and others. Now, the supporting formula of the EVIL's PDF can be attained via the following expansion formulae based on some easy mathematical steps:

$(1-z)^a = \sum_{l=0}^{\infty} (-1)^l \binom{a}{l} z^l$;

$\frac{1}{(1-z)^a} = \sum_{l=0}^{\infty} \binom{l+a-1}{l} z^l$ for $|z| < 1$, $a > 0$; and

$e^{-z} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} z^l$.

Thus, the PDF in equation (6) can be expanded into the following form

$$f^E(x;v) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \psi_{(k,l)} \frac{\alpha\delta}{x^2}(1+\delta/x)^{-(\alpha(\beta(k+1)+l)+1)} \quad \ldots(9)$$

where

$$\psi_{(k,l)} = \frac{\theta\beta}{1-e^{-\theta/2}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j+k}}{i!\,k!} \left(\frac{\theta}{2}\right)^i (j+1)^k \binom{2i+1}{j}\binom{\beta(k+1)+l}{l} \quad \ldots(10)$$

The density mixture representation in the equation (9) can be expanded and represented as a linear form of the PDF baseline distribution as $f^E(x;v) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\psi_{(k,l)}}{\beta(k+1)+l} m(x;\alpha(\beta(k+1)+l),\delta)$ where $m(x;\alpha(\beta(k+1)+l),\delta)$ is the PDF of the traditional IL distribution with parameters $\alpha(\beta(k+1)+l)$ and δ. Then, the $f^E(x;v)$ in (9) with $\psi^*_{(k,l)} = \frac{\psi_{(k,l)}}{\beta(k+1)+l}$ will be

$$f^E(x;v)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)}\ m(x;\alpha(\beta(k+1)+l),\delta)$$

Then

$$f^E(x;v)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)}\frac{\alpha(\beta(k+1)+l)\delta}{x^2}(1+\delta/x)^{-(\alpha(\beta(k+1)+l)+1)} \quad \ldots (11)$$

By integrating the PDF in the equation (11), the CDF of EVIL can be represented as a linear form of the CDF baseline IL distribution, i.e.

$$F^E(x;v)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)}M(x;\alpha(\beta(k+1)+l),\delta)$$

where $M(x;\alpha(\beta(k+1)+l),\delta)$ is the CDF of the baseline IL distribution with $\alpha(\beta(k+1)+l)$ and δ. Then, the $F^E(x;v)$ will be

$$F^E(x;v)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)}\left(1+\frac{\delta}{x}\right)^{-\alpha(\beta(k+1)+l)} \quad \ldots (12)$$

4. Essential Statistical Properties

The most useful statistical aspects are offered in this section.

4.1. Moments and inverse moments

The non-central $\boldsymbol{r^{th}}$ ($\boldsymbol{r=1,2,\ldots}$)$\boldsymbol{The}$ moment can be attained via the density linear representation in the equation (**11**) as follows

$$E(X^r)=\int_0^\infty x^r f^E(x;v)\,dx$$
$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)}\int_0^\infty x^r\frac{\alpha(\beta(k+1)+l)\delta}{x^2}(1+\delta/x)^{-(\alpha(\beta(k+1)+l)+1)}dx$$
$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)}\alpha(\beta(k+1)+l)\delta\int_0^\infty x^{r-2}\,(1+\delta/x)^{-(\alpha(\beta(k+1)+l)+1)}dx$$

Setting $y=\frac{\delta}{x}\to x=\frac{\delta}{y}\to dx=-\delta y^{-2}dy$. When $x\to 0$ then $y\to\infty$ and vice versa, thus the integration will be $\delta^{r-1}\int_0^\infty\frac{y^{-r}}{(1+y)^{\alpha(\beta(k+1)+l)+1}}dy$. Now, based on the second type beta function $B(a,b)=\int_0^\infty\frac{z^{a-1}}{(1+z)^{a+b}}dz$ It can easily be attained that

$$\delta^{r-1}\int_0^\infty\frac{y^{-r}}{(1+y)^{\alpha(\beta(k+1)+l)+1}}dy=\delta^{r-1}B(1-r,\alpha(\beta(k+1)+l)+r).$$

After substituting the integration's outcome in $E(X^r)$, the final form of the r^{th} The moment of EVIL can be attained as

$$E(X^r)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)}\alpha(\beta(k+1)+l)\delta^r B(1-r,\alpha(\beta(k+1)+l)+r)$$

That equivalent to

$$E(X^r)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_{(k,l)}\,\alpha\delta^r B(1-r,\alpha(\beta(k+1)+l)+r) \quad \ldots (13)$$

Noting that the beta function $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$where $\Gamma(0)=-\gamma,\Gamma(-r)=\frac{(-1)^r}{r!}\sum_{j=1}^r\frac{1}{j}-\frac{(-1)^r}{r!}$ and γ is Euler's constant, see [25]. Further, for more details about how to handle moments with $\Gamma(0)$ and $\Gamma(-r)$, the interested can see [9],[17], and [18].

Particularly, the mean $(E(X))$ and variance $V(X)=E(X^2)-(E(X))^2$ of the EVIL distribution can be attained from $E(X^r)$ with $r=1$ and $r=2$ as

$$E(X)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_{(k,l)}\,\alpha\delta\Gamma(0) \quad \ldots (14)$$

$$E(X^2)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_{(k,l)}\,\alpha\delta^2\Gamma(-1)(\alpha(\beta(k+1)+l)+1) \quad \ldots (15)$$

and

$$V(X)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_{(k,l)}\,\alpha\delta^2\Gamma(-1)(\alpha(\beta(k+1)+l)+1)-\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_{(k,l)}\,\alpha\delta\Gamma(0)\right)^2 \quad \ldots (16)$$

Similarly to equation (13), the r^{th} inverse moment is given by

$$E(X^{-r})=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_{(k,l)}\frac{\alpha}{\delta^r}B(1+r,\alpha(\beta(k+1)+l)-r) \quad \ldots (17)$$

Further, the $r^{th}(r=1,2,\ldots)$ central moment can be calculated based on r^{th} non-central moment in equation (13) from the formula

$$E(X-E(X))^r=\sum_{m=0}^r(-1)^{r-m}\binom{r}{m}(E(X))^m E(X^{r-m})$$

with some mathematical steps as

$$E(X-E(X))^r=\sum_{m=0}^r(-1)^r\binom{r}{m}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_{(k,l)}\right)^{m+1}\alpha^{m+1}\delta^r\gamma^m B(1-r+m,\alpha(\beta(k+1)+l)+r-m) \quad \ldots (18)$$

4.2. Incomplete moments

The incomplete moments can be attained from equation (11) by $I_X(t)=\int_0^t x^r f^E(x;v)\,dx$. Now

$$I_X(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)}\int_0^t x^r\frac{\alpha(\beta(k+1)+l)\delta}{x^2}(1+\delta/x)^{-(\alpha(\beta(k+1)+l)+1)}dx$$

Recall $\psi^*_{(k,l)}=\psi_{(k,l)}\frac{1}{\beta(k+1)+l}$, $I_X(t)$ will be

$$I_X(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_{(k,l)}\,\alpha\delta\int_0^t x^{r+\alpha(\beta(k+1)+l)-1}(x+\delta)^{-\alpha(\beta(k+1)+l)-1}\,dx$$

Then

$$I_X(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \psi_{(k,l)} \frac{\alpha t^r (t/\delta)^{\alpha(\beta(k+1)+l)}}{\alpha(\beta(k+1)+l)+r} F_{2:1}(a;b;c;d) \quad \dots (19)$$

where $F_{2:1}(a;b;c;d) = \frac{1}{B(a,c-b)}\int_0^1 \frac{u^{b-1}(1-u)^{c-b-1}}{(1-du)^a} du$ is the hyper-geometric function with

$a = \alpha(\beta(k+1)+l)+1; b = \alpha(\beta(k+1)+l)+r;$
$c = \alpha(\beta(k+1)+l)+r+1; d = -\frac{t}{\delta}$. For more details about $F_{2:1}(a;b;c;d)$, see [9].

4.3. Conditional moments

The conditional moments can be attained by applying the form $\boldsymbol{C_X(t) = E(E(X^r|X > x)) = \frac{\Delta_r(x)}{1-F(x)}}$, where $\boldsymbol{\Delta_r(x) = \int_x^\infty u^r f(u)\, du}$ and $\boldsymbol{F(x)}$ is the CDF of the considered distribution. Now, based on the equation (**11**), the $\Delta_r(x)$ related to EVIL distribution is

$$\Delta_r(x) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \psi^*_{(k,l)} \int_x^\infty u^{r-2}\, \alpha(\beta(k+1)+l)\delta\, (1+\delta/u)^{-(\alpha(\beta(k+1)+l)+1)} du$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \psi_{(k,l)} (-1)^{r-1}\alpha\delta^r\, B_{\frac{-\delta}{x}}(1-r, -\alpha(\beta(k+1)+l))$$

where $B_z(u,v) = \int_0^z y^{u-1}(1-y)^{v-1}\, dy$ represent the incomplete beta function.

Then, based on obtained $\Delta_r(x)$ and the CDF in equation (12), the conditional moments of EVIL random variable is

$$C_X(t) = \frac{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \psi_{(k,l)}(-1)^{r-1}\alpha\delta^r B_{\frac{-\delta}{x}}(1-r, -\alpha(\beta(k+1)+l))}{1-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \psi^*_{(k,l)}(1+\delta/x)^{-\alpha(\beta(k+1)+l)}} \quad \dots (20)$$

4.4. Characteristic and generating functions

The characteristic function can be obtained by applying its definition to the Maclaurin series, as $\boldsymbol{\Phi_X(it) = E(e^{itX}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r)}$. Based on equation (**13**), the characteristic function is given by

$$\Phi_X(it) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} \psi_{(k,l)} \frac{\alpha(it\delta)^r}{r!} B(1-r, \alpha(\beta(k+1)+l)+r) \quad \dots (21)$$

Similarly, the moment generating function $\varphi_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$ can be attained by

$$\Phi_X(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} \psi_{(k,l)} \frac{\alpha(t\delta)^r}{r!} B(1-r, \alpha(\beta(k+1)+l)+r) \quad \dots (22)$$

Based on equation (22), the cumulant generating function can be attained by

$$K_X(t) = \ln(\varphi_X(t)) \quad \dots (23)$$

4.5. Quantile function

The quantile function (symbolized by $\boldsymbol{\Theta(q)}$ or $\boldsymbol{x_q}$) of the EVIL can be determined from equation (5) via solving $\boldsymbol{x_q = F_X^{-1}(q);\ 0 < q < 1}$ as

$$x_q = \frac{\delta}{\left(1 + \left\{-\ln\left(1 - \sqrt{\frac{-2\ln(1-(1-e^{-\theta/2})q)}{\theta}}\right)\right\}^{-1/\beta}\right)^{1/\alpha} - 1} \quad \dots (24)$$

Accordingly, the EVIL random variable can be simulated from equation (24) by substituting q with u, where U represents a standard Uniform random variable. Setting $q = 1/2$ makes it simple to calculate the median. Other measures, such as skewness, kurtosis, and inter-quantile range, can also be easily computed.

4.6. Residual life function and its reverse

The extra lifetime that a system or component has given rise to till time t in life experiments is referred to as the component's residual life (RL) function. To be more precise, if X is the life of a system, then $(X_t = X - t|X > t)$ is known as RL function. The general mathematical form of the r^{th} $(r = 1, 2, \dots)$ moments of the RL of X is given by

$RL_r(t) = E((X-t)^r|X > t) = \frac{1}{1-F(t)}\int_t^\infty (x-t)^r f(x)\, dx$

and it can be reformed by applying the binomial expansion of $(x-t)^r$ as

$\boldsymbol{RL_r(t) = \frac{1}{1-F(t)}\sum_{s=0}^{r}(-t)^{r-s}\binom{r}{s}\int_t^\infty x^r f(x)\, dx}$.

Then with equation (11),

$$RL_r(t) = \frac{1}{1-F(t)}\sum_{s=0}^{r}(-t)^{r-s}\binom{r}{s}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)} \int_t^\infty x^r \frac{\alpha(\beta(k+1)+l)\delta}{x^2}(1+\delta/x)^{-(\alpha(\beta(k+1)+l)+1)} dx$$

Now, with $F(t)$ as in the equation (12), $RL_r(t)$ will be

$$RL_r(t) = \frac{1}{1-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi^*_{(k,l)}(1+\delta/x)^{-\alpha(\beta(k+1)+l)}} \sum_{s=0}^{r}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_{(k,l)}(-t)^{r-s}\binom{r}{s}\alpha\delta(-\delta)^{r-1} B_{\frac{-\delta}{x}}(1-r, -\alpha(\beta(k+1)+l)) \quad \dots (25)$$

On the other side, the general mathematical form of the r^{th} moments of the reversed residual life (RRL) function of X is given by

$RRL_r(t) = E((t-X)^r|X \le t) = \frac{1}{F(t)}\int_0^t (t-x)^r f(x)\, dx$

and similarly it can be reformed by applying the binomial expansion of $(t-x)^r$ as

$RRL_r(t) = \frac{1}{F(t)} \sum_{s=0}^{r} (-t)^{r-s} \binom{r}{s} \int_0^t x^r f(x)\, dx.$
$= \frac{1}{F(t)} \sum_{s=0}^{r} (-t)^{r-s} \binom{r}{s} I_X(t)$

Now, based on equations (12) and (19), the $RRL_r(t)$ of EVIL random variable can be obtained as

$$RRL_r(t) = \frac{1}{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \psi^*_{(k,l)} (1+\delta/x)^{-\alpha(\beta(k+1)+l)}} \sum_{s=0}^{r} (-t)^{r-s} \binom{r}{s} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \psi_{(k,l)} \frac{\alpha t^r (t/\delta)^{\alpha(\beta(k+1)+l)}}{\alpha(\beta(k+1)+l)+r} F_{2:1}(a;b;c;d) \quad \dots (26)$$

4.7. Order Statistics

Consider $X_{1:n} \le \dots \le X_{n:n}$ represent the O.S. (order statistics) of a random sample with size n. The PDF of $X_{i,n}$ is

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x)$$

where $B(.,.)$ is the beta function. Further, the first $(i = 1)$ and last $(i = n)$ O.S. can be obtained particularly by

$f_{1:n}(x) = n\, f(x) \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} F^j(x)$ and

$f_{n:n}(x) = n\, f(x)\, F^{n-1}(x)$. Consequently, the density function of O.S. $X_{i,n}$ of the EVIL is

$$f_{i:n}(x; v) = \frac{\theta\beta\alpha\delta(1+\delta/x)^{-(\alpha\beta+1)}}{B(i, n-i+1)(1-(1+\delta/x)^{-\alpha})^{\beta+1} x^2} e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta/x)^{\alpha}-1)^{-\beta}}\right)^2 - ((1+\delta/x)^{\alpha}-1)^{-\beta}} \left(1-e^{-((1+\delta/x)^{\alpha}-1)^{-\beta}}\right) \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(-1)^j}{(1-e^{-\theta/2})^{j+i}} \left(1 - e^{-\frac{\theta}{2}\left(1-e^{-((1+\delta/x)^{\alpha}-1)^{-\beta}}\right)^2}\right)^{j+i-1} \quad \dots (27)$$

With $i = 1$ and $i = n$, the first and last O.S. $(X_{1,n}, X_{n,n})$ of the EVIL can be obtained easily.

5. Parameter Estimation

The maximum likelihood methodology, with its attractive features (like asymptotic consistency, efficiency, and invariance), is the most widely used parameter estimation method. In this section, the MLEs (maximum likelihood estimates) of the unknown four parameters are derived.

Consider a random sample of size n, denoted by $\underline{x} = x_1, x_2, \dots, x_n$, drawn from the EVIL distribution with vector parameters $v = (\theta, \beta, \alpha, \delta)$. Based on equation (6), the likelihood function $L(v|\underline{x}) = \prod_{i=1}^{n} f(x_i; v)$ and its natural logarithm function $\ell(v|\underline{x}) = \ln\left(L(v|\underline{x})\right) = \sum_{i=1}^{n} \ln\left(f(x_i; v)\right)$ are given by

$$L(v|\underline{x}) = \prod_{i=1}^{n} \frac{\theta\beta\alpha\delta(1+\delta/x_i)^{-(\alpha\beta+1)}}{(1-e^{-\theta/2})\, x_i^2} (1-(1+\delta/x_i)^{-\alpha})^{-(\beta+1)} e^{-((1+\delta/x_i)^{\alpha}-1)^{-\beta}} \left(1-e^{-((1+\delta/x_i)^{\alpha}-1)^{-\beta}}\right) e^{-\frac{\theta}{2}\left(1-e^{-\left((1+\delta/x_i)^{\alpha}-1\right)^{-\beta}}\right)^2} \quad \dots (28)$$

$$\ell(v|\underline{x}) = n \ln\left(\frac{\theta\beta\alpha\delta}{1-e^{-\theta/2}}\right) - (\alpha\beta+1)\sum_{i=1}^{n} \ln(1+\delta/x_i) - 2\sum_{i=1}^{n} \ln(x_i) - (\beta+1)\sum_{i=1}^{n} \ln(1-(1+\delta/x_i)^{-\alpha}) - \sum_{i=1}^{n} ((1+\delta/x_i)^{\alpha}-1)^{-\beta} + \sum_{i=1}^{n} \ln\left(1-e^{-((1+\delta/x_i)^{\alpha}-1)^{-\beta}}\right) - \frac{\theta}{2}\sum_{i=1}^{n}\left(1-e^{-((1+\delta/x_i)^{\alpha}-1)^{-\beta}}\right)^2 \quad \dots (29)$$

To get the MLEs of four parameters, set the partial derivatives of the equation (29) w.r.t. four parameters, i.e. $\frac{\partial \ell(v|\underline{x})}{\partial\theta}, \frac{\partial \ell(v|\underline{x})}{\partial\beta}, \frac{\partial \ell(v|\underline{x})}{\partial\alpha}, \frac{\partial \ell(v|\underline{x})}{\partial\delta}$ to zero and solve numerically. The Newton-Raphson-type algorithm can be implemented by the R (optim function) to support the MLE computations.

6. Numerical Illustrations and Discussion Results

This section assesses the performance of the MLEs by conducting a simulation study. Furthermore, the EVIL's performance is assessed along with some competitive models in modeling Iraq's real medical data, where many researchers are interested in analyzing data in the context of medical and health aspects in Iraq from a statistical perspective, see [26] and [27]. All numerical illustrations are created using the R programming language.

6.1. Simulation study

The simulation study is carried out with different groupings of parameters' default settings based on 3000 i.i.d. random samples prepared via the formula given in equation (24) with $q = u$; $u \in U(0,1)$ and sample sizes $n = 10, 20, 40, 80$, and 160 to consider small, moderate, and large sizes. The combinations of four parameters are:

Set I: $\theta = 9.02, \beta = 1.04, \alpha = 4.08, \delta = 0.04$.
Set II: $\theta = 9.99, \beta = 3.01, \alpha = 0.86, \delta = 0.79$.
Set III: $\theta = 7.01, \beta = 8.05, \alpha = 7.82, \delta = 0.07$.
Set IV: $\theta = 0.02, \beta = 0.74, \alpha = 0.19, \delta = 0.68$.

After getting the MLEs of the parameters, the average estimate (AE) is calculated and mean squared error (MSE) is used to measure the performance, where

$$AE(\hat{v}) = \frac{1}{3000}\sum_{i=1}^{3000} \hat{v}_i$$

$$MSE(\hat{v}) = \frac{1}{3000}\sum_{i=1}^{3000} (\hat{v}_i - v)^2;\ v = (\theta, \beta, \alpha, \delta).$$

The numerical outcomes are list in Tables 1 and 2. The empirical results show that:

1. As the sample size increases, the AE values usually tend to be closer to the default values,
2. For all sample sizes, the parameter δ has the lowest MSE values in sets (I, II, and III) while the parameter α has the lowest MSE value in set (IV).
3. The MSE values are declining as the sample size grows, demonstrating the estimators' consistency.

Table 1. The AE values with different sets

Set	n	$AE(\hat{\theta})$	$AE(\hat{\beta})$	$AE(\hat{\alpha})$	$AE(\hat{\delta})$
I	10	9.3272568	1.2140721	4.2552875	0.0384223
	20	9.3261184	1.1123465	4.2787431	0.0384227
	40	9.3122368	1.0653220	4.2639267	0.0387394
	80	9.3193250	1.0460010	4.2330110	0.0391310
	160	9.3214002	1.0343467	4.2150733	0.0393569
II	10	10.4902996	3.4889571	0.8492207	0.8067470
	20	10.4724742	3.2272231	0.8538030	0.8049374
	40	10.4482809	3.1021474	0.8572288	0.8027410
	80	10.4194277	3.0538569	0.8561745	0.8038165
	160	10.3927251	3.0215923	0.8571249	0.8024865
III	10	6.8996327	9.0505430	7.8947800	0.0691200
	20	7.0414790	8.5905596	7.8792838	0.0693681
	40	7.1489249	8.3143374	7.8688610	0.0695764
	80	7.2349473	8.1717102	7.8577268	0.0697527
	160	7.2453750	8.0844610	7.8503100	0.0698440
IV	10	0.0377430	0.8282348	0.2103618	0.6608633
	20	0.0340295	0.7884248	0.1975212	0.6985753
	40	0.0570852	0.7665647	0.1936243	0.7048774
	80	0.0423245	0.7558861	0.1913399	0.7064106
	160	0.0507360	0.7494004	0.1905984	0.7047389

Table 2. The MSE values with different sets

Set	n	$MSE(\hat{\theta})$	$MSE(\hat{\beta})$	$MSE(\hat{\alpha})$	$MSE(\hat{\delta})$
I	10	0.6977567	0.1926621	0.3055428	0.0000696
	20	0.5238595	0.0656210	0.2173719	0.0000371
	40	0.4224387	0.0269648	0.1321345	0.0000190
	80	0.3273056	0.0120599	0.0818645	0.0000114
	160	0.3244364	0.0056004	0.0562178	0.0000073
II	10	2.6411064	1.3221658	0.0044199	0.0084813
	20	1.8348931	0.4664284	0.0030975	0.0057129
	40	1.4197431	0.2001338	0.0018512	0.0036013
	80	1.1791273	0.0917828	0.0010832	0.0022531
	160	1.0148527	0.0432570	0.0006803	0.0015642
III	10	1.9264651	6.5282461	0.0905044	0.0000082
	20	0.9633139	2.8316124	0.0417546	0.0000037
	40	0.5853404	1.2801518	0.0175237	0.0000016
	80	0.3420327	0.5758293	0.0073396	0.0000007
	160	0.2364847	0.2685284	0.0033817	0.0000003
IV	10	0.1924904	0.0339382	0.0056131	0.0557960
	20	0.1908387	0.0151081	0.0020499	0.0422774
	40	0.1838692	0.0073916	0.0009363	0.0283209
	80	0.1787471	0.0039085	0.0005195	0.0211283
	160	0.1331865	0.0021549	0.0003182	0.0167277

6.2. Real data modeling

The real data represent the crude birth rate (CBR) (per 10000 people) in Iraq, that is available at the Ministry of Health-Republic of Iraq, Annual Statistical Report 2022. The CBR data related to 18 governorates are:

"2.33, 2.9, 2.16, 2.92, 2.3, 2.53, 2.67, 2.61, 3.2, 2.33, 2.67, 2.56, 2.94, 2.03, 2.95, 3.36, 3.05, 2.09".

The descriptive statistics of CBR indicate that the data is right-skewed and platykurtic, with skewness and kurtosis coefficients equal to 0.08 and -0.90, respectively. The EVIL fitting is compared to five extended IL distributions belonging to different common flexible generator families related to [1-5]. Accordingly, the competitive distributions are Beta Inverse Lomax (BIL),Kumaraswamy Inverse Lomax (KuIL), Exponentiated Generalized Inverse Lomax (EGIL), Weibull Inverse Lomax (WIL), and Gompertz Inverse Lomax (GoIL). The negative log-likelihood (NLL) with the following Information Criteria (IC):
Bayesian (BIC), Hanan and Quinn (HQIC), Akaike (AIC), and Consistent Akaike (CAIC) are employed to compare the fitted competitive distributions with the following mathematical forms:

$\text{BIC} = p\ln(n) - 2\ell$;
$\text{HQIC} = 2p\ln(\ln(n)) - 2\ell$;
$\text{AIC} = 2p - 2\ell$; and
$$\text{CAIC} = \frac{2np}{n-p-1} - 2\ell\,.$$
where
n: sample size,
p: number of model's parameters, and
ℓ: natural log-likelihood function assessed at MLEs.
The model with the lowest values for the considered ICs is the best data fitting. It is essential to point out that all of the fitted competitive distributions' p-values for the test (K-S) (Kolmogorov-Smirnov) goodness of fit are more than 0.05 (significant value), and the traditional IL is also included in the comparison process. The outcomes of IC and MLE values are displayed in Table 3. Comparing EVIL to the other six competitive distributions, evidence that IC′s lowest values are linked with it, making it the most appropriate distribution for modeling CBR data.

Table 3. Values of the MLEs and ICs of CBR data

Dist.	MLEs				ICs				
	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\delta}$	NLL	BIC	HQIC	AIC	CAIC
EVIL	0.733	4.215	0.233	8.066	8.008	27.577	24.507	24.015	27.092
BIL	24.833	60.416	0.299	11.413	8.057	27.676	24.606	24.114	27.191
KuIL	16.163	189.16	0.056	16.163	9.825	31.211	28.140	27.649	30.726
EGIL	42.12	18.032	0.951	8.309	8.308	28.178	25.107	24.616	27.693
WIL	5.344	0.300	0.575	7.207	8.085	27.732	24.661	24.170	27.247
GoIL	0.01	12.611	0.181	13.467	8.654	28.870	25.799	25.308	28.385
IL	---	---	0.367	7.099	36.69	79.172	77.636	77.391	78.191

7. Conclusions

A novel version of the extended inverse Lomax distribution is introduced. Several of its essential mathematical and statistical features are presented. Closed forms of quantile and cumulative distribution functions, with increasing, decreasing, and bathtub hazard functions, are among the most important mathematical features of this new distribution. Parameter estimation is conducted by the maximum likelihood method and assessed through a simulation study. The average estimates and mean squared errors are computed. The simulation's outcomes revealed that the estimates were consistent and flexible. The distribution's capability is also investigated to model real-life Iraqi crude birth rate data. The proposed distribution's performance surpasses that of the traditional exponential distribution and exponential baseline related to families of Weibull-G, exponentiated generalized-G, and Gompertz-G in terms of information criteria and goodness of fit. Consequently, in contexts with real-life data, this distribution is recommended over comparable distributions for modeling data. Future directions include exploring additional statistical features of this new distribution, such as the stress-strength reliability model, as well as broadening its application to various sampling techniques, such as ranked set sampling and median ranked set sampling. As advanced steps toward investigating the statistical inference of the parameters through other estimation methods, further research could expand the scope to include ordinary least squares, weighted least squares, and Bayesian methods.

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