Al-Nahrain Journal of Science
ANJS, Vol. 28(3), September, 2025, pp. 125 – 129

Small Quasi Primary Submodules

Adwia J. Abdul-Alkalik

[1] Directorate General of Education in Diyala, Ministry of Education, Diyala, Iraq.

Article's Information

Received: 18.02.2025
Accepted: 24.06.2025
Published: 15.09.2025

Keywords:
Quasi Primary Submodules,
Small Primary Submodules,
Small Quasi Primary Submodules

Abstract

In this research, a new concept (small quasi-primary submodules) has been presented, which is a generalization of a previously studied concept (quasi-primary submodules) and a comprehensive study of it in terms of features and properties. We have also given some examples and observations about it. Its relationship with previously studied concepts was studied, and the equivalence, under certain conditions, between these concepts and the new concept was demonstrated.

http://doi.org/10.22401/ANJS.28.3.16
*Corresponding author: adwiaj@yahoo.com

1. Introduction

The concept of primary submodules has been previously studied by several researchers and is presented in sources [1], [2]. This concept was generalized by a group of researchers, for example, A. J. Abdul-Alkalik in 2005 [2]. A generalization was presented called quasi-primary submodules. And the second generalization by the same researcher was called small primary submodules. In this research, we have generalized the concept of quasi-primary submodule, and its definition is as follows: we call a non-zero submodule $\boldsymbol{L}$ of $\boldsymbol{K}$ is **small** quasi-primary if $[\boldsymbol{L}:<\boldsymbol{k}>]$ is a primary ideal of Q, for each submodule $<\boldsymbol{k}>\ll \boldsymbol{K}$. In the third section of this research, we have demonstrated important issues and theorems that describe the most important properties of this type of partial measurements.

2. Basics of Research

Definition (2.1) [2]: A proper submodule L of Q-module K is called a quasi-primary if $[L:K]$ is a primary ideal of Q, for each submodule L of K.

Definition (2.2) [3]: a submodule L of $Q-module\ K$ small primary submodule if whenever $e \in Q, k \in K, <k>$ is small in K and $ek \in L$, then either $k \in L$ or $e \in \sqrt{[L:K]}$.

Definition (2.3)[4]: A submodule L of K is called small (notationally, $L \ll K$) if $L + H = K$ for all submodules H of K implies $H = K$.

Definition (2.4) [5]: A proper submodule L of an Q-module K is called a quasi-small prime if $[L:<l>]$ is a prime ideal of Q, for each submodule $<l>\ll K$.

Definition (2.5) [6]: An Q-module K is called a hollow module if every non-zero submodule of K is small in K .

Definition (2.6) [7]: A proper submodule C of an Q-module K is called a small prime if whenever $a \in Q, l \in K$ with $<l>\ll K$ such that $al \in C$ implies either $l \in C$ or $a \in [C:K]$.

Definition (2.7) [9]: An Q -module K is called a multiplication if for each submodule L of K there is an ideal I of Q such that $L = IK$.

Proposition (2.8) [9]: A module M of R is multiplication if every non-empty submodule B of M such that $B = (B:_R \mathrm{M})\,\mathrm{M}$.

3. Small Quasi Primary Submodules

Definition (3. 1): A proper $\boldsymbol{L}$ of Q- module $\boldsymbol{K}$ is **small** quasi primary if $[\boldsymbol{L}:<\boldsymbol{l}>]$ is primary ideal of $\boldsymbol{Q}$, for each ***small submodule*** $<\boldsymbol{l}>$; that is whenever $\boldsymbol{xyl} \in \boldsymbol{L}$ for $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{Q}$ and $\boldsymbol{l} \in \boldsymbol{K}$ such that $<\boldsymbol{l}>\ll \boldsymbol{K}$, then either $\boldsymbol{xl} \in \boldsymbol{L}$ or $\boldsymbol{y}^{\boldsymbol{m}}\, \boldsymbol{l} \in \boldsymbol{L}$ for some $\boldsymbol{m} \in \boldsymbol{Z}_+$.
A proper ideal $\boldsymbol{C}$ of $\boldsymbol{Q}$ is small quasi-primary if $\boldsymbol{C}$ is a small quasi-primary submodule of ***an*** $\boldsymbol{Q}$-module $\boldsymbol{Q}$.

Remarks And Examples (3.2):

1. Every quasi-small prime submodule is small quasi-primary, but the converse is not true,

for example: $(\bar{0})$ is small quasi-primary in Z_8, but not a quasi-small prime because $(\bar{2}) \ll Z_8$ and $[(\bar{0}):(\bar{2})] = 4$ Z is not a prime ideal of Z.

2. $(\bar{0})$ is small quasi-primary in Z_{32}, as $(\bar{8}) \ll Z_{32}$ and $[(\bar{0}):(\bar{8})] = 4Z$, a primary ideal .
3. Assume $K = Z$ is a Z-module; then each non-zero submodule L of Z is small quasi-primary. Since $< 0 >$ is the only small submodule of Z.
4. All primary submodules are small quasi-primary. However, the converse is not true. For example, if we consider $L = 30Z$, it is evident that L is not primary. But it is a small quasi-primary by (3).
5. Every quasi-primary submodule is small quasi-primary. However, the converse is not true. For example, consider $K = Z_{12}$ as a $Z-$module, $L = (\bar{6})$ is small quasi-primary (since L is quasi-small prime by [5]. However L is not quasi quasi-primary submodule of , since $2.3.\bar{1} \in L$, but $2.\bar{1} \notin L$ and $3^m.\bar{1} \notin L$, *for each* $m \in Z_+$.
6. If K is a hollow Q-module, then every small quasi primary submodule L of a module K is quasi-primary submodule.

Proof:
As L is a small quasi primary in K, $[L: < l >]$ is a primary ideal for each submodule $, < l > \ll K$. *Since* K is a hollow Q-module, each submodules is small and hence $[L: < l >]$is a primary ideal for each submodule of an Q-module K. Thus, L is quasi-primary submodule, [2].

7. If C is a small prime submodule of an Q-module K, then C is a small quasi primary submodule in K .

Proof:
Because C is a small prime submodule, it is also a quasi-small prime submodule, [5]. Therefore, C is a small quasi-primary submodule, (3.2, 1).

8. If C is a small primary submodule of a Q-module K, then it is a small quasi-primary submodule in K .

Theorem (3.3):
Assume that L is a non-zero submodule of module K.These are comparable:
I] L It is a small quasi-primary.
II] for each submodule $U \ll K$ where H, G are ideals of Q, $HGU \subsetneq L$, implies either $HU \subsetneq L$ or $G^m U \subsetneq L$ *for some* $m \in Z_+$.

Proof:
i. $\longrightarrow$ (II): Let $HGU \subsetneq L$. If $HU \nsubseteq L$ and $G^m U \nsubseteq L$ then $\exists\ l, t \in U$, $a \in H$, $b \in G$ and $al \notin L$, $b^m t \notin L$. Since $< l > \subsetneq U \ll K$ and $< t > \subsetneq U \ll K$, it follows $< l > \ll K$ and $< t > \ll K$ as well. However, L is a small quasi-primary, therefore $ab\,l \in L$ and $al \notin L$ imply that $b^m l \in L$. Also, ab t $\in L$ and $b^m t \notin L$ indicate that $at \in L$. Therefore, $HU \subsetneq L$ or $G^m U \subsetneq L$ *for some* $m \in Z_+$.
ii. $\longrightarrow$(I): Let $a, b \in Q$, $< l > \ll K$ such that abl $\in$ L Since. $ab < l > \subsetneq L$, it follows that either $a < l > \subsetneq L$ or $b^m < l > \subsetneq L$. This means that $al \in L$ or $b^m l \in L$. Hence, L is small quasi primary.

Corollary (3.4):
Assume that L is a non-zero submodule of module K.These are comparable:

I] L is small quasi primary.
II] $\forall\ a, b \in Q$, $U \ll K$ such that $abU \subsetneq L$ implies either $aU \subsetneq L$ or $b^m U \subsetneq L$ for some $m \in Z_+$.

Corollary (3.5):

Assume that L be a non-zero submodule of a module K.These are comparable:
I] L is a small quasi primary.

II] If $(u) \ll K$ and H, G is are ideals of Q such that $HGu \subsetneq L$, implies either $Hu \subsetneq L$ or $G^m u \subsetneq L$ *for some* $m \in Z_+$. We may now deliver the following result.

Theorem (3.6):
Let L be a non-zero submodule of a Q- module K.These are comparable:
1. L is a small quasi-primary
2. $\sqrt{[L: < qk >]} = \sqrt{[L: < k >]}$ .For all $< k > \ll K$, $k \in K$, there exists $q \in Q$ such that $q \notin \sqrt{[L: < k >]}$.

Proof:
(1) $\longrightarrow$(2): It is obvious that $\sqrt{[L: < k >]} \subsetneq \sqrt{[L: < qk >]}$. Let $a \in \sqrt{[L: < qk >]}$ for all $q \notin \sqrt{[L: < k >]}$ and $k \in K$, $< k > \ll K$, therefore $a^m qk \subsetneq L$ for some $m \in Z_+$. It follows that $a^m q \in [L: < k >]$ is a primary ideal of Q and $q \notin \sqrt{[L: < k >]}$. Hence $(a^m)^n \in [L: < k >]$. for some $n \in Z_+$. Thus, $a \in \sqrt{[L: < k >]}$. So $\sqrt{[L: < qk >]} \subsetneq \sqrt{[L: < k >]}$. Therefore $\sqrt{[L: < qk >]} = \sqrt{[L: < k >]}$.

(2) $\longrightarrow$(1): Suppose that $a, q \in Q, k \in K$ and $< k > \ll K$ such that $\in [L: < k >]$, and q $\notin \sqrt{[L: < k >]}$. Then $a \in [L: < qk >] \subsetneq \sqrt{[L: < qk >]}$. But $\sqrt{[L: < qk >]} = \sqrt{[L: < k >]}$ by (2). Thus, a$\in \sqrt{[L: < k >]}$. Then L is small quasi-primary .

Proposition (3.7):
Assume K is an Q-module and let I is an ideal of Q with $I \subsetneq annK$. Then L is a small quasi-primary Q-submodule of K, iff L is a small quasi-primary $Q/I-$ submodule of K.

Proof:
Let $\bar{a} \in Q/I$, $k \in K$, $< k > \ll K$ and $\bar{a}k \in L$. But $\bar{a}k = ak$, since $I \subsetneq annK$. As a result, we obtain the desired outcome.

Proposition (3.8):
Assume $\mu: K \longrightarrow H$ be a Q-epimorphism. If L is a small quasi primary submodule of a module H, then μ^{-1}(L) is a small quasi primary submodule of K.

Proof:
We need to show that $[\mu^{-1}(L): < k >]$ is a primary ideal if $0 \neq < k > \ll$ K. Let t and r $\in$ Q, with t r$\in [\mu^{-1}(L) : < k >]$. And thus t r $< k > \subsetneq \mu^{-1}(L)$. Since μ (t r $< k >$) $\subsetneq \mu$ (μ^{-1}(L)), it follows that t r [μ ($< k >$] $\subsetneq L$. Sice μ is an epimorphism, μ (μ^{-1}(L)) = L. But $< k > \ll$ K , so $\mu < k > \ll$ H,[8] . L is small quasi-primary submodule of a module H . Thus, tr $\in$ [L : $\mu(< k >)$] which is a primary ideal. Hence either t $\in$ [L : $\mu(< k >)$] or $r^m \in$ [L : $\mu(< k >)$] for some m$\in$Z+ and so t $\mu(< k >) \subsetneq L$ or r^m $\mu(< k >) \subsetneq L$. Therefore, either $\mu(t < k >) \subsetneq L$ or $\mu(r^m < k >) \subsetneq L$. Thus, either $t < k > \subsetneq \mu^{-1}(L)$ or $r^m < k > \subsetneq \mu^{-1}(L)$. that is either t $\in [\mu^{-1}(L) : < k >]$ or $r^m \in [\mu^{-1}(L) : < k >]$. Hence $[\mu^{-1}(L) : < k >]$ is a primary ideal $\forall\, 0 \neq < k > \ll$ K . Therefore μ^{-1}(L) is a small quasi-primary submodule of K.
We may now deliver the following result:

Proposition (3. 9):
Assume K is a Q-module, S a multiplicative subset of Q, and L is a small quasi-primary of K. Then L_S is a small quasi-primary submodule of K_S .

Proof:
Assume that $a/s \in Q_S$ and $x/t \in K_S$ with $ax/st \in L_S$ such that $(x/t) \ll K_S$. So $\exists\, u \in S$ such that $uax \in L$. However, because $(x/t) \ll K_S$, [7] concludes that $(x) \ll K$. So that $(ux) \ll K$. Since L is small quasi-primary of K. Then either ux $\in$ L or $(a)^m \in [L: K]$ where m $\in$ Z+. Therefore. either $ux/ut = x/t \in L_S$ or $(a/s)^m \in [L{:}K]_S \subsetneq [L_S : K_S]$ for some $m \in Z+$. Therefore. L_S is a small quasi-primary submodule of K_S.

Proposition (3.10):
Assume L is a submodule of Q-module K. Then L is a small quasi-primary submodule of K if and only if $[L{:}_K I]$ is a small quasi-primary submodule of K for each ideal I of Q .

Proof:
To prove $[L{:}_K I]$ is a small quasi- primary submodule of K , we must show that [$[L{:}_K I] : (k)$] is a primary ideal, $\forall\, 0 \neq (k) \ll K$. If $a, b \in Q$ and $a\, b \in [[L{:}_K I] : (k)]$ and suppose a $\notin [[L{:}_K I] : k)]$, then $abkI \subsetneq L$ and $akI \nsubseteq L$. Hence, $abI \subsetneq [L: (k)]$. However, is a primary ideal, which means that a I $\nsubseteq [L: (k)]$. Thus, $b^m \in [L: (k)]$ for some $m \in Z_+$ and $b^m k \subsetneq L \subsetneq [L{:}_K I]$ .So, $b^m \in [[L{:}_K I] : (k)]$ and hence $[L{:}_K I]$ is a small quasi-primary submodule of K . To get the opposite, take $= Q$.

Remark (3. 11):
If L is a small quasi-primary submodule of K, then [L : K] is not a primary ideal of Q. For $[L: (k)]$ example: $K = Z_{24}$ as a Z –module , $L = (\bar{6})$ is small quasi-primary. But $6Z = [L: K]$ is not primary ideal of Z.

Proposition (3.12):
Let L be a non-zero submodule of a faithful finitely generated multiplication Q- Q-module K. Then L is small quasi-primary submodule of K iff [L:K] is a small quasi-primary ideal of Q.

Proof:
$\Rightarrow$ Let $abc \in [L: K]$ where $a, b, c \in Q$ such that $(c) \ll Q$. Then $< abc > K \subsetneq L$. But $(c) \ll Q$, so $< abc > \ll Q$, [8]. Since K is a finitely generated faithful multiplication module, so $cK \ll K$ [8]. But L is a small quasi-primary submodule, it follows that either $acX \subsetneq L$ or b^m cK $\subsetneq$ L for some m $\in Z_+$. .Thus, either $ac \in [L: K]$ or $b^m c \in [L: K]$. Hence $[L: K]$ is a small quasi-primary ideal of Q.
$\Leftarrow$] Let $a, b \in Q, k \in K, < k > \ll K$ such that $abk \in L$, thus, $ab < k > \subsetneq L$. Since K is a multiplication module, so $< k > = IK$ for some idael I of Q. Hence $abIK \subsetneq L$ and so abI$\subsetneq$ [L:K]. But $IK \ll K$, then $I \ll Q$. Since $[L: K]$ is a small quasi-primary ideal of Q, so either Ia $\subsetneq$ [L:K] or $b^m I \subsetneq [L: K]$ for some m $\in Z_+$. Hence either $IaK \subsetneq [L: K]K$ or $b^m IK \subsetneq$ [L:K] X for some m $\in Z_+$. Since K a multiplication module, so either $ak \in L$ or $b^m k \in$ L. Then L is a small quasi-primary submodule of K.

Proposition (3.13):
Let L be a non-zero submodule of a faithful finitely generated multiplication Q- module K. Then $[L:K]$ is a small quasi-primary ideal of Q iff $L = IK$ for some small quasi primary ideal I of Q.

Proof:
⇒ Since [L:K] is a small quasi-primary ideal of Q and K multiplication Q- Q-module, then L= [L:K]K, it follows that L = IK and I= [L: K] is a small quasi-primary ideal of Q.

⇐; Suppose that L = IK for some small quasi-primary ideal I of Q. Since K is a multiplication Q- Q-module, so we have L= [L:K]K= IK. But K is a faithful finitely generated multiplication, then I= [L:K], [9]. Then [L:K] is a small quasi-primary ideal of Q.

Theorem(3.14):
Let L be a non-zero submodule of a faithful finitely generated multiplication Q- Q-module K,.These are comparable:

1. L is small quasi-primary.
2. [L:$_Q$ K] is a small quasi primary ideal of Q.
3. $L = IK$ for some ideal I of Q.

Proof :
(1) ⇔ (2) , by Proposition (2.12).
(2) ⇔ (3) , by Proposition (2.13).

Proposition (3.15):
Let Q be a ring and U,V be two submodules of an Q –module K such that U is not contained in V. If N is a small quasi- primary submodule of K then U ∩ V is a small quasi-primary submodule of U .

Proof:
Since U ⊄ N ,U ∩ V is a proper submodule of U . Let a,q ∈ Q, u ∈ U, (u) ≪ U such that aqu ∈ U ∩ V, then aqu ∈ V. But V is a small quasi-Primary submodule of K and (u) ≪ K ,so either au ∈ V or q^m u∈ V. Since u ∈ U, so either au ∈ U ∩ V or q u∈ U ∩ V. Thus, V ∩ U is a small quasi-primary submodule of U .

Remark (3.16):
Let K_1, K_2 be two Q-modules and let $K = K_1 \oplus K_2$. If L_1, L_2 are small quasi-primary submodules of K_1, K_2 so it is not necessary $L = L_1 \oplus L_2$ is a small quasi-primary submodule of K. For example, Let $K = Z \oplus Z$ as a Z –module, $L = 2Z \oplus 3Z$ is not a small quasi-primary since (0) ≪ K and [L : (0)] = 6Z, which is not a primary ideal of Z. But 2Z ,3Z are small quasi-primary.

Proposition (3.17):
Let L_1, L_2 be two Q-modules, and define L as $L_1 \oplus L_2$.

1- L_1 is a small quasi-primary of K iff $L_1 \oplus K_2$ is a small quasi-primary submodule of K.
2- L_2 is a small quasi-primary of K iff $K_1 \oplus L_2$ is a small quasi-primary submodule of K.

Proof: 1) :
⇒ *Assume* $a, q \in Q$, $k \in K_1$, $< k > \ll K_1$ and $aqk \in L$. Thus, $aq(k+0) \in L_1 \oplus K_2$, and since $< k > \ll K_1$, then $< k+0 > \ll K$, [8]. However, since $L_1 \oplus K_2$ is a small quasi-primary submodule of K, either $a(k+0) \in L_1 \oplus K_2$ or $q^m(k+0) \in L_1 \oplus K_2$ *for some* $m \in Z_+$. Thus, either a $k \in L_1$ or q^m ($k \in L_1$. Therefore, L_1 is a small quasi-primary of K.

⇐; Assume a, b ∈ Q, k ∈ K, *and* $< k > = < p > \oplus < y > \ll K$, p ∈ L_1 and y ∈ L_2 so that ab ∈ [$L_1 \oplus K_2$: $< k >$]. Since k = p+y, ab((p+y) ∈ $L_1 \oplus K_2$. Hence, abp + aby = q+ z, for some q∈ L_1 and z ∈ K_2 . Thus, abp _ q =z_ aby ∈ $K_1 \cap K_2 = 0$, so abp =q ∈ L_1 and $< p > \subsetneq L_1$, which is a direct summnd of K, then $< p > \ll K$. But L_1 is a small quasi-primary ofK_1 , then either a p ∈ L_1 or $b^m p \in L_1$. So, either ak = a (p+y) ∈ $L_1 \oplus K_2$ or $b^m k = b^m(p+y) \in L_1 \oplus K_2$ *for some* $m \in Z_+$. Hence, $K_1 \oplus L_2$ is a small quasi-primary submodule of K. A similar method can be used to show (2).

Proposition (3.18):
Let K_1 *and* K_2 be two Q-modules, and set $K = K_1 \oplus K_2$. If $L = L_1 \oplus L_2$ is a small quasi-primary submodule of K, then L_1 and L_2 are small quasi-primary of K_1 and K_2 ..

Proof:
Assume $t, y \in Q$, $k \in K_1$, $< k > \ll K_1$ and $ty < k > \subsetneq L_1$. Then $ty < k > \oplus 0 \subsetneq L_1 \oplus L_2$ and so $ty \in [L_1 \oplus L_2 : < k > \oplus 0]$. However, since $(k) \ll K_1$ and $(0) \ll K_2$, states that $(k, 0) \ll K_1 \oplus K_2$ by [8]. But $L_1 \oplus L_2$ is a small quasi-primary submodule of K. Therefore, $t \in [L_1 \oplus L_2 : < k > \oplus 0]$ or $y^m \in [L_1 \oplus L_2 : < k > \oplus 0]$ *for some* $m \in Z_+$. Thus, either t ($< k > \oplus 0) \subsetneq L_1 \oplus L_2$ or y^m ($< k > \oplus 0) \subsetneq L_1 \oplus L_2$ and thus either (tk, 0) ∈ $L_1 \oplus L_2$ or $(y^m k, 0) \in L_1 \oplus L_2$. Thus, either tk ∈ L_1 or $y^m k \in L_1$ implying that t ∈ $[L_1 : k]$ or $y^m \in [L : k]$ for some $m \in Z_+$. Hence, $[L_1 : k]$ is a small quasi-primary ideal. Therefore L_1 is a small quasi-primary of K_1.

L_2 is a quasi-small primary of K_2 by a similar proof.

4. Conclusions
The most important findings in this research are the following properties and theorems: All (primary -

quasi-small prime quasi primary - small primary) submodules are small quasi primary. And L is small quasi primary iff for each submodule $U \ll K$ where H, G are ideals of Q, $HGU \subsetneq L$, implies either $HU \subsetneq L$ or $G^m U \subsetneq L$ *for some* $m \in Z_+$. Also, L is a small quasi-primary iff $\sqrt{[L: <qk>]} = \sqrt{[L: <k>]}$ .For all $<k> \ll K$, $k \in K$, there exists $q \in Q$ such that q$\notin \sqrt{[L: <k>]}$.

Acknowledgments: I would like to thank Allah first for enabling me to write this research, and secondly, the journal's staff for helping me in many matters since receiving my research.

Conflict of Interest: The author confirms that there are no conflicts of interest.

Funding: No funds have been received for this work.

References

[1] Smith, P. F.; "Primary module over commutative rings". Glasgow J . Math. , 43(1): 103-111, 2001.
[2] Adwia , J. A. ; "Primary Modules ". M. Sc, Thesis, College of Education for Pure Science Ibn -Al-Haitham , University of Baghdad, Baghdad, 2005.
[3] Adwia , J. A. ; "On Small Primary Modules". Iraqi J. Sci., 62(4): 1307-1313, 2021.
[4] Inoue, T. ; " Sum of Hollow Modules". Osaka J . Math. 20 (2), 331- 336,1983.
[5] Nuhad , S. ; Husain, A. ; " On Quasi- Small Prime Sumodules". Iraqi J. Sci., 63(4): 1692-1699, 2022.
[6] Patrick, F. ;" Hollow Modules and Local Endomorphism Rings". Pac J Math. , 53 (2): 379-385 , 1974.
[7] Layal, S. M. ; "Small Prime Modules and Small Prime Submodules". Al-Nahrain J. . Sci. , 15(4): 191-199, 2012.
[8] Iman, A. , I. ; "Some Generalization of Projective Modules". Ph. D,. Thesis, College of Science, University of Baghdad, , Baghdad, 2004.
[9] Smith, P. F.; "Some Remarks on Multiplication Modules ". Arch J. Math. , 50 (3): 223-235, 1988.